FREQUENCY DISTRIBUTIONS IN
BIOMOLECULAR SYSTEMS AND GROWING
NETWORKS

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A basic topic of any statistical inference of biomolecular systems is characterization of the distributions of object frequencies for a population - so called frequency distributions. Based on huge datasets of such systems several common statistical facts on frequency distribution have been discovered. From the mathematical point of view these are: skewness to the right; regular variation at infinity; upward/downward convexity; continuity by parameters (stability); unimodality, etc. of frequency distributions.

Some well-known distributions are widely used in large-scale biomolecular systems. But the variety of such systems requires to generate new ones that satisfy the empirical facts above. There are two ways to obtain new useful frequency distributions for such systems:

1. To construct them (statistically, or based by intuition);

2. To use well-justified stochastic models that describe the behavior of such systems and derive corresponding distributions.

In the first way, many statistical distributions appear (Power Law, Pareto, multivariate, binomial, Poisson differential, etc.).

The basis of the second way is the standard stochastic birth-death process with various forms of coefficients.

The present investigation is devoted to mathematical analysis of common statistical properties of frequency distributions that arise in biomolecular systems. These properties have been mathematically described. Based on statistical facts we suggest new frequency distributions for the needs of large-scale biomolecular systems. Both of the techniques mentioned above will be used.

From the point of view of common statistical facts about frequency distributions in biomolecular models we focus on the class of so-called stable laws. The notion of stable law has been introduced in monograph of P. Levy ”Calcul des probabilities” (1925). Only after the publications of B. Mandelbrot in 1956-1967 it has become clear that stable laws have applications in biology (and economics).

First we show that three-parametric Right-side stable laws generate densities which can be considered as continuous approximations of frequency distributions in biomolecular systems.

Second, considering the structure of growing biomolecular networks we notice that many of them behave as follows. They can be divided into ”fractals” with same probabilistic nature. On these ”fractals” the local frequency distributions of corresponding random variables on objects’ appearance are distributions of ”weakly dependent”, identically distributed random variables. So, the number of objects here is represented as growing sums of identically distributed random variables. That is why the limit results of the theory of sums of independent, identically distributed random variables may be applied in order to estimate the frequency distribution of such growing networks. Indeed, in the theory of sums of identically distributed random variables
the limit results in cases of independent and different type weakly dependent random variables, as a rule, are the same.

In order to derive the stable approximation of frequency distributions in growing networks with the help of limit theorems for sums of independent, identically distributed random variables we need to find the norming and centring constants. Here there are two cases.

If the frequency distribution on "fractals" is known, then according to general theory of limit theorems one only needs to solve the computational problem of moments, or to find some solutions of functional equations for this frequency distribution.

If the frequency distribution is unknown, then the problem is statistical. We estimate the exponent of regular variation of the unknown frequency distribution and some of the parameters.

Thus, we described our approach on stable approximation of frequency distributions in biomolecular systems which is related with the first way of obtaining of new frequency distributions.

The basis of the second way is the standard stochastic birth-death process with various forms of coefficients. In this way Yule, Waring, Lotka-Zipf, etc. distributions have been obtained for biomolecular applications. There are many possibilities of the standard birth-death process for biology that have not been discovered previously.

In the recent investigation of J.Astola and E.Danielian "Regularly varying skewed distributions generated by birth-death process" (2004) an attempt has been made to expand the list of well-known frequency distributions that are obtained as the stationary distributions of standard birth-death process with different forms of coefficients.

In the present work we continue to discover new possibilities of birth-death process for large-scale biomolecular systems. In this way, we find out a large class of skewed, regularly varying, convex, etc. distributions which include all the known ones obtained by this way and satisfy the statistical facts above. But these new distributions do not always have simple "enough" expressions that would be suitable for applications. In order to solve this problem of simplification of resulting frequency distributions we suggest so-called dediscretization approach. It allows to construct continuous, very "smooth" (even infinite differentiable) analogs of the before obtained frequency distributions.

This problem is related with interpolation problems for regularly varying functions. That is why we include description and solutions of some such new problems in the present work.

After construction of analogs of birth-death process’ stationary distributions forming the class of suggested frequency distributions we found that it is possible to derive simple approximations of analogs which also satisfy known statistical facts. Now, the reverse operation (discretization) in some cases transforms simple approximations into simple-form distributions which can be suggested as frequency distributions in biomolecular systems.

Finally note that the continuous analogs and their approximations may be considered independently of the birth-death process.

Having a "rich" class of frequency distributions on "fractals" of growing biomolecular networks obtained as it was described above by the second way we return to the stable approxima-
tion. It requires a study of some parameters of these frequency distributions and their analogs.

For the obtained new class, in particular, it is necessary to check out the fulfilment of common statistical facts, to introduce different skewness measures and be able to compare the skewness of any two different distributions. These problems are also in the center of our attention in the present investigation.

We devote this book to Helena Astola who has typeset the manuscript. The typesetting of the text has been a very demanding task because of the large number of complex formulas. Moreover during the preparation we have made numerous changes that have required rewriting and she has always been ready to quickly implement them.

Jaakko Astola  
Eduard Danielian
0.2 Introduction

A basic topic of any statistical interference of evolutionary large-scale complex biomolecular systems, including biomolecular networks with growing size over the time, is characterization of the distributions of object frequencies for a population, i.e. so-called frequency distribution.

The great variety and diversity of such systems do not allow to figure out and suggest a universal approximation for the frequency distribution, say \( \{p_n\} \), i.e. suggest a universal model, which might be suitable in all possible situations.

Based on huge datasets of great number of large-scale biomolecular systems it has been possible to extract only some common information (statistical facts) being applicable almost to all situations for empirical frequency distributions.

Below we deal with random variable \( \xi \) which takes the value \( n, n = 0, 1, 2 \) with probability \( p_n \). So, \( p_n \geq 0, n = 0, 1, 2, \ldots \), and \( \sum p_n = 1 \).

Let us introduce common statistical facts on \( \{p_n\} \).

1. \( \{p_n\} \) has a skew to the right.
2. \( \{p_n\} \) shows a power law behavior as \( n \to +\infty \).
3. \( \{p_n\} \) satisfies some convexity properties.
4. There is only one index \( n \) such that \( p_n > p_{n-1} \) and \( p_n > p_{n+1} \), i.e. \( \{p_n\} \) is unimodal.

Any distribution satisfying the statistical facts above has a chance to be approved by biologist in order to be applied, at least, in one among great variety of large-scale biomolecular systems.

There exists imagination, heuristic faith on ”smoothness” of continuous analogs of in quest of desired frequency distributions, which biologists try to check out experimentally. So, the next requirement on \( \{p_n\} \)’s continuous analog is its ”smoothness”.

In Biology these statistical facts are not explained from the mathematical point of view. The conceptions of a power law behavior and convexity are understood based on log-log plot of \( \{p_n\} \) (log \( p_n \) versus log \( n \)). The conception of skewness in biomolecular systems is based on intuition. Its quantitative aspects are not even exploited. Some measures of skewness for couple of concrete empirical frequency distributions are declared (Power Laws, Pareto Distributions).

Many statistical frequency distributions in macroevolution theory are proposed. Mostly they are found around a Power Law which plays an essential role, in particular, in self-organized large-scale biomolecular systems because of its scale-invariant property.

In self-organized systems, particularly, in growing biomolecular networks knowing the frequency distribution in some fractals we may derive the form of distribution for the whole system. The scale-invariant property gives such a possibility.

But the same possibility gives also so-called semi-group property.

The large subclass of distribution functions from the class of distribution functions with densities satisfying semi-group property is four-parametric family of Stable Laws. According to
fact 1. three-parametric family of right-side Stable Laws attracts our attention. So, right-side Stable Laws whose densities satisfy all statistical facts have a good chance to be suggested for approximation of \( \{p_n \} \) with the help of their densities.

This is the first central idea being developed in the present investigation.

Attempts on discovering the mechanism of biomolecular systems’ functioning is in progress. They are based on standard birth-death stochastic process with various forms of intensities (coefficients). The stationary distributions of this process are considered as approximation of empirical frequency distributions. In this way families of Yule, Waring, Lotka-Zipf, Kolmogorov-Waring distributions have been obtained. Not always the suggested stationary distributions of the birth-death process are applicable for approximation of frequency distributions in biomolecular systems. For instance, the family of (Kolmogorov-Waring distributions) - (Waring distributions) introduced by V.Kuznetsov does not satisfy the statistical fact 2.

In the recent investigation of J. Astola and E. Danielian "Regularly varying skewed distributions generated by birth-death process" (2004) the successful attempt has been made to expand the list of before known frequency distributions obtained as the stationary distributions of birth-death process with general assumptions on coefficients.

In the present work we continue to discover new possibilities of birth-death process for large-scale biomolecular systems being unknown before. In this way we find out a large class of skewed, regularly varying, convex, etc. distributions which include all before known ones obtained as stationary distributions of the birth-death process with general assumptions on coefficients.

This is the second central idea being developed in the present investigation.

But the obtained new frequency distributions often are not simple ”enough”, or otherwise suitable for biomolecular applications expressions. So, the problem of simplification of the frequency distributions’ forms arises. One of the ways of simplification suggested by us is the dediscretization approach. Sometimes special constructions of continuous analogs of frequency distributions lead to more simple expressions. By the dediscretization approach it is possible to construct very ”smooth” (even infinite differentiable) analogs - densities of the before obtained frequency distributions. Now, for the analogs’ study instead of combinatorial methods one may use the effective tool of Mathematical Analysis.

The problem of simplification of frequency distributions by using the dediscretization approach is related with different interpolation problems for regularly and slowly varying functions. Thus the dediscretization and solutions of such interpolation problems are needed also.

This is the third central idea being developed in the present investigation.

The material in this work is located as follows.

In Chapter 1, we analyze from the mathematical point of view the known statistical facts on empirical frequency distributions. We discuss the skewness conception for finite-parametric families of frequency distributions on examples of Pareto and Waring distributions. In order to compare the skewness of two distributions preliminary we need a conception of Regular Variation.
In particular, the power law-like behavior is interpreted for the frequency distribution \( \{p_n\} \) as regular variation at infinity with some exponent \((-\rho), \rho \in [1, +\infty)\).

Next, we consider the convexity properties of \( \{p_n\} \) based on log-log plot of \( \{p_n\} \). The Stability conception is investigated on examples of two known families of frequency distributions mentioned above.

In Chapter 2, we give necessary information on Stable Laws’ definition and properties to prepare the reader to understand the first central idea.

Stable densities concentrated on \([0, +\infty)\) vary regularly at infinity with exponent \((-\rho), \rho \in (1, 2)\). Then their distribution functions have only (right) tail which varies regularly at infinity with exponent \((-\rho + 1)\). They satisfy all other known statistical facts and at once may be suggested as infinite differentiable analogs of empirical frequency distributions in large-scale biomolecular models. Other right-side stable densities are concentrated on \((-\infty, +\infty)\). But their left tails are extremely small in comparison with the right tails. Therefore we may give a manner how to construct with the help of them infinite differentiable analogs of \( \{p_n\} \).

But more perspective is to use three-parametric family of symmetric stable densities in order to construct infinite differentiable analogs of \( \{p_n\} \). It is possible if we consider the random variables being the absolute values of symmetrically distributed stable random variables.

Chapter 3 includes a substantiation of a possibility of approximation by standard Right-side Stable Law, as a result of Limit Theorems for sums of independent, identically distributed random variables, to a frequency distributions of events’ occurrence numbers for large class of growing biomolecular networks. In this the Problems of General Theory of Stable Laws that are essential to us are formulated for our purposes and the solutions to them are given. The limit distributions are described by their Canonical Representations (in distinction to Chapter 2, where they are characterized by series expansions of stable densities) in terms of Laplace-Stieltjes Transforms. Taking into account that this transform exists for Right-side Stable Laws, the Method of Laplace-Stieltjes Transforms in Limit Theorems is developed for a case, when the frequency distribution in fractals of growing network belongs to the domain of normal attraction of standard Right-side Stable Laws.

Having the theoretical background the stable approximation is applied to well-known in bioinformatics frequency distributions: Power Laws, Pareto, Waring Distributions. This application requires the knowledge of some characteristics of considering distributions. That is why, in particular, a manner of evaluation of moments of Waring Distributions is suggested.

In case when a frequency distribution on fractals of growing network is not known, the arising statistical problems are discussed.

Chapter 4 is devoted to new empirical frequency distributions discovered by using the mechanism of standard birth-death process with various forms of coefficients. We start from assumptions of moderate growth of coefficients which are the most general in bioinformatics. The assumptions of all known birth-death biomolecular models are included into suggested ones. The obtained stationary distributions are verified from the point of view of general statistical
facts being extracted in bioinformatic from various datasets.

A class of regularly varying stationary distributions is found and its properties are deeply investigated.

It is important to notice that the arising class of distributions includes a family of Hypergeometric Distributions introduced and carefully discussed in this Chapter. It appears that the family of Waring Distributions is a particular case of introduced family. For regularly varying Hypergeometric Distributions a Generating Function, factorial moments, mean value, and variance are evaluated.

In Chapter 5 a new dediscretization approach is suggested which allows to derive infinite differentiable analogs of discovered in Chapter 4 frequency distributions. The advantages of this approach are verified on several examples. Namely, two typical examples of regularly varying at infinity sequences of frequency distributions with exponent \((-1)\) are considered. The linear and asymptotically linear cases of coefficients of generating the considering frequency distributions standard birth-death process are investigated in detail.

The substantiation of dediscretization procedure is done: we show that the obtained infinite differentiable analogs are proper distribution functions, vary regularly at infinity and posses some convex properties.

Next, the obtained class of infinite differentiable analogs is enlarged conserving the most important properties. It allows to make further reducement of the class, which leads to new very simple form distribution functions. These distribution functions are not included in the initial class. It at once gives new frequency distribution with simple form, which may be suggested for applications in biomolecular systems.

The last Chapter 6 includes information on selected topics on Regularly Varying Functions. We make the reader familiar with the main problems of the theory, especially with the problems which are connected to the contents of this monograph. All formulated result either are new and proved by authors or are known but the methods of their establishment are new.

In particular, Tauberian Theorems for Laplace-Stieltjes Transform and for Generating Function are proved.

New type of Representation Theorem for Slowly Varying Functions is introduced. It leads to a new Interpolation Theorem, which allows to solve the interpolation problem arising during the dediscretization procedure realization in Chapter 5.

In the text Chapters, Definitions and Figures are numerated by only one number. Sections, Theorems, Lemmas, Corollaries, Remarks have double numeration: they are characterized by a number of Chapter and by additional number showing their location inside a Chapter. The triple numeration posses Subsections and Formulas: these are double numeration of Section and the third additional number which characterizes their location inside a Section.
Chapter 1

Mathematical Interpretation of Statistical Facts in Biomolecular Systems

1.1 Regularities of Biomolecular Systems

Statistics of events in a wide class of physical, biological, social and engineered large-scale systems share frequency distributions having the following characterization in common.

\[
\text{There are few redundant and many rare classes.} \quad (1.1.1)
\]

This and many other regularities in application to the large-scale biomolecular systems, in particular, to the growing biomolecular networks are the base on which the frequency distributions are chosen and mathematical models are constructed in the present investigation.

As it was already mentioned a basic topic of any statistical or probabilistic (stochastic) inference of evolutionary large-scale biomolecular systems consists in characterization of different events’ frequency distribution. All regularities must be reflected in the form of empirical frequency distribution, and, in reverse - follow from this form.

Everywhere we denote by \( \{p_n\} \) with \( p_n \geq 0, \ n = 0, 1, 2, \ldots, \sum_{n=0}^{\infty} p_n = 1 \), the empirical frequency distributions.

1.1.1 Regularities

In general, the development of any evolutionary large-scale complex biomolecular system is a result of two fundamental phenomena (regularities).

Those are: Darwin natural selection; Random mutation.
In addition to these both regularities one may mention: The adaptivity; The robustness; The diversity of large-scale biomolecular systems in the observed biological complexity growth around us.

The wide diversity of such systems doesn’t allow to suggest universal mathematical model explaining the mechanism of their dynamics.

Next, the important regularity of many large-scale biomolecular systems is their self-organization, which is partly responsible for their observed biological convexity growth, as well as adaptivity and robustness. Since self-organization is important for future consideration, therefore some information on it is necessary. The self-organization conception was transferred from the phase transition systems theory to biology phenomena [1-3]. Such systems spontaneously self-organized themselves in fractals.

We say that the random variable $\xi$ of events exhibits the Power Law $\{p_n\}$ if

$$p_n = P(\xi = n) = c(\rho)n^{-\rho}, \quad 1 \leq \rho + \infty, \quad n = 1, 2, \cdots$$

where the normalization factor $c(\rho)$ takes the value

$$c(\rho) = \left(\sum_{n \geq 1} n^{-\rho}\right)^{-1}$$

and $P$ denotes probability. The lower bound of parameter $\rho$ in (1.1.2) equals to 1. It is quite understandable because already for the value $\rho = 1$ the series at the right-hand-side of (1.1.3) diverges

$$\sum_{n \geq 1} n^{-\rho} \text{ (harmonic series)}.$$  

A Power Law (1.1.2)-(1.1.3) was found to describe various events in the vicinity of critical points in physical and chemical phase transition systems.

Generalizing such critical systems observation, S.Kauffman [1] applied the self-organization conception to large-scale biomolecular systems as one of their regularities playing an essential role. For such systems Power Law is of interest also because of its genetic property in the case of self-organization, which can be explained as follows. The selection process cannot avoid the order exhibited by most members of the system. Here it is of importance that Power Law is scale-invariant, implying that the knowledge of statistical properties of any part of the complex system allows to extrapolate these properties to whole system.

The scale-invariant property means that the replacement of variable $n$ in $p_n$ by a new variable $m = s \cdot n$ with arbitrary positive integer $s$ doesn’t change the functional form of frequency distribution. For Power Law we have

$$p_m = \frac{1}{c(\rho)} p_s \cdot p_n \text{ (see, (1.1.2))}.$$
1.1.2 Statistical Facts on Frequency Distribution

Let us talk about a known statistical fact especially related with empirical frequency distributions. The observed on various data sets of large-scale biomolecular systems the frequency distributions combine several specific properties (regularities). These are: A long left tail; Skewness to the right; Monotonically/convex skewed shape; Shape with only one maximal point; etc. (see, [4]).

Note that, for instance, the specific regularity (1.1.1) in large-scale biomolecular systems has been established for connectivity number of metabolic networks [5], for the rates of protein synthesis in protein sets of prokaryotic organisms [6], for the expressed genes in the eukaryotic cells [7-8], etc.

The conception of skewness (see, (1.1.5)) in biomolecular systems until now is based on intuition and on the shapes of graphs of empirical frequency distributions. The quantitative aspects of the skewness conception are not even exploited. There are only some declared measures of skewness for some concrete empirical frequency distributions.

The Power Law is very popular in biology. But in reality the Power Law may not always be appropriate. As it was mentioned [4], the log-log plot of most empirical frequency distributions \( \{p_n\} \) systematically deviated from the straight line and show the upward/downward convexity [5], [9-14].

The last sentence needs no explanation.

In the log-log plot (log \( p_n \) versus log \( n \)) a Power Law asymptotically for large values of argument is represented by the straight line. Indeed, due to to (1.1.2),

\[
\frac{\log p_n}{\log n} = (-\rho) + \frac{\ln c(\rho)}{\log n} = (-\rho) + o(1), \quad n \to +\infty.
\]

If the last mentioned sentence is true, then the empirical frequency distribution doesn’t coincide with the Power Law.

Based on data from many large-scale biomolecular systems the following conclusion has been made [5], [9-11], [13], [15]:

The frequency distribution exhibits the power law behavior for large values of argument.

This statistical fact has been interpreted in mathematical sense in [16].

1.2 The Classical Frequency Distributions

In this Section we present known families of frequency distributions. They play a role of examples on which common statistical facts shall be improved.
Sometimes it is more convenient to deal with continuous analogs of frequency distributions in terms of probability density functions. In case of known frequency distributions considered below the construction of such densities is not complicate.

### 1.2.1 The Power Laws

The one-parametric family of Power Laws is defined by formulas (1.1.2)-(1.1.3). For the Power Law (1.1.2)-(1.1.3) its continuous analog-density \( f(x) \) takes the form

\[
f(x) = \hat{c}(\rho) \cdot x^{-\rho}, \quad 1 < \rho < +\infty, \quad 1 \leq x < +\infty,
\]

where the normalization factor \( \hat{c}(\rho) \) is determined from the equality for densities, \( \int_{-\infty}^{\infty} f(x)dx = 1 \). This equality in our case, due to (1.2.1), gives

\[
\hat{c}(\rho) \cdot \int_{1}^{\infty} \frac{dx}{x^\rho} = \frac{\hat{c}(\rho)}{\rho - 1} = 1,
\]

or

\[
\hat{c}(\rho) = \rho - 1.
\]

The scale-invariant property (1.1.4) and the following one taken separately are characterization properties for the Power Law.

For any \( m = 1, 2, \ldots \) and \( n = 1, 2, \ldots \)

\[
\frac{p_m}{p_n} = m^{-\rho} \text{ doesn’t depend on } n.
\]

For the Power Law’s continuous analog (1.2.1)-(1.2.2) the property (1.2.3) transforms into the following one: for any \( x \in \mathbb{R}^+ \) and \( y \in \mathbb{R}^+ \), \( \mathbb{R}^+ = (0, +\infty) \),

\[
\frac{f(x,y)}{f(x)} = y^{-\rho} \text{ doesn’t depend on } x.
\]

In terms of absolutely continuous \( \{p_n\} \) ‘s analog the scale-invariant property takes the form

\[
f(z) = \frac{1}{\rho - 1} f(x) \cdot f(y),
\]

where \( z = x \cdot y, \quad 1 \leq x < +\infty, \quad 1 \leq y < +\infty \) and (1.2.1)-(1.2.2) are used.

The formulas (1.2.3) and (1.1.4) generate new ideas about fruitful generalizations in different directions for large-scale biomolecular system.

### 1.2.2 Pareto Distributions

Taking into account that the Power Law may not always be appropriate many new statistical frequency distributions have been proposed, [14], [17]-[19], in particular, the following family of Pareto Distributions \( \{p_n\} \) :

\[
p_n = c(\rho, b) \cdot (n + b)^{-\rho}, \quad -1 < b < +\infty, \quad 1 < \rho < +\infty, \quad n = 1, 2, \ldots,
\]
which includes in the case \( b = 0 \) the one-parametric family of Power Laws.

The normalization factor \( c(\rho, b) \) for distribution (1.2.6) takes the form

\[
c(\rho, b) = \left( \sum_{n \geq 1} (n + b)^{-\rho} \right)^{-1} \quad \text{(see, [7])}. \tag{1.2.7}
\]

Similarly to (1.1.2) the lower bound of parameter \( \rho \) in (1.2.6), i.e. the number 1, cannot be included into the range of definition in (1.2.6) because the series at the right-hand-side of (1.2.7) diverges in case \( \rho = 1 \). Obviously, the lower bound of parameter \( b \) in (1.2.6) is not allowed because for \( n = 1 \) from (1.2.6) in case \( b = -1 \) we obtain \( c(\rho, b) = 0 \).

The Pareto Distributions form a two-parametric family of frequency distributions.

For Pareto Distribution (1.2.6)-(1.2.7) its continuous analog-density \( f_b(x) \) takes the form

\[
f_b(x) = \hat{c}(\rho, b) \cdot (x + b)^{-\rho}, \quad -1 < b < +\infty, \quad 1 < \rho < +\infty, \quad 1 \leq x < +\infty, \tag{1.2.8}
\]

where, due to general equality for densities \( \int_1^{+\infty} f_b(x) \, dx = 1 \), from (1.2.8) we have

\[
\hat{c}(\rho, b) = (\rho - 1) \cdot (1 + b)^{\rho - 1}. \tag{1.2.9}
\]

In the case of \( b \neq 0 \) in (1.2.6) the scale-invariant property for Pareto Distributions doesn’t take place. This fact is proved in [8].

### 1.2.3 Waring Distributions

The famous two-parametric family of Waring Distributions [17] takes the form

\[
p_0 = (1 + \sum_{n \geq 1} \prod_{K=1}^{n} \frac{p + K - 1}{q + K})^{-1}, \tag{1.2.10}
\]

\[
p_n = p_0 \cdot \prod_{K=1}^{n} \frac{p + K - 1}{q + K}, \quad n = 1, 2, \ldots, \tag{1.2.11}
\]

where

\[
0 < p < q < +\infty. \tag{1.2.12}
\]

The family of Waring Distributions is a subfamily of the three-parametric family of Kolmogorov-Waring Distributions

\[
p_0 = (1 + \sum_{n \geq 1} \Theta^n \prod_{K=1}^{n} \frac{p + K - 1}{q + K})^{-1}, \tag{1.2.13}
\]

\[
p_n = p_0 \cdot \Theta^n \cdot \prod_{K=1}^{n} \frac{p + K - 1}{q + K}, \quad n = 1, 2, \ldots, \tag{1.2.14}
\]
introduced by V.Kuznetsov [4], where either $0 < \Theta < 1$, $0 < p < +\infty$, $0 < q < +\infty$, or $\Theta = 1$ and (1.2.12) holds.

It is of interest that the probability $p_0$ in (1.2.13) may be represented in the form of two integrals’ ratio, and this ratio in case (1.2.10) is easily calculated. In order to show that let us consider the hypergeometric series (see, 9.100, p. 1039, [20])

$$F(\alpha, \beta; \gamma, z) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} \cdot z + \frac{\alpha \cdot (\alpha + 1) \cdot \beta \cdot (\beta + 1)}{\gamma \cdot (\gamma + 1) \cdot 1 \cdot 2} z^2 + \frac{\alpha \cdot (\alpha + 1)(\alpha + 2) \cdot \beta \cdot (\beta + 1)(\beta + 2)}{\gamma \cdot (\gamma + 1)(\gamma + 2) \cdot 1 \cdot 2 \cdot 3} z^3 + \ldots$$  \hspace{1cm} (1.2.15)

for positive values of arguments. The following statement about the convergence/divergence of the series (1.2.15) is known (see, 9.102, p. 1040, [20]).

A series (1.2.15) converges if $0 < z < 1$. Moreover:

1. The series converges for $1 < z < +\infty$ if $0 \leq \alpha + \beta - \gamma < 1$;
2. The series converges for $1 \leq z < +\infty$ if $\alpha + \beta - \gamma < 0$;
3. The series diverges for $0 < z \leq 1$ if $\alpha + \beta - \gamma \geq 1$.

Finally note that the following integral representation holds for $\gamma > \beta > 0$:

$$F(\alpha, \beta; \gamma, z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1} \cdot (1 - t)^{\gamma-\beta-1} \cdot (1 - tz)^{-\alpha} dt,$$  \hspace{1cm} (1.2.16)

where (see, 9.111, p. 1040, [20]),

$$B(x, y) = \int_0^1 t^{x-1} \cdot (1 - t)^{y-1} dt, \quad x \in R^+, y \in R^+.$$  \hspace{1cm} (1.2.17)

Having this initial information from (1.2.13) and (1.2.15)-(1.2.17) we obtain

$$\frac{1}{p_0} = F(p, 1; q + 1; \Theta) = \frac{1}{B(1, q)} \int_0^1 (1 - t)^{q-1} \cdot (1 - t\Theta)^{-p} dt = q \cdot \int_0^1 (1 - t)^{q-1} \cdot (1 - t\Theta)^{-p} dt,$$

or

$$p_0 = \frac{q^{-1}}{\int_0^1 (1 - t)^{q-1} \cdot (1 - t\Theta)^{-p} dt}.$$  \hspace{1cm} (1.2.18)

In the case of Waring Distributions ($\Theta = 1$, and (1.2.12) holds) from (1.2.18) we have

$$p_0 = \frac{q^{-1}}{\int_0^1 (1 - t)^{q-1}} = \frac{q-p}{q} = 1 - \frac{p}{q}.\text{ Thus,}$$

$$0 < p_0 = 1 - \frac{p}{q} < 1.$$  \hspace{1cm} (1.2.19)

The construction of the continuous analog of $\{p_n\}$ in this case is possible but requires the application of the dediscretization approach which shall be introduced later.
1.3 Regular Variation of Frequency Distributions

In the present Section we begin to discuss the mathematical background of the statistical facts introduced in Section 1.1. Here the power law-like behavior of frequency distribution is interpreted as its regular variation at infinity. We show that known families of frequency distributions vary regularly at infinity excepting the family of Kolmogorov-Waring Distributions without Waring Distributions.

1.3.1 Information

**Definition 1.** (see, for instance, [21]) A measurable function $R(t) > 0$ defined on $R^+$ varies regularly at infinity (as $t \to +\infty$) if for any $x \in R^+$:

1. The limit exists
$$\lim_{t \to +\infty} \frac{R(xt)}{R(t)} = \varphi(x);$$

2. $0 < \varphi(x) < +\infty$.

The Definition 1 implies that necessarily for $x \in R^+$
$$\varphi(x) = x^\rho \text{ with } \rho \in R^1 = (-\infty, +\infty)$$

and the convergence in (1.3.1) is uniform with respect to $x$ in any finite segment $[a, b]$, where $0 < a < b < +\infty$.

This is the so-called uniform convergence theorem on regularly varying function.

If $R(t)$ varies regularly at infinity and (1.3.3) holds with number $\rho$, then $\rho$ is called the exponent of regularly varying $R(t)$.

**Definition 2.** A measurable function $L(t) > 0$ defined on $R^+$ varies slowly at infinity (as $t \to +\infty$) if for any $x \in R^+$
$$\lim_{t \to +\infty} \frac{L(xt)}{L(t)} = 1.$$

From this point we use symbol $L$ to denote the slowly varying at infinity function (with upper and/or lower indexes if necessary).

It is clear that Definition 2 is a particular case of Definition 1. Thus, a function varying regularly at infinity with exponent $\rho = 0$ varies slowly at infinity. Obviously, any function varying regularly at infinity $R(t)$ with exponent $\rho$ can be expressed in the form
$$R(t) = t^\rho \cdot L(t) \text{ for all } t \in R^+. \tag{1.3.5}$$

In reverse, any function of type (1.3.5) with slowly varying $L(t)$ varies regularly at infinity with exponent $\rho$. 

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For a sequence of positive numbers Definitions 1-2 on regularly and slowly varying functions stay unchanged if in (1.3.1)-(1.3.2) and (1.3.4) we replace $x$ and $t$ by integers $s > 1$ and $n \geq 1$, respectively.

For the sequence $\{\Gamma_n\}$ the representation (1.3.5) is transformed into the following one

$$\Gamma_n = n^\rho \cdot L(n), \ n = 1, 2, \cdots, \quad (1.3.6)$$

where $\{L(n)\}$ denotes some slowly varying at infinity sequence.

At once the following question arises

*How to check out if the given measurable function $R(t) > 0$ defined on $R^+$ varies regularly at infinity?*

In order to answer on this question different types of *Criteria* are suggested. They may be also taken as definitions of regular variation.

The well-known Criterion of E. Seneta [21] was improved in [22]:

**Criterion 1.1** If for the measurable on $R^+$ function $R(t) > 0$ condition (1.3.1) holds on some set $S \subseteq R^+$ with $m(S) > 0$, where $m$ denotes the Lebesque measure, and condition (1.3.2) holds in some point $x \in R^+, \ x \neq 1$, then $R(t)$ varies regularly at infinity.

For the Probability Theory and in our investigation the particular case of regularly varying function $R(t)$ with the additional assumption on $R(t)$’s *monotony* is of great interest.

Remind that monotone, or continuous function is measurable and formulate the *Feller’s Criterion*: if for a monotone on $R^+$ function $R(t) > 0$ the conditions (1.3.1) and (1.3.2) hold in some set that is everywhere dense in $R^+$ set, then $R(t)$ varies regularly [23].

This Criterion has been improved in [22].

**Criterion 1.2** If for the monotone on $R^+$ function $R(t) > 0$ condition (1.3.1) holds on some convergent to the point $x \in R^+$ sequence $\{x_n\} \in R^+$, and condition (1.3.2) holds on some point $y \in R^+, \ y \neq 1$, then $R(t)$ varies regularly.

### 1.3.2 The Power Law - Like Behavior

The *characteristic* property (1.2.3) of the Power Law, due to *Definition 1* (on regular variation) and its corollary (1.3.3) (or to representation (1.3.5)), allows to interpret the empirical fact (1.1.7) in mathematical terms as *regular variation* at infinity of frequency distribution $\{p_n\}$. In other words, we assume that for the empirical frequency distribution $\{p_n\}$ for $s = 2, 3, \cdots$, the limit exists

$$\lim_{n \to +\infty} \frac{p_{sn}}{p_n} = s^{-\rho}, \quad 1 < \rho < +\infty. \quad (1.3.7)$$
This interpretation of the power law-like behavior has been discovered in our work [16].

Let us consider some examples.

It is easy to see that the Pareto Distribution \( (1.2.6)-(1.2.7) \), and its continuous analog \( (1.2.8)-(1.2.9) \), similarly to Power Law \( (1.1.2)-(1.1.3) \), and its continuous analog \( (1.2.1)-(1.2.2) \), vary regularly at infinity with exponent \( -(\rho) \), where \( \rho \in (1, +\infty) \).

Indeed, for instance, in case \( (1.2.6)-(1.2.7) \) for \( s = 2, 3, \cdots \) we proceed

\[
\lim_{n \to +\infty} \frac{p_{sn}}{p_n} = \lim_{n \to +\infty} \frac{(n + b)^\rho}{(sn + b)^\rho} = \frac{1}{s^\rho} \lim_{n \to +\infty} \frac{(1 + b \cdot n^{-1})^\rho}{(1 + b \cdot (sn)^{-1})^\rho} = s^{-\rho}.
\]

So, the condition \( (1.3.7) \) holds.

For the family of Waring Distributions defined by \( (1.2.10)-(1.2.11) \) with constraint \( (1.2.12) \) a similar conclusion may be made. Namely, any Waring Distribution \( \{p_n\} \) with parameters \( p \) and \( q \), \( 0 < p < q < +\infty \), varies regularly at infinity with exponent \( -\rho \), where

\[
\rho = 1 + q - p. \quad (\text{see, [16]})
\]

Indeed, for \( s = 2, 3, \cdots \) the limit exists

\[
\lim_{n \to +\infty} \prod_{K=n}^{sn-1} (1 + \frac{\rho}{p+K}) = \exp(\lim_{n \to +\infty} \sum_{K=n}^{sn-1} \ln(1 + \frac{\rho}{p+K})) = \exp(\rho \cdot \lim_{n \to +\infty} \sum_{K=n}^{sn-1} \frac{1}{p+K})
\]

\[
= \exp(\rho \cdot \lim_{n \to +\infty} \int_n^{sn-1} \frac{dx}{x+p}) = \exp(-\rho \cdot \lim_{n \to +\infty} \ln \frac{p+sn-1}{p+n}) = s^{-\rho}.
\]

Therefore, for \( s = 2, 3, \cdots \) we get

\[
\lim_{n \to +\infty} \frac{p_{sn}}{p_n} = \lim_{n \to +\infty} \frac{(p + n) \cdot (p + n + 1) \cdots (p + sn - 1)}{(q + n + 1)(q + n + 2) \cdots (q + sn)} = \lim_{n \to +\infty} \left\{ \prod_{K=n}^{sn-1} (1 + \frac{\rho}{p+K}) \right\}^{-1} = s^{-\rho},
\]

which proves the statement. Here the notation \( (1.3.8) \) was used.

It is important to notice that the part of the family of Komolgorov-Waring Distributions introduced by V.Kuznetsov, i.e. the subfamily

\[
(\text{Komolgorov-Waring Distributions}) \rightarrow (\text{Waring Distributions}),
\]

doesn’t satisfy empirical fact \( (1.1.7) \).

Indeed, due to \( (1.2.14) \), for \( 0 < \Theta < 1, p \in R^+, q \in R^+ \)

\[
p_n = \frac{p_0 \cdot p}{q + n} \cdot \Theta^n \cdot \prod_{K=1}^{n-1} \frac{p + K}{q + K}, \quad n = 1, 2, \cdots.
\]

Therefore, for \( s = 2, 3, \cdots \) we have

\[
\lim_{n \to +\infty} \frac{p_{sn}}{p_n} = \lim_{n \to +\infty} \Theta^{(n-1)} \frac{q + n}{q + sn} \prod_{K=n}^{sn-1} (1 + \frac{p - q}{q + K}) = \cdots
\]
\[
\begin{align*}
&\lim_{s \to +\infty} \Theta^s(n-1) \exp \left\{ \sum_{K=n}^{sn-1} \ln(1 + \frac{p-q}{q+K}) \right\} = \\
&\lim_{s \to +\infty} \Theta^s(n-1) \exp \left\{ \frac{1}{s} \sum_{K=n}^{sn-1} \frac{1}{q+K} \right\} = \\
&\lim_{s \to +\infty} \Theta^s(n-1) \exp \left\{ (p-q) \cdot \int_n^{sn-1} \frac{dx}{q+x} \right\} = \\
&\lim_{s \to +\infty} \Theta^s(n-1) = 1
\end{align*}
\]

which proves the statement.

### 1.3.3 Distribution Function of \( \{p_n\} \)

For the frequency distribution \( \{p_n\} \) let us denote
\[
q_n = p_n + p_{n+1} + \cdots, \quad n = 1, 2, \cdots.
\]

Let the random variable \( \xi \) have distribution \( \{p_n\} \). Let us write down the distribution function of random variable \( \xi \)
\[
P(\xi < x) = \sum_{n < x} p_n = \sum_{n \leq [x]} p_n, \quad x \in R^+,
\]
where \([x]\) denotes the entire part of positive number \( x \).

Then, the only (right) tail (1.3.10) of distribution function, due to notation (1.3.9), takes the form
\[
P(\xi \geq x) = \begin{cases} 
q_{[x]+1} & \text{if } x \text{ is not an integer,} \\
q_x & \text{if } x \text{ is an integer.}
\end{cases}
\]

For the continuous analog of \( \{p_n\} \)-density \( f(x) \) the distribution function \( F(x) \) takes the form \( F(x) = \int_0^x f(u) du, \ x \in R^+ \), and, as a result of this, the only (right) tail of \( F \) may be written as follows \( 1 - F(x) = \int_x^{+\infty} f(u) du, \ x \in R^+ \).

Note that in general for given sequence \( \{p_n\} \) distribution functions \( P(\xi < x) \) and \( F(x) \) are different.

If \( f(t) \) varies regularly with exponent \( -\rho \), then, due to L’Hopital rule, for \( x \in R^+ \)
\[
\lim_{t \to +\infty} \frac{1 - F(xt)}{1 - F(t)} = \lim_{t \to +\infty} \left\{ \int_{xt}^{+\infty} f(u) du / \int_t^{+\infty} f(u) du \right\} = \lim_{t \to +\infty} \frac{x \cdot f(xt)}{f(t)} = x^{-\rho+1}.
\]

Thus, \( 1 - F(t) \) varies regularly at infinity with exponent \( -\rho + 1 \).

This statement follows also from one general result (see, Theorem 1, VIII, 9 [23]).

It is remarkable that a more general statement than given above holds. Namely:

(a) The sequences \( \{p_n\} \) and \( \{q_n\} \) vary regularly at infinity with exponents \( -\rho \) and \( -\rho + 1 \) simultaneously;

(b) The functions \( f(x) \) and \( 1 - F(x) \) vary regularly at infinity with exponents \( -\rho \) and \( -\rho + 1 \) simultaneously.
Very often such statements are proved with the help of so-called Tauberian Theorems.

Let us apply the statement (a) to known frequency distributions. We may conclude the following. The distribution function of: Pareto Distribution (1.2.6)-(1.2.7) has regularly varying at infinity only (right) tail with exponent $(-\rho + 1)$; Waring Distribution (1.2.10)-(1.2.12) has regularly varying at infinity only (right) tail with exponent $p - q$ (see, (1.3.8)).

\section*{1.4 Shapes of Frequency Distributions}

\subsection*{1.4.1 On Shapes of Known Distributions}

In the present Section we consider some peculiarities of the shapes of empirical frequency distributions. These are: upward/downward convexity, the only point where the frequency distribution achieves it’s maximal value, etc.

These peculiarities are clear for known frequency distributions.

It is easy to see from the form (1.2.8)-(1.2.9) of the continuous analog of Pareto Distribution that this function is decreasing and downward convex. Therefore it takes the maximal value $c(\rho, b) \cdot (1 + b)^{-\rho}$ at point 1.

For the family of Waring Distributions with the help of (1.2.11) let us form a ratio

$$\frac{p_{n+1}}{p_n} = \frac{p + n}{q + n + 1} = 1 - \frac{q - p + 1}{q + n + 1}, \quad n = 0, 1, 2, \ldots.$$ \hfill (1.4.1)

Since $0 < p < q < +\infty$ and $\frac{p}{q} = \frac{p_{n+1}}{p_n} > 0$, so, from (1.4.1) we conclude that $\{p_n\}$ decreases and is downward convex. Therefore it takes the maximal value $p_0 = 1 - \frac{p}{q}$ (see, (1.2.19)) at point 0.

As it was shown in [16], for empirical frequency distributions the following two types of shapes of their continuous analogs exist (see, Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{shape.png}
\caption{Figure 1.}
\end{figure}
As a rule, a biologist prefers to deal with *log-log plot* of empirical frequency distribution instead of its shape. That is why one of the *statistical facts* which is an initial point of our conclusions sounds as follows.

*The log−log plot of most empirical frequency distributions \{p_n\} systematically deviated from the straight line and shows upward/downward convexity.*  

(1.4.2)

It means also that the deviations from the straight line must be ”not too large”.

### 1.4.2 The Log-Log Plot of Frequency Distributions

Let us begin from log-log plot of known frequency distributions.

For the Pareto Distribution (1.2.6)-(1.2.7) we have

\[
\log p_n = -\rho \log (n+b) + c \frac{\log n}{\log n} - \rho \log n + \log(1 + \frac{b}{n}) + c \frac{\log n}{\log n} = -\rho + c \frac{1}{\log n} + o\left(\frac{1}{\log n}\right),
\]

where \( c = c(\rho, b) \in \mathbb{R}^+ \). In cases \(-1 < b < 0\) and \(0 < b < +\infty\) it is easy to verify that the statistical fact (1.4.2) holds at least starting from some integer \( n_0 > 1 \).

Below instead of sentence ”denote \( x \) by \( y \)” we use notation \( y := x \).

For the family of Komolgorov-Waring Distributions we deal with \( \ln p_n \) versus \( \ln n \). Due to (1.2.10)-(1.2.14), we write down

\[
x_n := \ln p_n - n \ln \Theta = \frac{1}{\ln n} \left\{ c_1 - \ln(q + n) + \sum_{K=1}^{n-1} \ln \left( \frac{p + K}{q + K} \right) \right\} =
\]

\[
= -1 + \frac{1}{\ln n} \left\{ c_1 - \ln(1 + \frac{q}{n}) + \sum_{K=1}^{n-1} \ln(1 + \frac{p - q}{q + K}) \right\}, \quad n = 2, 3, \cdots,
\]

(1.4.3)

where \( c_1 \) is some constant. Here we use a slightly different form of \( \{p_n\} \):

\[
p_n = \frac{p_0 p}{q + n} \cdot \Theta^n \cdot \prod_{K=1}^{n-1} \frac{p + K}{q + K}, \quad n = 1, 2, \cdots.
\]

Denote \( \tilde{f}(x) = \ln(1 + \frac{p - q}{q + x}) \), \( x \in [1, +\infty) \). The function \( \tilde{f} \) is bounded, increases if \( p < q \), decreases if \( p > q \), and equals to zero if \( p = q \). By [24] p. 284,

\[
\sum_{K=1}^{n} \tilde{f}(K) = \int_{1}^{n} \tilde{f}(x) dx + a + \alpha_n, \quad n = 1, 2, \cdots,
\]

where \( a \) is a constant, \( \lim_{n\to+\infty} \alpha_n = 0 \). Further, for \( n = 1, 2, \cdots \) and \( p \neq q \) we obtain

\[
\int_{1}^{n} x d\tilde{f}(x) = \int_{1}^{n} x d \{\ln(p + x) - \ln(q + x)\} =
\]
\[ p \ln(p + 1) - q \ln(q + 1) - p \ln(p + n) + q \ln(q + n). \]

Therefore, taking into account (1.4.3), for \( n = 2, 3, \ldots \) we proceed

\[
x_n = -1 + \frac{1}{\ln n} \left\{ c_2 \cdot (1 + o(1)) + \int_1^{n-1} \hat{f}(x)dx \right\} =
\[
= -1 + \frac{1}{\ln n} \left\{ c_2 \cdot (1 + o(1)) - \int_1^{n-1} x \hat{f}(x) \right\} =
\[
= (p - q - 1) + \frac{c_3}{\ln n} (1 + o(1)), \quad c_2 = \text{const}, \quad c_3 = \text{const}.
\]

Hence, for \( n = 2, 3, \ldots \)

\[
\frac{\ln p_n}{\ln n} = p - q - 1 + \left\{ \frac{n}{\ln n} \ln \Theta + o(1) \quad \text{if } 0 < \Theta < 1, p \in R^+, q \in R^+, \right. \]
\[
\left. \frac{c_3(1+o(1))}{\ln n} \quad \text{if } \Theta = 1, 0 < p < q < +\infty. \right. \quad (1.4.4)
\]

Due to (1.4.4), in case of (Komolgorov-Waring Distributions) - (Waring Distributions) the deviations of \( \ln p_n \) versus \( \ln n \) from the straight line \( y = p - q - 1 = \text{const} \) are extremely large for large \( n \). But the case of Waring Distributions at least starting from some \( n_0 > 1 \) satisfies empirical fact (1.4.2).

Let us turn to the general case of frequency distribution.

Since the frequency distribution \( \{ p_n \} \) varies regularly at infinity with some exponent \((-\rho)\), therefore the following representation holds

\[ p_n = n^{-\rho} \cdot L(n), \; n = 1, 2, \ldots, \rho \in [1, +\infty). \quad (1.4.5) \]

The given range of changes \([1, +\infty)\) for \( \rho \) shall be substantiated later. Now we may only mention that for known families of empirical frequency distributions the conclusion \( \rho \in (1, +\infty) \) always takes place. The new possibility \( \rho = 1 \) is discovered in [16].

Due to (1.4.5), let us write down the log-log plot of distribution \( \{ p_n \} \):

\[ \frac{\log p_n}{\log p} = -\rho + \frac{\log L(n)}{\log n}, \; n = 1, 2, \ldots. \quad (1.4.6) \]

The ”pathological” cases when the limit \( \lim_{n \to +\infty} L(n) \) doesn’t exist is investigated in [22] in continuous case of slowly varying function \( L(t) \). Those are functions of ”oscillating” type. Thus, it is natural to assume that the limit exists

\[ 0 \leq L = \lim_{n \to +\infty} L(n) \leq +\infty. \quad (1.4.7) \]

If (1.4.7) holds, then, as a rule, the situations \( L = 0 \) and \( L = +\infty \) for biomolecular applications have only theoretical interest. For known families of empirical frequency distributions in the next Section we’ll show that the limit (1.4.7) exists with \( L \in R^+ \).

Let us show that

\[ \lim_{n \to +\infty} \frac{\log L(n)}{\log n} = 0. \quad (1.4.8) \]
We need to check out (1.4.8) only in case $L = +\infty$ (see, (1.4.7)). In other cases (1.4.8) is obvious.

By the known property of regularly varying sequences (see, for instance, [21]) for any $\varepsilon \in (0, 1)$ there is an integer $n_0 > 1$ such that for $n = n_0, n_0 + 1, \ldots$ we have

$$L(n) < n^\varepsilon \text{ or } \log L(n) < \varepsilon \log n.$$  

(1.4.9)

Therefore, due to (1.4.9), $0 \leq \lim_{n \to +\infty} \log L(n)/\log n \leq \varepsilon$ for all $\varepsilon \in (0, 1)$. Tending $\varepsilon \downarrow 0$ we obtain (1.4.8).

### 1.4.3 New Assumption and Its Consequences

If $\{p_n\}$ varies regularly at infinity, then, due to (1.4.8) and (1.4.6), the "systematical deviation of log-log plot from the straight line" being mentioned in (1.4.2) is possible to observe practically only in finite intervals $(a, b)$ with $0 < a < b < +\infty$. In addition to this the situation at infinity is clear.

Then, it seems that the following additional assumption might be useful:

The sequence $\{L(n)\}$ is convex. 

(1.4.10)

We are going to discuss the assumption (1.4.10) for limit cases in (1.4.7) and next pass to conclusions:

(1) If $L = +\infty$ in (1.4.4), then $\{L(n)\}$ is upward convex.

(2) If $L = 0$ in (1.4.4), then $\{L(n)\}$ is downward convex.

It is enough to prove the statement (1). Indeed, the statement (2) is reduced to statement (1) if we take instead of $\{L(n)\}$ the sequence $\{1/L(n)\}$.

Let us prove the statement (1). Let us assume the opposite, i.e. that $\{L(n)\}$ is downward convex. Let us draw a "broken" line passing through points $(1, L(1)), (2, L(2)), \ldots, (n, L(n)), \ldots$ on the plane, say $L_0(t), t \in [1, +\infty)$. The function $L_0(t)$ satisfies condition: $L_0(n) = L(n), n = 1, 2, \ldots$.

The graph of linear function $y = cx + (L(1) - c)$ is drawn in such a way that it intersects the graph of function $y = L_0(x)$ in some point $x_0 \in (1, +\infty)$. Then, we may evaluate number $c$. It is clear that $c \in R^+$. Since $L_0(x)$ is downward convex (this conclusion is a consequence of $\{L(n)\}$'s downward convexity), therefore $L_0(x) > cx + (L(1) - c)$ for all $x \in (x_0, +\infty)$. From the last inequality we obtain $\lim_{n \to +\infty} \frac{L(n)}{n} \geq \lim_{x \to +\infty} \frac{L_0(x)}{x} > c - \varepsilon > 0$ for any $\varepsilon \in (0, c)$. This inequality contradicts the limit relationship $\lim_{n \to +\infty} \frac{L(n)}{n} = 0$.

Let us pass to conclusions.
If \( \{L(n)\} \) is downward convex, then we have
\[
\frac{\log L(n)}{\log n} = \frac{\log(1/L(n))^{-1}}{\log n} = -\frac{\log(1/L(n))}{\log n},
\]
(1.4.11)
where the sequence \( \{1/L(n)\} \) is upward convex.

In (1.4.6) the convexity of log-log plot is related with the second term at the right-hand-side. Due to (1.4.11), in all cases we deal with the expression of the form
\[
\pm\frac{\log L(n)}{\log n}, \quad n = 1, 2, \ldots,
\]
(1.4.12)
where \( \{L(n)\} \) is upward convex. Since \( \{L(n)\} \) is upward convex, therefore because of the upward convexity of \( \log x \) the sequence \( \{\log L(n)\} \) is upward convex too. Since the sequence \( \left\{\frac{1}{\log n}\right\} \) is downward convex, therefore from (1.4.12) we conclude:

If the sequence \( \{L(n)\} \) is downward convex, then
\[
\left\{\frac{\log p_n}{\log n}\right\}
\]
may be either downward convex or upward convex in some interval \((1, x_0)\) with \(1 < x_0 < +\infty\), and downward convex in \((x_0, +\infty)\).

Similar conclusions may be made in case of upward convex sequence \( \{L(n)\} \).

### 1.5 Universal Measures of Skewness

#### 1.5.1 Introduction

One of the basic properties of empirical frequency distributions and their continuous analogs arising in large-scale biomolecular systems is their skewness to the right. This property has been discovered by an experimental way based on observation of various data sets of such systems. The necessity of frequency distributions’ skewness comparison is dictated even in order to construct suitable approximations for empirical frequency distributions. Thus, we need so-called measures of skewness. It would be wonderful if we were able to build universal measures that would work for any frequency distribution. But in general it is impossible.

In general inside any class of distribution functions which cannot be described with the help of finite or numerable set of parameters the problem of skewness comparison seems to be very complicated.

Note that any moment of distribution function in some sense characterizes its skewness. Thus, it may serve as a measure of skewness. The classical Moment Problem deals with situations when a given class of distribution functions is determined with the help of moments values. In this case taking the moments as measures of skewness one may try to pass to skewness comparison inside this class. Otherwise it is possible to extract subclasses with, for instance,
fixed first $n$ moment’s values and compare the skewness of such distributions with the help of the value of the $[n+1]$-st moment. It leads to specific Moment Problem and Extremal Problems connected with the Moment Problem. Then it is possible to introduce some ordering in skewness inside considering class. One of the fruitful methods for such extremal problem solution is the Sign-Changes Approach developed in [25-26].

Let us make a remark. If we have a class of distributions being described by finite number of parameters, then any parameter may be taken as a measure of skewness. Only the following restriction has to be satisfied. We must present some physical, biological, mathematical, etc. interpretation of given parameters.

In order to give a simple example let us formulate

Definition 3. Two distribution functions $F_1$ and $F_2$ in $R^+$ are said to differ only by location parameters if

$$F_2(x) = F_1(\alpha x + \beta), \ x \in R^+,$$

where $\alpha$ is positive and $\beta$ is real, i.e. $\alpha \in R^+, \beta \in R^1$.

Then we say that $F_1$ and $F_2$ are of the same type. Here $\alpha$ is a scale factor and $\beta$ is a shifting (centring) constant.

In terms of densities (if exist) (1.5.1) is written as follows

$$f_2(x) = \alpha \cdot f_1(\alpha x + \beta),$$

where $f_i, i = 1, 2,$ is a density of $F_i$.

Any distribution function $F$ with the help of location parameters generates a two-parametric family of distribution functions

$$\left\{ F\left(\frac{x - \beta}{\alpha}\right) : \alpha \in R^+, \beta \in R^1 \right\}$$

of the same type. Here $\alpha$ and $\beta$ having some interpretation may be taken as measures of skewness for the family (1.5.3). It is natural to postulate:

Larger the parameter $\beta$, larger the skewness;
Larger the parameter $\alpha$, smaller the skewness.

Finally note that the continuous analogs of the Power Laws (1.2.1)-(1.2.2) and of the Pareto Distributions (1.2.8)-(1.2.9), due to Definition 3, are of the same type.

1.5.2 First Universal Measure of Skewness

The situation with measures of skewness is more pleasant for regularly varying frequency distributions. In this situation we may find a connection between the notions of skewness and of regular variation. For a frequency distribution varying regularly at infinity
undoubtedly as a *universal* measure of skewness has to be taken the number \((-\rho)\), where \(\rho \in [1, +\infty]\) denotes the exponent of the frequency distribution’s regular variation.

It is of interest to point out that in [9] *intuitively* the parameter \(\rho \in [1, +\infty]\) of Pareto Distribution (1.2.6)-(1.2.7) has been declared for this distribution as a *measure* of skewness. Now we know that this parameter with a sign “-” is the *exponent* of the Pareto Distribution’s *regular variation*.

Let us explain our choice. Let \(\{p_n\}\) and \(\{p_n'\}\) be distributions of random variables \(\xi\) and \(\xi'\) respectively. We assume that distributions \(\{p_n\}\) and \(\{p_n'\}\) vary regularly at infinity with exponents \((-\rho)\) and \((-\rho')\) respectively, where \(1 < \rho < \rho' < +\infty\). It is natural to postulate that

*The skewness of \(\{p_n\}\) is more than the skewness of \(\{p_n'\}\).* (1.5.4)

The statement (1.5.4) is based on following arguments. Denote

\[ q_n = p_n + p_{n+1} + \cdots, \quad q_n' = p_n' + p_{n+1}' + \cdots, \quad n = 1, 2, \cdots. \]

Starting from some point the graph of \(\{p_n\}\) is located "upper" than the graph of \(\{p_n'\}\), i.e. the probabilistic "mass" of \(\{p_n\}\) in \([n, +\infty)\) equals to \(q_n\) for large values of \(n\) is "larger" than the similar "mass" of \(\{p_n'\}\), i.e. \(q_n'\). In particular, it leads to the following statement which is well-known in Theory of Regularly Varying Functions [21].

For \(\alpha \in (\rho - 1, \rho' - 1)\) the moment \(m_\alpha := \sum_{n \geq 1} n^\alpha p_n = \int_0^{+\infty} x^\alpha dP(\xi < x) = +\infty\), but \(m_\alpha := \sum_{n \geq 1} n^\alpha p_n' = \int_0^{+\infty} x^\alpha dP(\xi' < x) < +\infty\). At the same time for \(\alpha \in (0, \rho - 1)\) the moments \(m_\alpha\) and \(m'_\alpha\) are finite.

This is the main reason for statement (1.5.4).

The *exponent* of regular variation characterizes the behavior of distribution at infinity. Thus, it is *amazing* (for the first look) that the exponent can be chosen as an *universal* measure of skewness. Now, we may even conclude that in some sense the skewness is a characterization of distribution *at infinity*!

In the case \(\rho = \rho'\) we need additional measure of skewness in order to compare the skewness of \(\{p_n\}\) and \(\{p_n'\}\). The last conclusion in this case shows that the role of *slowly varying sequences* \(\{L(n)\}\) and \(\{L'(n)\}\) from representations \(p_n = n^{-\rho} L(n)\) and \(p_n' = n^{-\rho'} L'(n)\), \(n = 1, 2, \cdots\) for skewness comparison of \(\{p_n\}\) and \(\{p_n'\}\) increases.

**Definition 4.** For the regularly varying at infinity function \(R(t)\) of type (1.3.5) the function \(L(t)\) is called a *slowly varying component*.

**Definition 5.** We say that the slowly varying component \(L(t)\) exhibits the constant *slowly varying component* \(L\) if the limit exists \(\lim_{t \to +\infty} L(t) = L \in R^+\).

**Definitions** 4 and 5 automatically are expanded on *regularly varying sequence*, in particular, on empirical frequency distribution \(\{p_n\}\).
1.5.3 Second Universal Measure of Skewness

Let us consider the regularly varying at infinity empirical frequency distribution \( \{p_n\} \):

\[
p_n = n^{-\rho} \cdot L(n), \quad n = 1, 2, \cdots, \quad \rho \in (1, +\infty)
\]  \hspace{1cm} (1.5.5)

with exponent \( \rho \) and slowly varying component \( \{L(n)\} \).

Let us assume that \( \{L(n)\} \) exhibits constant slowly varying component \( L \), i.e. the limit exists

\[
\lim_{n \to +\infty} L(n) = L \in R^+.
\]  \hspace{1cm} (1.5.6)

In this situation together with the first universal measure of skewness \( \rho \) we take \( L \) as the second universal measure of skewness.

Let us explain our choice. Let \( \{p_n\} \) and \( \{p'_n\} \) are distributions of types (1.5.5)-(1.5.6) and

\[
p'_n = n^{-\rho}L'(n), \quad n = 1, 2, \cdots, \lim_{n \to +\infty} L'(n) = L' \in R^+.
\]

It is natural to postulate that:

\[
\text{For regularly varying frequency distributions } \{p_n\} \text{ and } \{p'_n\} \\
\text{with the same exponent } (-\rho) \text{ and with constant slowly varying} \\
\text{components } L \text{ and } L' \text{ respectively the skewness of } \{p_n\} \\
\text{is more than the skewness of } \{p'_n\} \text{ if } L > L'.
\]  \hspace{1cm} (1.5.7)

The statement (1.5.7) is based on following arguments. Starting from some point \( x_0 > 1 \) the graph of \( \{p_n\} \) is located "upper" than the graph of \( \{p'_n\} \). Let us draw the continuous analogs of \( \{p_n\} \) and \( \{p'_n\} \) of the type presented, for instance, for \( \{p_n\} \) in Figure 2(a). These analogs we denote by \( g(x) \) and \( g'(x) \) respectively. The linear continuous analog of \( \{p_n\} \) is drawn in Figure 2(b).
If for two continuous analogs of type (a) we have coincidences of values in some interval, then we assume that in this interval there is only one intersection of the graphs of these continuous analogs. From this point of view we may conclude:

For two distributions the number of intersections of their continuous analogs of type (a) and linear continuous analogs of type (b) (see, Figure 1) coincide.

The square of the shaded area in Figure 2(a) equals to

\[ \int_{0}^{\infty} g(x)dx = \sum_{n \geq 1} p_n = 1. \]  \hfill (1.5.8)

Let us show that there is at least one intersection of the graphs of functions

\[ y = g(x) \text{ and } y = g'(x). \]  \hfill (1.5.9)

Let us assume the opposite, i.e. there are no intersections of these graphs. The graph of function \( y = g(x) \) is "upper" than the graph of function \( y = g'(x) \). Due to (1.5.8), it means that

\[ \int_{0}^{\infty} g'(x)dx = \sum_{n \geq 1} p'_n < 1, \]

which contradicts the equality \( \sum_{n \geq 1} p'_n = 1 \).

Below we denote by \( x_0 \) the last intersection point of the graphs of functions (1.5.9) (the number of intersections is finite). Obviously, \( x_0 \) is a positive integer.
Basing on these facts in the next Section we'll prove the following statement in several particular cases of the shape of frequency distributions.

Under our conditions for some \( \alpha \in (0, \rho - 1) \) the inequalities hold

\[
0 < m'_\alpha < m_\alpha < +\infty. \tag{1.5.10}
\]

Note that the finiteness of moments of distributions \( \{p_n\} \) and \( \{p'_n\} \) of orders \( \alpha \in (0, \rho - 1) \) in (1.5.10) follows from the regular variation of these distributions with exponent \( -\rho \).

The statement (1.5.10) gives the ground for declaring constant slowly varying component as the second universal measure of skewness. Thus, for two-parametric families of regularly varying with constant slowly varying component frequency distributions two universal measures of skewness are introduced, and the problem of the skewness comparison inside such families is completely solved.

### 1.6 Moment Inequalities

In the present Section the statement (1.5.10) is established.

We will proceed in two different ways. The first one is based on simple property of regularly varying sequences, gives only the proof of statement (1.5.10) for values on \( \alpha \) being "close" to \( \rho - 1 \). But this way is of methodological interest. The second one directly proves the statement (1.5.10) for \( \alpha \in (0, 1 - \rho) \) being "close" to zero.

#### 1.6.1 The First Method

For the regularly varying frequency distributions \( \{p_n\} \) and \( \{p'_n\} \) with the same exponent \( -\rho \), where \( \rho \in (1, +\infty) \), and with constant slowly varying components, say \( L \) and \( L' \), respectively, the equalities hold \( m''_{\rho-1} = m_{\rho-1} = +\infty \), and (remind that \( m'_\alpha < +\infty, m_\alpha < +\infty \) for \( \alpha \in (0, \rho - 1) \)) as a result of this

\[
\lim_{\alpha \to \rho - 1} m_\alpha = \lim_{\alpha \to \rho - 1} m'_\alpha = +\infty. \tag{1.6.1}
\]

Now, for "small" value of \( \varepsilon \in (0, \rho - 1) \) and for large \( n \) let us compare the sums

\[
\sum_{m \geq n} m^{\rho - \varepsilon - 1} \cdot p_m \quad \text{and} \quad \sum_{m \geq n} m^{\rho - \varepsilon - 1} \cdot p'_m.
\]

For \( n > x_0 \) (remind that \( x_0 \) is the last intersection point of graphs of functions (1.5.9)) if \( L > L' \), then corresponding terms of the first sum are larger than of the second one. Therefore, forming the difference of these sums we get

\[
A_{n, \varepsilon} := \sum_{m \geq n} m^{\rho - \varepsilon - 1} \cdot p_m - \sum_{m \geq n} m^{\rho - \varepsilon - 1} \cdot p'_m = \sum_{m \geq n} m^{\rho - \varepsilon - 1} \cdot (p_m - p'_m) > 0. \tag{1.6.2}
\]
For given $\delta \in (0, 1)$ there is an integer $n_0 > x_0$ such that for all $n \geq n_0$ the inequality holds

$$\frac{p_n - p'_n}{p_n} > \delta(1 - \frac{L'}{L})p_n.$$  \hspace{1cm} (1.6.3)

It follows from the regular variation of $\{p_n\}$ and $\{p'_n\}$ and their constant slowly varying components’ $L$ and $L'$ existence.

From (1.6.1)-(1.6.3) for $n \geq n_0$ we obtain

$$A_{n, \varepsilon} > \delta(1 - \frac{L'}{L}) \sum_{m \geq n} m^{\rho - \varepsilon - 1}p_m \to +\infty \text{ as } \varepsilon \downarrow 0.$$  \hspace{1cm} (1.6.4)

Due to (1.6.2), replacing index $n$ in $A_{n, \varepsilon}$ by index $x_0$ we increase the sum (1.6.2), i.e. $A_{x_0, \varepsilon} > A_{n, \varepsilon}$ for $n > x_0$. That is why instead of (1.6.4) we may write down

$$A_{x_0, \varepsilon} \to +\infty \text{ as } \varepsilon \downarrow 0.$$  \hspace{1cm} (1.6.5)

Finally, let us write down the inequalities

$$0 < \sum_{m \leq x_0} m^{\rho - \varepsilon - 1}p_m < \sum_{m \leq x_0} m^{\rho - 1}p_m := C < +\infty$$

and

$$0 < \sum_{m \leq x_0} m^{\rho - \varepsilon - 1}p'_m < \sum_{m \leq x_0} m^{\rho - 1}p'_m := C' < +\infty$$

for all $\varepsilon \in (0, \rho - 1)$. These inequalities together with (1.6.5) lead to the following $m_{\rho - \varepsilon - 1} - m'_{\rho - \varepsilon - 1} \to +\infty$ as $\varepsilon \downarrow 0$. Therefore there is $\varepsilon \in (0, \rho - 1)$ such that $m'_\alpha < m_\alpha$ for $\alpha \in (\rho - 1 - \varepsilon, \rho - 1)$.

### 1.6.2 The Second Method

In the case of only one intersection point of the graphs of functions (1.5.9) it is easy to verify the inequalities (1.5.10). Indeed, it is easy to see that, due to (1.5.8) and to similar equality $\sum_{n \geq 1} p'_n = 1$ for $\{p'_n\}$, we have

$$0 < \sum_{n > x_0} (p_n - p'_n) = \sum_{n \leq x_0} (p_n - p'_n).$$  \hspace{1cm} (1.6.6)

Here $x_0$ being a positive integer is the only intersection point of graphs of functions (1.5.9). So, for any $\alpha \in R^+$, by using (1.6.6), we obtain

$$\sum_{n > x_0} n^\alpha (p_n - p'_n) > x_0^\alpha \sum_{n > x_0} (p_n - p'_n) = x_0^\alpha \sum_{n \leq x_0} (p'_n - p_n) > \sum_{n \leq x_0} n^\alpha (p'_n - p_n) > 0.$$  

It leads for $\alpha \in (0, \rho - 1)$ to the inequality $(m_\alpha =) \sum_{n \geq 1} n^\alpha p_n > \sum_{n \geq 1} n^\alpha p'_n (= m'_\alpha)$, and proves (1.5.10) in the case of only one intersection.
Let us consider the more complicate case of more than one intersections of graphs of functions \((1.5.9)\). In this case for distributions \(\{p_n\}\) and \(\{p'_n\}\) let us build two new distributions \(\{\bar{p}_n\}\) and \(\{\bar{p}'_n\}\) respectively. We conserve the last intersection value \(x_0\) and probabilities \(p_{x_0+1}, p_{x_0+2}, \cdots\) and \(p'_{x_0+1}, p'_{x_0+2}, \cdots\) by putting \(p_n = \bar{p}_n\) and \(p'_n = \bar{p}'_n\) for \(n = x_0 + 1, x_0 + 2, \cdots\). Next, we put \(\bar{p}_1 = p_1 + p_2 + \cdots + p_{x_0}, \bar{p}_2 = \cdots \bar{p}_{x_0} = 0, \text{ and } \bar{p}'_1 = \bar{p}'_2 = \cdots = \bar{p}'_{x_0-1} = 0, \bar{p}'_{x_0} = p'_1 + p'_2 + \cdots + p'_{x_0}.\) This operation leads to distributions \(\{p_n\}\) and \(\{p'_n\}\) that, obviously, vary regularly with the same exponent \((-\rho)\) and exhibit constant slowly varying components \(L\) and \(L'\), similar to distributions \(\{p_n\}\) and \(\{p'_n\}\). It is clear that:

The skewness of \(\{\bar{p}_n\}\) is less than the skewness of \(\{p_n\}\).
The skewness of \(\{\bar{p}'_n\}\) is larger than the skewness of \(\{p'_n\}\).

The following inequalities hold

\[
\bar{m}_\alpha \leq m_\alpha \quad \text{and} \quad \bar{m}'_\alpha \geq m'_\alpha \quad \text{for all } \alpha \in (0, \rho - 1),
\]

where \(\bar{m}_\alpha = \sum_{n \geq 1} n^\alpha \bar{p}_n\) and \(\bar{m}'_\alpha = \sum_{n \geq 1} n^\alpha \bar{p}'_n\). Thus, if we prove that \(\bar{m}_\alpha > \bar{m}'_\alpha\) for all \(\alpha \in (0, \rho - 1)\), then, due to \((1.6.7)\), we may establish \((1.5.10)\).

Without lost of generality from the very beginning let us assume that \(\{p_n\}\) and \(\{p'_n\}\) are of the constructed type. The preliminary preparations are over.

Let us write down the analog of \((1.6.6)\) in this case

\[
0 < \sum_{n > x_0} (p_n - p'_n) = p'_{x_0} - p_1.
\]

There is \(y_0 \in (x_0, +\infty)\) such that for a given \(\alpha \in (0, \rho - 1)\) we have

\[
y_0^\alpha \sum_{n > x_0} (p_n - p'_n) = \sum_{n > x_0} n^\alpha (p_n - p'_n).
\]

Thus, in order to prove \((1.5.10)\) for a given \(\alpha\), due to \((1.6.8)\), the inequality \(y_0^\alpha (p'_{x_0} - p_1) > x_0^\alpha p'_{x_0} - p_1\), or

\[
(y_0^\alpha - x_0^\alpha)p'_{x_0} > (y_0^\alpha - 1)p_1
\]

has to be established. Because of \((1.6.9)\) we have \(p_1 < p'_{x_0}\). Therefore by continuity the inequality \((1.6.9)\) takes place at least for ”small” values of \(\alpha\).

### 1.6.3 One Particular Situation

We already postulated that for the regularly varying distribution which exhibits constant slowly varying component the exponent and the constant slowly varying component have to be taken as universal measures of skewness. It may be applied to \(m\)-parametric families of frequency distributions with \(m > 1\) (the number of parameters equals to \(m\)).
Below we require that any distribution from considering \textit{m-parametric} family varies regularly with the exponent whose absolute values belong to \((1, +\infty)\) and this distribution exhibits constant slowly varying component.

If we compare the \textit{skewness} of two distributions \(\{p_n(\rho, L)\}\) and \(\{p_n(\rho, L')\}\) of such a type two \textit{different} two-parametric families where the distributions have the same exponent \(\rho\) but \textit{different} constant slowly varying components \(L\) and \(L', L > L'\), then the moment’s inequality (1.5.10), may not be true for all values \(\alpha \in (0, \rho - 1)\). More precisely, for \(\alpha\) from the “middle” part of interval \((0, \rho - 1)\). Even if we compare the \textit{skewness} of distributions \(\{p_n(\rho, L)\}\) and \(\{p_n(\rho, L')\}\) of the \textit{same} \(m\)-parametric family with \(m > 2\), then the mentioned situation may arise also in this case. But, as a rule, in \textit{two-parametric} families of such a type for distributions \(\{p_n(\rho, L)\}\) and \(\{p_n(\rho, L')\}\) from the \textit{same} family if \(L > L'\), then the moments’ inequality (1.5.10) is fulfilled for all \(\alpha \in (0, \rho - 1)\).

Let us explain this \textit{particular situation}. As a rule, the above \textit{two-parametric} families of frequency distributions are built in such a way that the following conditions hold.

1. For \(n = 1, 2, \cdots\) the representation takes place \(p_n(\rho, L) = (g_n(\rho, L)/g(\rho, L)), g_n(\rho, L) \geq 0\), where \(g(\rho, L)\) denotes a normalization factor.

2. For fixed \(\rho\) and \(L\) the sequence \(\{g_n(\rho, L)\}\) increases and is \textit{downward convex} by \(n\).

3. For any \(n = 1, 2, \cdots\) the function \(g_n(\rho, L)\) is an \textit{increasing} function by \(\rho\) and \(L\) separately.

4. For \(L > L'\) the sequence \(\{g_n(\rho, L) - g_n(\rho, L')\}\) is \textit{decreasing}. Here the graph \(\{g_n(\rho, L)\}\) is located “upper” than the graph of \(\{g_n(\rho, L')\}\) and they have no intersections. As a result of this we have

\[
x_0 = (g(\rho, L)/g(\rho, L')) > 1.
\]  

(1.6.10)

Let us denote by \(p(t; \rho, L)(g(t; \rho, L))\) and \(p(t; \rho, L')(g(t; \rho, L'))\) the linear continuous analogs for \(\{p_n(\rho, L)\}\) \(\{g_n(\rho, L)\}\) and \(\{p_n(\rho, L')\}\) \(\{g_n(\rho, L')\}\), respectively.

Let us show that the graphs of functions \(y = p(t; \rho, L)\) and \(y = p(t; \rho, L'), L > L', \text{ intersect each other only once}\). Indeed, there is \(x_1 > x_0\) (see, (1.6.10) such that

\[
g(t; \rho, L) < x_1 \cdot g(t; \rho, L').
\]  

(1.6.11)

At the same time

\[
g(t; \rho, L) > g(t; \rho, L').
\]  

(1.6.12)

In Figure 3 all possible situation for the location of graph of functions \(y = g(t; \rho, L)\) and \(y = xg(t; \rho, L)\) are presented. Here \(x \in [1, x_1]\) and is fixed.
According to condition 1.-4. for \( x_0 \in (1, x_1) \) the graphs of functions \( y = g(t; \rho, L) \) and \( y = g(t; \rho, L') \cdot x_0 \) may intersect each other no more than once. The same is true for graphs of functions \( y = (g(t; x, L)/g(\rho, L)) = p(t; x, L) \) and \( y = g(t; x, L') \cdot x_0/g(\rho, L') = p(t; x, L') \).

But we already know that the graphs intersect each other no less than once. Thus, we conclude that there is only one intersection of these graphs.

That is why, due to the proved in (1.6.2) statement, the moments’ inequality in our case holds for all \( \alpha \in (0, \rho - 1) \).

### 1.7 Constant Slowly Varying Components

We already know that well-known families of empirical frequency distributions (Power Laws, Pareto and Waring Distributions) are formed by regularly varying at infinity distributions.

In the present Section we are going to show that the slowly varying component of this distributions exhibit constant slowly varying component. Moreover, below these components shall be expressed in terms of special functions of Mathematical Analysis.

Let \( \{p_n\} \) be a frequency distribution varying regularly at infinity with exponent \((-\rho)\), where \( \rho \in (1, +\infty) \). Then the following representation holds

\[
p_n = n^{-\rho} \cdot L(n), \quad n = 1, 2, \ldots,
\]

where the slowly varying sequence \( \{L(n)\} \) we call a slowly varying component for \( \{p_n\} \).

The following more broaden interpretation to slowly varying component is preferable.
Definition 6. The positive functions $f$ and $g$ defined on $\mathbb{R}^+$ are said to be asymptotically equivalent (at infinity) if the limit exists $\lim_{t \to +\infty} \frac{f(t)}{g(t)} = 1$.

The equivalency shall be expressed as follows $f(t) \approx g(t)$, $t \to +\infty$.

Definition 7. Any asymptotically equivalent to $\{L(n)\}$ (taken from (1.7.1)) slowly varying sequence $\{L_1(n)\}$ is called a slowly varying component (in a wide sense) for distribution $\{p_n\}$ of form (1.7.1).

Due to Definitions 6 and 7 we may write $p_n \approx n^{-\rho} \cdot L_1(n)$, $n \to +\infty$.

1.7.1 Power Laws

For the Power Law (1.1.2)-(1.1.3) we have $L(n) = n^{-\rho} \cdot p_n = c(\rho)$, $n = 1, 2, \cdots$, where $c(\rho) = \frac{1}{\sum_{m \geq 1} m^{-\rho}}$. Thus the Power Law exhibits the constant slowly varying component

$$L = \frac{1}{\sum_{m \geq 1} m^{-\rho}}. \quad (1.7.2)$$

In order to evaluate the constant $L$ for practical needs from (1.7.2) different representations and estimations of the series $\sum_{m \geq 1} m^{-\rho}$ are useful. For instance, by [24], p.284, for $n = 1, 2, \cdots$ we have

$$\sum_{m=1}^{n} \frac{1}{m^\rho} = \int_{1}^{n} \frac{dx}{x^\rho} + a + \alpha_n = \frac{1}{\rho - 1}(1 - \frac{1}{n^{\rho - 1}}) + a + \alpha_n, \quad (1.7.3)$$

where $0 < a < 1$ is some constant and $\alpha_n \to 0$ as $n \to +\infty$. At the same time

$$\sum_{m \geq n} \frac{1}{m^\rho} \approx \int_{n}^{\infty} \frac{dx}{x^\rho} = \frac{1}{(\rho - 1)n^{\rho - 1}}, \quad n \to +\infty. \quad (1.7.4)$$

On the other hand, if $\rho$ is an integer, then due to 4.272.11, p. 550, [20],

$$\sum_{m=1}^{n} \frac{1}{m^\rho} = \frac{1}{(\rho - 1)!} \int_{0}^{1} (\ln \frac{1}{x})^{\rho - 1} 1 - \frac{x^n}{1 - x} dx. \quad (1.7.5)$$

According to (1.7.3)-(1.7.5) we obtain $\sum_{m \geq 1} \frac{1}{m^\rho} \in (\frac{1}{\rho - 1}, \frac{1}{\rho - 1} + 1) = (\frac{1}{\rho - 1}, \frac{\rho}{\rho - 1})$ and $\sum_{m \geq 1} \frac{1}{m^\rho} = \lim_{n \to +\infty} \int_{0}^{1} (\ln \frac{1}{x})^{\rho - 1} 1 - \frac{x^n}{1 - x} dx$ for integer $\rho$.

Next, in 4.316.1, p. 568, [20], the following equality is presented

$$\int_{0}^{1} \ln(1 - ax^r)(\ln \frac{1}{x})^p dx = -\frac{1}{r^{p+1}} \Gamma(p + 1) \sum_{K \geq 1} \frac{a^K}{K^{p+2}}, \quad p > -1, \ a < 1, \ r > 0.$$

Here $\Gamma(\cdot)$ is the Euler’s Gamma Function.

Using this equality we may present the series $\sum m^{-\rho}$ in the form

$$\sum_{m \geq 1} \frac{1}{m^\rho} = \frac{1}{\Gamma(\rho - 1)} \lim_{n \to 1} \int_{0}^{1} \ln(1 - ax)(\ln \frac{1}{x})^{\rho - 2} dx.$$


1.7.2 Pareto Distributions

It is clear that for Pareto Distribution (1.2.6)-(1.2.7) we have

\[ L(n) \approx \left(\frac{n}{n+b}\right)^\rho \sum_{m \geq 1} \frac{1}{(m+b)^\rho}, \quad n \to +\infty, \]

where \(-1 < b < +\infty\), \(1 < \rho < +\infty\), and \(\lim_{n \to +\infty} L(n) = \frac{1}{\sum_{m \geq 1} (m+b)^{-\rho}}\).

Thus, Pareto Distribution exhibits the constant slowly varying component

\[ L = \frac{1}{\sum_{m \geq 1} (m+b)^{-\rho}}, \quad (1.7.6) \]

which for \(b = 0\) coincides with the particular case of Power Law.

From this point we consider the regular case \(b \in \mathbb{R}^+\). It is important to mention that the series \(\zeta(\rho, b) = \sum_{m \geq 0} \frac{1}{(m+b)^\rho} = \frac{1}{L} + \frac{1}{\rho} \) with \(1 < \rho < +\infty\), \(b \in \mathbb{R}^+\) is a well-known Riemann’s Zeta Function because we may now use properties of this function. We may even write down the sum of series in (1.7.6) in the form of some improper integral.

Indeed, for \(\rho \in (1, +\infty)\) and \(\nu \in \mathbb{R}^+\) we have (see, 4.272.12, p. 551, [20])

\[ \sum_{m \geq 0} \frac{1}{(\nu + 2m)^\rho} = \frac{1}{\Gamma(\rho)} \int_0^1 (\ln \frac{1}{x})^{\rho-1} \frac{x^{\nu-1}}{1-x^2} dx. \quad (1.7.7) \]

By putting \(\nu = b\) and \(\nu = b + 1\) in (1.7.7), and combining after that the obtained two sums, we come to the following equality

\[ \sum_{m \geq 0} \frac{1}{(b+m)^\rho} = \frac{1}{\Gamma(\rho)} \int_0^1 (\ln \frac{1}{x})^{\rho-1} \cdot x^{b-1} + x^b \frac{1}{1-x^2} dx = \frac{1}{\Gamma(\rho)} \int_0^1 (\ln \frac{1}{x})^{\rho-1} \cdot \frac{x^{b-1}}{1-x} dx. \quad (1.7.8) \]

That is why, due to (1.7.6) and (1.7.8), for the constant \(L\) we obtain the following representation

\[ L = \frac{\Gamma(\rho)}{\int_0^1 (\ln \frac{1}{x})^{\rho-1} \cdot \frac{x^{b-1}}{1-x} dx - \frac{1}{b}}, \quad 1 < \rho < +\infty, \quad b \in \mathbb{R}^+. \quad (1.7.9) \]

Now, let us consider the case \(\rho \in (1, +\infty), -1 < b < 0\). By introducing new variable \(c = 1+b, 0 < c < 1\) we obtain (see, (1.7.6)) \(L = \frac{1}{\sum_{m \geq 0} (m+c)^{-\rho}} = (\zeta(\rho, c))^{-1}, \) or \(\zeta(\rho, c) = \frac{1}{L}\). Moreover, by (1.7.8) with constant \(c\) instead of constant \(b\), we have \(\zeta(\rho, c) = \frac{1}{\Gamma(\rho)} \int_0^1 (\ln \frac{1}{x})^{\rho-1} x^{c-1} \frac{1}{1-x} dx\), which implies: in case \(1 < \rho < +\infty\) and \(-1 < b < 0\)

\[ L = \frac{\Gamma(\rho)}{\int_0^1 (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} dx}. \quad (1.7.9') \]

Excluding the case of Power Law (\(b = 0\)), formulas (1.7.9) and (1.7.9') give a complete description of the constant slowly varying component of Pareto Distribution in terms of some improper integral.
The only problem arising here is the following one. The chosen second universal measure of skewness in this case has a complicate expression and we are not able to write the parameter $b$ in terms of $\rho$ and $L$, i.e. in the form $b = \varphi(\rho, L)$. So, it is difficult to write $\{p_n\}$ in terms of $\rho$ and $b$. That is why in this concrete case it seems to be more convenient and natural to suggest another measure of skewness instead of $L$. For instance, it may be the parameter $b$ itself. Due to (1.7.2), we conclude:

*The increasement of parameter $b$ is equivalent to the increasement of constant slowly varying component $L$.*

Easily verify that the conditions 1.-4. from Section 1.6 are fulfilled.

That is why the following statement may be formulated.

**Theorem 1.1.** For two distributions $\{p_n(\rho, b)\}$ and $\{p_n(\rho, b')\}$ of the form (1.2.6)-(1.2.7) with $0 < b' < b < +\infty$ the inequality $m'_\alpha := \sum_{n \geq 1} n^\alpha p_n(\rho, b') < \sum_{n \geq 1} n^\alpha p_n(\rho, b) := m_\alpha$ holds for any $\alpha \in (0, \rho - 1)$.

### 1.7.3 Waring Distributions

Let us consider the *family* of Waring Distributions given by formulas (1.2.10)-(1.2.11) under constraint $0 < p < q < +\infty$. Since we already evaluated $p_0 = 1 - \frac{p}{q}$ (see, (1.2.9)), therefore the probabilities $p_n, n = 1, 2, \cdots$, take the form

$$p_n = \frac{p_0 \cdot p}{q + n} \prod_{m=1}^{n-1} \frac{p + m}{q + m} = \frac{(\rho - 1) \cdot \frac{p}{q}}{q + n} \cdot \frac{1}{\prod_{m=1}^{n-1} \frac{q + m}{p + m}} = (\rho - 1) \cdot \frac{\Gamma(q + 1)}{\Gamma(p + 1)} \cdot \frac{1}{\Gamma(p + q + n) \cdot \Gamma(p + n - q)} \cdot \prod_{m=1}^{n-1} \frac{m}{m + q} \cdot \frac{1}{m + p} \cdot \frac{1}{\prod_{m=1}^{n-1} \frac{q + m}{p + m}} =$$

(1.7.10)

We are going to apply the following formula (see, 8.322, p. 936, [20])

$$\Gamma(z) = \lim_{n \to +\infty} \frac{n^z}{z} \cdot \prod_{K=1}^{n} \frac{K}{z + K}, z \in R^+, \quad \text{where } \Gamma(\cdot) \text{ is the Euler’s Gamma Function, or, due to formula } z \cdot \Gamma(z) = \Gamma(z + 1), \quad \text{the formula}$$

$$\prod_{K=1}^{n} \frac{K}{z + K} \approx \frac{\Gamma(z + 1)}{n^z}, \quad n \to +\infty. \quad (1.7.11)$$

So, from (1.7.10)-(1.7.11) for $n = 1, 2, \cdots$ we obtain

$$p_n \approx (\rho - 1) \cdot \frac{\Gamma(q + 1)}{\Gamma(p + 1)} = \frac{1}{\Gamma(p)} \cdot \frac{\Gamma(q + 1)}{\Gamma(p)} \cdot \frac{\rho - 1}{\rho - p} \cdot \frac{\Gamma(p + n - q)}{\Gamma(p + n - q + 1)} \cdot \frac{\Gamma(p + n - q + 1)}{\Gamma(p + n - q)}, \quad n \to +\infty, \quad (1.7.12)$$

which implies $L(n) \approx (\rho - 1) \cdot \frac{\Gamma(p + n - q + 1)}{\Gamma(p)}, \quad n \to +\infty$, or

$$L = \lim_{n \to +\infty} L(n) = (\rho - 1) \cdot \frac{\Gamma(p + n - q + 1)}{\Gamma(p)}. \quad (1.7.13)$$
The formula (1.7.12) says that the Waring Distribution with parameters $\rho \in (1, +\infty)$ and $p \in R^+$ varies regularly at infinity with exponent $(-\rho)$. This fact has been proved by direct method in Section 1.3. But this method didn’t give the asymptotical behavior of the slowly varying component $\{L(n)\}$.

The formula (1.7.13) says that the Waring Distribution with parameters $\rho \in (1, +\infty)$ and $p \in R^+$ exhibits the constant slowly varying component of the form (1.7.13). In order to evaluate the constant $L$ for practical needs we may use the tables of Gamma Function.

For large values of $\rho$, due to 8.327, p. 937, [20], we may suggest formula

$$\Gamma(\rho) \approx z\rho^{\rho - \frac{1}{2}}e^{-\rho}\sqrt{2\pi}, \rho \to +\infty.$$  

Applying this formula to (1.7.13) we come to the equivalency

$$L = L(\rho) \approx (\rho - 1) \cdot \frac{(\rho + p)}{\rho}^{\rho - \frac{1}{2}} \cdot (\rho + p)^p e^{-p} \approx \rho^{p+1} \cdot (1 + \frac{p}{\rho}) e^{-p} \approx \rho^{p+1}, \rho \to +\infty.$$  

For large values of $p$, due to relationship 8.328, p. 937, [20], $\lim_{p \to +\infty} \frac{\Gamma(p+\rho)}{\Gamma(p)} e^{-p \ln p} = 1$, from (1.7.13) we obtain $L = L(p) \approx (\rho - 1) \cdot p^p, p \to +\infty$.

Let us represent (1.7.13) in the form

$$L = (\rho - 1) \cdot \frac{\Gamma(\rho + p)}{\Gamma(p)\Gamma(\rho)} \cdot \Gamma(\rho) = B(\rho, p) \cdot \Gamma(\rho),$$  

where $B(x, y)$ is well-known Beta Function (see, 8.380.1, p. 948, [20])

$$B(x, y) = \int_0^1 (1 - t)^{x-1} \cdot t^{y-1} dt, x \in R^+, y \in R^+,$$  

and the functional relation $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is used (see, 8.384.1, p. 950 [20]).

According to (1.7.14)-(1.7.15) we conclude

**Theorem 1.2** For two distributions $\{\hat{p}_n(\rho, p)\}$ and $\{\hat{p}_n(\rho, p')\}$ of the form (1.7.10) with $0 < p' < p < +\infty$, where in (1.7.10) $q = \rho + p - 1$, the inequality

$$\hat{m}'_\alpha := \sum_{n \geq 1} n^\alpha \hat{p}_n(\rho, p') < \sum_{n \geq 1} n^\alpha \hat{p}_n(\rho, p) := \hat{m}_\alpha$$

holds for any $\alpha \in (0, \rho - 1)$. 

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1.8 Continuity Theorem and Stability Problems

In the present Section stability problems for finite-parametric families of frequency distributions are discussed. Here we introduce the notion of Generating Functions and demonstrate the fruitfulness of Continuity Theorem for them.

Continuity Theorem (see, XI.6, p. 262, [27])

Let for given \( K \) the sequence \( \{p_n^{(K)}\} \) satisfies conditions \( p_n^{(K)} \geq 0, n = 1, 2, \ldots, \sum_{n \geq 1} p_n^{(K)} = 1 \).

In order for fixed \( n \): \( p_n^K \to p_n \) as \( n \to +\infty \), it is necessary and sufficient that for every \( x \in [0, 1) \)

\[
P_K(x) := \sum_{n \geq 1} p_n^{(K)} \cdot x^n \to \sum_{n \geq 1} p_n \cdot x^n := P(x) \quad \text{as} \quad K \to +\infty.
\]

Here \( P_K(x) \) and \( P(x) \) are so-called Generating Functions of \( \{p_n^{(K)}\} \) and \( \{p_n\} \).

In order to prove the convergence of \( \{p_n^K\} \) as \( K \to +\infty \) with the help of Continuity Theorem one needs to find the expressions for corresponding Generating Functions. Sometimes it is a complicate problem which reduces the ability of the approach. In some cases the distributions are defined only by their properties and we do not know suitable expressions for them, but we know suitable expressions for some type of their Integral Transforms (Characteristic Function, Laplace-Stieltjes Transform, Generating Function, etc.). Then we may manage only with the help of Continuity Theorem for this type of Integral Transform. For instance, for four-parametric family of Stable Laws, their densities are known only in terms of convergent series having small practical usage. But Characteristic Functions of Stable Laws have simple enough expressions.

In the present Section we consider two-parametric families of frequency distributions (Pareto, Waring Distributions) for which Generating Functions in bioinformatics are of interest. We express these Generating Functions in terms of some special functions of Mathematical Analysis.

1.8.1 Generating Functions

Let us consider finite-parametric family of frequency distributions \( \{p_n\} \). If parameters are \( c_1, c_2, \ldots, c_m \), and \( \vec{c} = (c_1, c_2, \ldots, c_m) \), then \( p_n = p_n(c_1, c_2, \ldots, c_m) = p_n(\vec{c}), n = 0, 1, 2, \ldots \).

The Generating Function of distribution \( \{p_n(\vec{c})\} \) is defined as follows: for any \( x \in [0, 1) \)

\[
P(x; \vec{c}) = \sum_{n \geq 0} p_n(\vec{c}) x^n.
\]

So, due to the Continuity Theorem, if \( \vec{c} \to \vec{c}' \), i.e. \( c_1 \to c'_1, c_2 \to c'_2, \ldots, c_m \to c'_m \), then \( P(x; \vec{c}) \to P(x; \vec{c}') \) and reverse.

Consider examples of well-known two-parametric families of frequency distributions.

Let us consider the family of Pareto Distributions (1.2.6)-(1.2.7)

\[
p_n(\rho, b) = c(\rho, b) \cdot (n + b)^{-\rho}, \quad n = 1, 2, \ldots, \quad (1.8.1)
\]

\[
c(\rho, b) = \frac{1}{\sum_{n \geq 1} (n + b)^{-\rho}} = \frac{\Gamma(\rho)}{I(\rho, b)}, \quad (1.8.2)
\]

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where

\[ \Gamma(\rho) = \int_0^\infty e^{-t} \cdot t^{\rho-1} dt, \quad (1.8.3) \]

\[ I(\rho, b) = \int_0^1 (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} dx. \]

We have the following restriction on parameters: \(-1 < b < +\infty, 1 < \rho < +\infty.\)

The second expression for \(c(\rho, b)\) in (1.8.1) is taken from formula (1.7.8) and is valid under more strong restriction on parameter \(b : 0 < b < +\infty.\)

For the Pareto Distribution (1.8.1)-(1.8.2) let us start once more. One of the special functions of Mathematical Analysis \(\phi(x, \rho, b), 0 \leq x < 1, b \neq 0, -1, -2, \ldots, \rho \in R^+,\) is defined in the form of convergence series (see, 9550, p. 1075, [20])

\[ \phi(x, \rho, b) = \sum_{n \geq 0} \frac{1}{(n + b)^\rho} \cdot x^n. \quad (1.8.4) \]

For \(\phi(x; \rho, b)\) the following integral representations hold (see, 9.556, p. 1075, [20])

\[ \phi(x, \rho, b) = \frac{1}{\Gamma(\rho)} \int_0^\infty \frac{t^{\rho-1}e^{bt}}{1-xe^{-t}} dt = \frac{1}{\Gamma(\rho)} \int_0^\infty \frac{t^{\rho-1}e^{-(b-1)t}}{e^t - x} dt. \quad (1.8.5) \]

under restrictions: \(b \in R^+\) and either \(0 \leq x < 1, \rho \in R^+; \) or \(x = 1, 1 < \rho < +\infty.\)

Let us form for Pareto Distribution \(\{p_n(\rho, b)\}\) its Generating Function

\[ P(x; \rho, b) = \sum_{n \geq 1} p_n \cdot x^n = \frac{\sum_{n \geq 1} (n + b)^{-\rho} x^n}{\sum_{n \geq 1} (n + b)^{-\rho}}, \quad 0 \leq x < 1. \quad (1.8.6) \]

Due to (1.8.4)-(1.8.6), in case \(b \in R^+,\) we obtain

\[ P(x; \rho, b) = \frac{\sum_{n \geq 0} (n + b)^{-\rho} x^n - b^{-\rho}}{\sum_{n \geq 0} (n + b)^{-\rho} - b^{-\rho}} = \frac{\int_0^\infty \frac{t^{\rho-1}e^{bt}}{1-xe^{-t}} dt - \frac{1}{b}}{\int_0^\infty \frac{t^{\rho-1}e^{bt}}{1-e^{-t}} dt - \frac{1}{b}} = \frac{\int_0^\infty \frac{t^{\rho-1}e^{-(b-1)t}}{e^t - x} dt - \frac{1}{b}}{\int_0^\infty \frac{t^{\rho-1}e^{-(b-1)t}}{e^t - 1} dt - \frac{1}{b}}. \quad (1.8.7) \]

Consider the case \(-1 < b < 0.\) Introducing new variable \(c = 1 + b, \) \(0 < c < 1,\) we obtain

\[ P(x; \rho, b) = \frac{\sum_{m \geq 0} (m + c)^{-\rho} x^n}{\sum_{m \geq 0} (m + c)^{-\rho}} = \frac{\int_0^\infty \frac{t^{\rho-1}e^{-(1+c)t}}{1-xe^{-t}} dt}{\int_0^\infty \frac{t^{\rho-1}e^{-(1+c)t}}{1-e^t} dt} = \frac{\int_0^\infty \frac{t^{\rho-1}e^{-bt}}{e^{c} - x} dt}{\int_0^\infty \frac{t^{\rho-1}e^{-bt}}{e^c - 1} dt}. \quad (1.8.8) \]

It remains the case \(b = 0,\) which corresponds to Power Laws. Denote \(c(\rho, 0) = c(\rho).\)

The function \((1/c(\rho))\) for \(1 < \rho < +\infty\) is the well-known in Mathematical Analysis
Weierstrass’ Zeta function \( \zeta(\rho) \) (see, 9.522.1, p. 1073, [20]). For \( \zeta(\rho), 1 < \rho < +\infty \), the following integral representation holds (see, 9.513.2, p. 1072, [20])

\[
\zeta(\rho) = \frac{2^\rho}{(2^\rho - 1)\Gamma(\rho)} \int_0^\infty \frac{t^{\rho-1}e^t}{e^{2t} - 1} \, dt = \frac{1}{c(\rho)}.
\]

Next, in 4.316.1, p. 568, [20] the following equality is presented

\[
\int_0^1 \ln(1 - xt^n)(\ln \frac{1}{t})^{\rho-2} \frac{dt}{t} = -\frac{1}{\alpha^{\rho-1}} \Gamma(\rho - 1) \sum_{n \geq 1} \frac{x^n}{n^{\rho}}, \quad 1 < \rho < \infty, 0 < x < 1, 0 < \alpha < +\infty.
\]

In particular, \( \sum_{n \geq 1} \frac{x^n}{n^{\rho}} = \frac{1}{\Gamma(\rho)} \int_0^1 \ln(1 - xt)(\ln \frac{1}{t})^{\rho-2} \frac{dt}{t} \) for \( \alpha = 1 \).

The last equality together with representation of \( \zeta(\rho), 1 < \rho < +\infty \), implies

\[
P(x; \rho, 0) = (\rho - 1) \frac{2^\rho - 1}{2^\rho} \int_0^1 \ln(1 - xt)(\ln \frac{1}{t})^{\rho-2} \frac{dt}{t} - 1, \quad 0 \leq x < 1, 1 < \rho < +\infty. \quad (1.8.9)
\]

The formulas (1.8.7)-(1.8.9) give the complete representation in form of improper integrals for the Generating Function of Pareto Distribution.

Next, we need in hypergeometric function \( F(p, 1; q + 1; z) \), \( 0 < p < q < +\infty, 0 < z \leq 1 \), given by series (see, (1.2.15))

\[
F(p, 1; q + 1; z) = 1 + \frac{p}{q+1} z + \frac{p(p+1)}{(q+1)(q+2)} z^2 + \frac{p(p+1)(p+2)}{(q+1)(q+2)(q+3)} z^3 + \cdots =
\]

\[
= 1 + \sum_{n \geq 1} z^n \prod_{K=1}^n \frac{p + K - 1}{q + K}. \quad (1.8.10)
\]

Due to (1.2.16)-(1.2.17), this function exhibits the following integral representation

\[
F(p, 1; q + 1; z) = \frac{1}{B(1, q)} \int_0^\infty (1 - t)^q(1 - tz)^{-p} \, dt,
\]

where \( B(x, y) \) is the Beta Function (1.2.17). Since \( B(1, q) = \int_0^\infty (1 - t)^{q-1} \, dt = q^{-1} \), therefore

\[
F(p, 1; q + 1; z) = q \int_0^\infty (1 - t)^{q-1}(1 - tz)^{-p} \, dt. \quad (1.8.11)
\]

Let us consider the family of Waring Distribution (1.2.10)-(1.2.12)

\[
\hat{p}_0(p, q) = 1 - \frac{p}{q}, \quad \hat{p}_n(p, q) = \left( 1 - \frac{p}{q} \right) \prod_{K=1}^n \frac{p + K - 1}{q + K}, \quad n = 1, 2, \cdots,
\]

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and form *Generating Function* for it

\[
\hat{P}(x; p, q) = \sum_{n=0}^{\infty} \hat{p}_n(p, q)x^n = (1 - \frac{p}{q}) \left\{ 1 + \sum_{n=1}^{\infty} x^n \prod_{K=1}^{n} \frac{p + K - 1}{q + K} \right\}. \tag{1.8.12}
\]

In accordance with (1.8.10)-(1.8.12) under the restrictions \(0 < p < q < +\infty, 0 < x \leq 1\) we obtain

\[
\hat{P}(x; p, q) = (1 - \frac{p}{q})F(p, 1; q + 1; z) = (q - p) \int_{0}^{\infty} (1 - t)^{q-1}(1 - tz)^{-p} dt. \tag{1.8.13}
\]

In some cases of parameters \(p\) and \(q\) it is possible to evaluate the integral at the right-hand-side of (8.8.13).

**Example** Let \(p = \frac{1}{2}, q = 1\). Due to 9.121.24, p. 1041, [20],

\[
F\left(\frac{1}{2}, 1; 2; 4z(1 - z)\right) = \frac{1}{1 - z}, \quad 0 < z < \frac{1}{2}; \quad 0 < z(1 - z) \leq \frac{1}{4}.
\]

Let us make the replacement of variable: \(4z(1 - z) = y, \quad 0 < y < 1\).

Since \(\frac{4}{dx}(4z(1 - z)) = 4(1 - 2z) > 0\) for \(0 < z < \frac{1}{2}\), therefore this function strictly increases in \((0, \frac{1}{2})\). So the reverse function \(z = z(y)\) exists in \((0, \frac{1}{2})\) and increases. Let us find \(z = z(y)\).

Solving the equation \(z^2 - z + \frac{y}{4} = 0\) we get two roots \(z_1 = \frac{1}{2}(1 + \sqrt{1 - y})\) and \(z_2 = \frac{1}{2}(1 - \sqrt{1 - y})\).

But only the second one satisfies condition \(0 < z < \frac{1}{2}\) for \(y \in (0, 1)\). That is why we have

\[
F\left(\frac{1}{2}, 1; 2y\right) = \frac{1}{1 - \frac{1}{2}(1 - \sqrt{1 - y})} = \frac{2}{1 + \sqrt{1 - y}}, \quad 0 < y < 1.
\]

It means that, due to (1.8.11), we obtain \(\hat{P}(x; \frac{1}{2}, 1) = \frac{1}{1 + \sqrt{1 - x}}\).

Now, taking into account the expressions for *Generating Functions* \(P(x; \rho, b)\) and \(\hat{P}(x; p, q)\) (see, (1.8.7) and (1.8.13)) we conclude: for \(0 \leq x < 1\) the limits exist

\[
\lim_{(\rho, b) \rightarrow (\rho', b')} P(x; \rho, b) = P(x; \rho', b'), \quad \lim_{(p, q) \rightarrow (p', q')} \hat{P}(x; p, q) = \hat{P}(x; p', q'),
\]

Due to the *Continuity Theorem*, these relationships are equivalent to the following ones

\[
\lim_{(\rho, b) \rightarrow (\rho', b')} p_n(\rho, b) = p_n(\rho', b'), \quad n = 1, 2, \ldots, \tag{1.8.14}
\]
\[
\lim_{(p, q) \rightarrow (p', q')} \hat{p}_n(p, q) = \hat{p}_n(p', q'), \quad n = 0, 1, 2, \ldots. \tag{1.8.15}
\]

### 1.8.2 Stability Problems

Denote by \(F(x; \bar{c}), 0 \leq x \leq +\infty\), the *distribution function*, which corresponds to distribution \(\{p_n(\bar{c})\}\), i.e.

\[
F(x; \bar{c}) = \sum_{n=0}^{\lfloor x \rfloor} p_n(\bar{c}), \quad \tag{1.8.16}
\]
where \([x]\) denotes the integer part of positive number \(x\). Denote

\[
\delta(\vec{c}; \vec{c}') = \delta(\vec{c}' ; \vec{c}) = \sup_{0 \leq x \leq +\infty} |F(x; \vec{c}) - F(x; \vec{c}')|.
\] (1.8.17)

Here \(F(x; \vec{c}')\) corresponds to \(\{p_n(\vec{c}')\}\), and \(\vec{c}' \neq \vec{c}\), i.e. at least for one index \(i_1\) from the set \(\{1, 2, \ldots, m\}\) we have \(c_{i_1} \neq c'_{i_1}\).

The limit relationships \(p_n(\vec{c}) \to p_n(\vec{c}')\), \(n = 1, 2, \ldots\), as \(\vec{c} \to \vec{c}'\) (compare also to (1.8.14) and (1.8.15)), are equivalent to

\[
\lim_{\vec{c} \to \vec{c}'} F(x; \vec{c}) = F(x; \vec{c}') \quad \text{for every} \quad x \in [0, +\infty),
\] (1.8.18)

which is the most simple form of stability result - the continuity by parameters.

According to (1.8.14) and (1.8.15) such results hold for the families of Pareto and Waring Distributions.

Let us fix all parameters of considering \textit{m-parametric} family of frequency distributions excluding one of them, for instance (without loss of generality) the first one, i.e. let us put \(c_1 \neq c'_1\) and \(c_2 = c'_2, c_3 = c'_3, \ldots, c_m = c'_m\). For such a case for \textit{brevity} we denote

\[
\delta(c; c') = \delta(\vec{c}; \vec{c}') \quad \text{with} \quad c = c_1, c' = c'_1.
\]

The following type of statement is of interest

\textit{Uniformly on} \(c\) \textit{and} \(c'\) \textit{from finite} \([\bar{c}, \tilde{c}]\), \textit{where} \(\bar{c}\) \textit{and} \(\tilde{c}\) \textit{belong to the domain of definition of parameters} \(c\) \textit{and} \(c'\), \textit{the limit exists}

\[
\lim_{|c-c'| \to 0} \delta(c; c') = 0.
\] (1.8.19)

This type of statements are obtained in [16] for several \textit{finite-parametric} families of frequency distributions, in particular, for the family of Waring Distributions by parameters \(p\) and \(q\) separately.

Denote by \(\Omega\) the domain of vector \(\vec{c}\)'s changes, \(\Omega \subseteq \mathbb{R}^m\). The most general form of stability statement (by parameters) may be formulated as follows

\textit{Uniformly on} \(\vec{c}\) \textit{and} \(\vec{c}'\) \textit{from any compact convex bounded} \(K \subseteq \Omega\)

\textit{the limit exists}

\[
\lim_{\sum_{i=1}^m |c_i-c'_i| \to 0} \delta(\vec{c}; \vec{c}') = 0.
\] (1.8.20)

Remind that in \(\mathbb{R}^m\) the metric \(\rho(\vec{c}, \vec{c}') = \sqrt{\sum_{i=1}^m (c_i - c'_i)^2}\) is equivalent to others, in particular, to \(\hat{\rho}(\vec{c}, \vec{c}') = \sum_{i=1}^m |c_i - c'_i|\).

Our purpose at the rest of this \textit{Chapter} is to establish statements of type (1.8.20) for the families of Pareto and Waring Distributions. Note that for the Pareto Distributions even the result of type (1.8.19) has not been proven.
1.9 Simplifications of Stability Problem

1.9.1 First Simplification

In conditions of the limit relationship (1.8.20)’s fulfillment, obviously, for all \( n = 0, 1, 2, \cdots \) uniformly on \( \vec{c} \) and \( \vec{c}' \) the limit

\[
\lim_{\sum_{i=1}^{m} |c_i - c'_i| \to 0} |p_n(\vec{c}) - p_n(\vec{c}')| = 0
\]

for \( m \)-parametric family \( \{p_n(\vec{c})\} \) of frequency distributions exists.

In general, the reverse to (1.9.1) statement is not true. But under following additional conditions the reverse statement is true. Here they are:

1. For any \( n = 0, 1, 2, \cdots \) the probability \( p_n(\vec{c}), \vec{c} \in \Omega \), is represented in the form

\[
p_n(\vec{c}) = \frac{g_n(\vec{c})}{g(\vec{c})}, \quad g_n(\vec{c}) \geq 0.
\]

Obviously,

\[
\sum_{n \geq 0} g_n(\vec{c}) = g(\vec{c}),
\]

and as a consequence of (1.9.2)

\[
g_n(\vec{c}) < g(\vec{c}), n = 0, 1, 2, \cdots.
\]  

2. There is \( \vec{c}^* \in K \), where \( K \) is compact convex bounded set from \( \Omega \), such that for all \( n = 0, 1, 2, \cdots \) we have \( g_n(\vec{c}^*) = \max_{\vec{c} \in K} g_n(\vec{c}) \).

3. There is \( \vec{c}_* \in K \) such that \( g(\vec{c}_*) = \min_{\vec{c} \in K} g(\vec{c})(>0) \).

4. The sequence \( \{p_n(\vec{c}^*)\} \) varies regularly at infinity with exponent \( (-\rho) \), where \( \rho \in (1, +\infty) \), and exhibits the constant slowly varying component, say \( L^* \).

Let us prove the reverse statement, which is based on the following inequalities. Namely, \( \vec{c} \in K \) and \( \vec{c}' \in K \) we have

\[
\delta(\vec{c}, \vec{c}') \leq \sum_{n \geq 0} |p_n(\vec{c}) - p_n(\vec{c}')| \leq \sum_{n=0}^{N-1} |p_n(\vec{c}) - p_n(\vec{c}')| + \sum_{n \geq N} p_n(\vec{c}) + \sum_{n \geq N} p_n(\vec{c}') = \\
= \sum_{n=0}^{N-1} |p_n(\vec{c}) - p_n(\vec{c}')| + \sum_{n \geq N} \frac{g_n(\vec{c})}{g(\vec{c})} + \sum_{n \geq N} \frac{g_n(\vec{c}')}{g(\vec{c}')} \leq \\
\leq \sum_{n=0}^{N-1} |p_n(\vec{c}) - p_n(\vec{c}')| + \frac{2}{g(\vec{c}^*)} \sum_{n \geq N} g_n(\vec{c}^*) = \sum_{n=0}^{N-1} |p_n(\vec{c}) - p_n(\vec{c}')| + 2 \cdot \frac{g(\vec{c}^*)}{g(\vec{c}_*)} \sum_{n \geq N} p_n(\vec{c}^*).
\]
Here the conditions 1.-3. were used and \( N \) denotes any positive integer.

For \( \vec{c} \in K \) and \( \vec{d} \in K \) we may rewrite (1.9.4) in the form

\[
\delta(\vec{c}, \vec{d}) \leq \sum_{n=0}^{N-1} |p_n(\vec{c}) - p_n(\vec{d})| + A \cdot q_N(\vec{c}),
\]

(1.9.5)

where \( A \) is some positive constant and \( q_n(\vec{c}) = \sum_{m \geq n} p_m(\vec{c}) \), \( n = 0, 1, 2, \ldots \).

Due to the condition 4. and Theorem 1.1(i) from VIII.9, p. 273, [23], the sequence \( \{q_n(\vec{c})\} \) of positive numbers varies regularly at infinity with exponent \((-\rho + 1)\) and exhibits constant slowly varying component being equal to \( K^* \cdot (-\rho + 1)^{-1} := \alpha \in \mathbb{R}^+ \).

Therefore any given \( \varepsilon \in (0, 1) \) there is an index \( N_0 > 0 \) such that for all \( n \geq N_0 \) we have

\[
q_{N_0}(\vec{c}) < \frac{1}{A} (\alpha + 1) \frac{1}{N_0^{\rho-1}} < \frac{\varepsilon}{2}.
\]

(1.9.6)

Due to the initial assumption (1.9.1) for \( n = 0, 1, 2, \ldots \), for the above given \( \varepsilon \in (0, 1) \) and number \( N_0 > 0 \) we obtain

\[
\lim_{\sum_{i=1}^{m} |\vec{c}_i - \vec{c}_j| \to 0} \left| p_{K}(\vec{c}) - p_{K}(\vec{d}) \right| < \frac{\varepsilon}{2N_0}, \quad K = 0, 1, \ldots, N_0 - 1.
\]

(1.9.7)

The inequalities (1.9.5)-(1.9.7) imply \( \lim_{\sum_{i=1}^{m} |\vec{c}_i - \vec{c}_j| \to 0} \delta(\vec{c}; \vec{d}) < \varepsilon \) for any \( \varepsilon \in (0, 1) \).

Tending \( \varepsilon \downarrow 0 \) in the last inequality we come to (1.8.20).

The reverse statement is proved.

Under conditions 1.-3. the uniform convergence (by \( \vec{c} \) and \( \vec{d} \) from compact convex bounded \( K \))

\[
\lim_{\sum_{i=1}^{m} |\vec{c}_i - \vec{c}_j| \to 0} \left| g_n(\vec{c}) - g_n(\vec{d}) \right| = 0, \quad n = 0, 1, 2, \ldots,
\]

(1.9.8)

\[
\lim_{\sum_{i=1}^{m} |\vec{c}_i - \vec{c}_j| \to 0} \left| g(\vec{c}) - g(\vec{d}) \right| = 0
\]

(1.9.9)

together imply the uniform convergence (1.9.1) by \( \vec{c} \) and \( \vec{d} \) taken from \( K \).

Indeed, for \( n = 0, 1, 2, \ldots \) we proceed

\[
|p_n(\vec{c}) - p_n(\vec{d})| = \left| \frac{g_n(\vec{c})}{g(\vec{c})} - \frac{g_n(\vec{d})}{g(\vec{d})} \right| = \frac{1}{g(\vec{c})g(\vec{d})} \left| g(\vec{d}) g_n(\vec{c}) - g(\vec{c}) g_n(\vec{d}) \right| \leq \frac{1}{(g(\vec{c}))^2} \left\{ g(\vec{d}) \cdot |g_n(\vec{c}) - g_n(\vec{d})| + g_n(\vec{c}) \cdot |g(\vec{d}) - g(\vec{c})| \right\} \leq \frac{1}{(g(\vec{c}))^2} \left\{ g(\vec{c}) \cdot |g_n(\vec{c}) - g_n(\vec{d})| + g_n(\vec{c}) \cdot |g(\vec{d}) - g(\vec{c})| \right\} < \frac{g(\vec{c})}{(g(\vec{c}))^2} \left\{ |g_n(\vec{c}) - g_n(\vec{d})| + |g(\vec{c}) - g(\vec{d})| \right\},
\]

or

\[
|p_n(\vec{c}) - p_n(\vec{d})| < B \cdot \left\{ |g_n(\vec{c}) - g_n(\vec{d})| + |g(\vec{c}) - g(\vec{d})| \right\},
\]

(1.9.10)

where \( B \) is some positive constant. Here the condition 1. and relations (1.9.2)-(1.9.3) were used. The relationships (1.9.8)-(1.9.10) prove the statement.
1.9.2 Second Simplification

Constructing finite-parametric families of frequency distributions one has to be troubled on satisfaction of following requirements.

1. The family has to be determined with the help of minimal number of parameters.

In other words, it means that there is no any functional relation of equality type among chosen parameters. In particular, one of the parameters cannot be written down with the help of some others as their function.

It is so called independence of parameters on each others.

2. The domain of the change of any parameter must not be dependent on other parameters’ changes.

There can be relations of inequality type on among parameters which are themselves independent. Then one may try with the help of linear transformation to make the domains’ of changes of any parameter independent on other parameters’ changes.

Everybody knows the Gauss Method of bring the matrix (of linear transformation) to the triangle form.

It is so-called independence of domains of parameters on other parameters.

Example. The Family of Waring Distributions (1.2.10)-(1.2.11) is determined with the help of two independent parameters $p$ and $q$, but the domain of their changes has the form $\{(p, q) : 0 < p < q < +\infty\}$. In order to obtain independent domains for each parameter we make a linear transformation. We conserve $q$ and introduce as another parameter $\rho = q - p + 1$. Then we present the family of Waring Distributions in the form

\[
p_0 = \frac{\rho - 1}{q}, \tag{1.9.11}
\]

\[
p_n = \frac{\rho - 1}{q} \prod_{K=1}^{n} \left(1 - \frac{\rho}{q + K}\right) \tag{1.9.12}
\]

with two independent parameters $\rho$ and $q$, and with independent domains of their changes

\[1 < \rho < +\infty, \ 0 < q < +\infty. \tag{1.9.13}\]

Consider $m$-parametric family of frequency distributions $\{p_n(\vec{c})\}$ with independent $c_1, c_2, \cdots, c_m$ and their independent domains $< a_1, b_1 >, < a_2, b_2 >, \cdots, < a_m, b_m >$.

Here intervals $< \cdot, \cdot >$ are of types $(\cdot, \cdot), (\cdot, \cdot], [\cdot, \cdot), [\cdot, \cdot]$.

Obviously, $a_1 = \inf c_i$, $b_i = \sup c_i$ for $i = 1, 2, \cdots, m$.

It means that the set $\Omega$ (see, Section 1.8) is of the form $\Omega = X_{i=1}^{m} < a_i, b_i >$, where $X$ is a symbol of Cartesian product.
Now, the most general form of stability statement is formulated as follows.

Let for all \( i = 1, 2, \ldots, m \) take constants
\[
\bar{c}_i \in (a_i, b_i), \quad \bar{c}_i \in (a_i, b_i) \quad \text{with} \quad c_i \geq \bar{c}_i. \tag{1.9.14}
\]

Uniformly on \( \bar{c} = (c_1, c_2, \ldots, c_m) \) and \( \bar{c}' = (c'_1, c'_2, \ldots, c'_m) \), where \( c_i \leq c_i \leq \bar{c}_i \), \( c_i \leq c'_i \leq \bar{c}_i \) the limit exists (compare to (1.8.18))
\[
\lim_{\sum_{i=1}^{m} |c_i - c'_i| \to 0} \delta(\bar{c}, \bar{c}') = 0. \tag{1.9.15}
\]

It is clear that now conditions of general stability statement’s fulfillment became more simple than before and easy for verification.

Finite-parametric families of frequency distributions being used in bioinformatics, say \( \{p_n(\bar{c})\} \), satisfy conditions 1. and 4. In representation \( p_n(\bar{c}) = \frac{g_n(\bar{c})}{g(\bar{c})} \), n = 0, 1, 2, \ldots, as a rule, is monotone by each parameter \( c_i \) separately. These families even are built in such a way. Because of equality \( \sum p_n(\bar{c}) = 1 \) for them the normalization factor \( g(\bar{c}) \) arises and, due to equality, \( g(\bar{c}) = \sum g_n(\bar{c}) \), is monotone too.

Let us present new additional condition which may replace conditions 2.-3.

2) For any \( n = 0, 1, 2, \ldots \) \( g_n(\bar{c}) \) increases by \( c_1, c_2, \ldots, c_m \) separately.

So, the same is true also for \( g(\bar{c}) \).

If \( g_n(\bar{c}) \) decreases by some parameter \( c_i \) then we replace \( c_i \) by \((1/c_i)\).

We renumarate conditions: 1. is 1); 2) is just presented; 3) is 4.

In our case conditions 1)-3) imply conditions 1.-4.

Indeed, for \( \bar{c}^* = (c_1^*, c_2^*, \ldots, c_m^*) \) taken from condition 2. we have \( c^*_i = \bar{c}_i, \quad i = 1, 2, \ldots, m, \) because of increase of \( g_n(\bar{c}) \) by parameters and the form of the set \( K = \prod_{i=1}^{m} [c_i, \bar{c}_i] \).

Similarly, for \( \bar{c}_i = (c_1, c_2, \ldots, c_m) \) taken from condition 3., by same reasons, we have \( c_{*i} = c_i, \quad i = 1, 2, \ldots, m. \)

Now, the following conclusion in our case may be made

*General stability statement (1.9.15) takes place uniformly on \( \bar{c}, \bar{c}' \) with \( \bar{c}_i \leq c_i \leq \bar{c}_i, \quad i = 1, 2, \ldots, m, \) if uniformly on \( \bar{c}, \bar{c}' \) (1.9.8) and (1.9.9) hold.*

### 1.10 Stability for Pareto Distributions

In the present Section the general stability statement is proved for Pareto Distributions (1.2.6)-(1.2.7) (see, also (1.7.4) with constraints
\[
0 < b < +\infty, \quad 1 < \rho < +\infty. \tag{1.10.1}
\]
1.10.1 The Result and Discussion

For Pareto Distribution \( \{p_n(\rho, b)\} \) denote by \( F(x; \rho, b) \) corresponding distribution function and denote \( \delta(\rho, b; \rho', b') = \sup_{0 \leq x \leq +\infty} |F(x; \rho, b) - F(x; \rho', b')| \). Let the constants \( \rho \) and \( \bar{\rho} \), \( b \) and \( \bar{b} \) be fixed and satisfy inequalities \( 1 < \rho \leq \bar{\rho} < +\infty \), \( 0 < b \leq \bar{b} < +\infty \).

The following statement takes place.

**Theorem 1.3** Uniformly on \( \rho \in [\rho, \bar{\rho}], \; \rho' \in [\rho, \bar{\rho}], \; b \in [b, \bar{b}], \; b' \in [b, \bar{b}] \) the limit exists

\[
\lim_{|\rho - \rho'| + |b - b'| \to 0} \delta(\rho, b; \rho', b') = 0. \tag{1.10.2}
\]

Easily seen that for Pareto Distributions the conditions 1)-3) hold. Here we have, due to (1.2.6)-(1.2.7) and to the second expression for \( c(\rho, b) \) from formula (1.7.8),

\[
g_n(\rho, b) = \frac{1}{(n+b)^\rho}, \; n = 1, 2, \ldots, \tag{1.10.3}
\]

\[
((c(\rho, b))^{-1} =) \; g(\rho, b) = \sum_{m \geq 1} \frac{1}{(m+b)\rho} = \frac{I(\rho, b) - b^{-1}}{\Gamma(\rho)}, \tag{1.10.4}
\]

where

\[
I(\rho, b) = \int_0^1 (\ln \frac{1}{x})^{\rho-1} \frac{x^{-1}}{1-x} dx, \tag{1.10.5}
\]

and \( \Gamma(\cdot) \) is given by (1.8.3).

In order to establish Theorem 1.3, due to the consideration in previous Section, we have to prove that under the conditions of (1.10.2)’s fulfillment uniformly on \( \rho, \rho', b, b' \)

\[
\lim_{|\rho - \rho'| + |b - b'| \to 0} \left| \frac{1}{(n+b)^\rho} - \frac{1}{(n+b')^{\rho'}} \right| = 0, \; n = 1, 2, \ldots, \tag{1.10.6}
\]

\[
\lim_{|\rho - \rho'| + |b - b'| \to 0} \left| \frac{I(\rho, b) - b^{-1}}{\Gamma(\rho)} - \frac{I(\rho', b') - (b')^{-1}}{\Gamma(\rho')} \right| = 0, \tag{1.10.7}
\]

where (1.10.3)-(1.10.4) are used.

Below the proof of (1.10.6)-(1.10.7) shall be done step by step.

1.10.2 Preliminary Estimations

Since (1.10.6) is easy to prove, so, let us do it at once. Let us put

\[
b_+ = \max(b, b'), \; b_- = \min(b, b'), \; \rho_+ = \max(\rho, \rho'), \; \rho_- = \min(\rho, \rho').
\]
For $n = 1, 2, \cdots$ the inequalities hold
\[
\left| \frac{1}{(n+b)^\rho} - \frac{1}{(n+b')^\rho} \right| \leq \left| \frac{1}{(n+b)^\rho} - \frac{1}{(n+b')^\rho} \right| + \left| \frac{1}{(n+b)^\rho} - \frac{1}{(n+b)^\rho} \right| \leq \frac{1}{(n+b)^\rho} \cdot (1 - (1 - \frac{b - b'}{n+b})^\rho) + \frac{1}{(n+b)^\rho} \cdot (1 - (1 - \frac{\rho}{n+b})^\rho) \leq \frac{1}{(1+b)^2} \cdot \left\{ (1 - (1 - \frac{b - b'}{n+b})^\rho) + (1 - (1 - \frac{\rho}{n+b})^\rho) \right\}.
\] (1.10.8)

Since for any $n = 1, 2, \cdots$ the terms at the right-hand-side of (1.10.8) tend to zero as $|\rho - \rho'| + |b - b'| \to 0$, therefore (1.10.7) takes place uniformly on parameters $\rho, \rho'$ and $b, b'$ from $[\bar{\rho}, \rho]$ and $[b, \bar{b}]$ respectively.

Next, let us write down the inequalities
\[
\left| \frac{I(\rho, b) - b^{-1}}{\Gamma(\rho)} - \frac{I'(\rho', b') - (b')^{-1}}{\Gamma(\rho')} \right| = \frac{1}{\Gamma(\rho)} \cdot |\Gamma(\rho')(I(\rho, b) - \frac{1}{b}) - \Gamma(\rho')(I(\rho', b') - \frac{1}{b'})| \leq \frac{I(\rho, b)}{\Gamma(\rho)} |\Gamma(\rho') - \Gamma(\rho)| + \frac{1}{\Gamma(\rho')} \left\{ |I(\rho, b) - I(\rho', b)| + |I(\rho, b') - I(\rho', b')| \right\} + \frac{b \cdot b' \cdot \Gamma(\rho)}{\Gamma(\rho')} |\Gamma(\rho') - \Gamma(\rho)|. \] (1.10.9)

According to (1.10.5) and (1.8.3) $I(\rho, b) \leq I(\bar{\rho}, \bar{b})$, $\Gamma(\rho) \geq \Gamma(\rho')$. So, from (1.10.10) we get
\[
\left| \frac{I(\rho, b) - b^{-1}}{\Gamma(\rho)} - \frac{I(\rho', b') - (b')^{-1}}{\Gamma(\rho')} \right| \leq \frac{I(\bar{\rho}, \bar{b}) + \frac{\bar{b}^{-1}}{\Gamma(\rho')} \cdot (\Gamma(\rho') - \Gamma(\rho)) + \frac{1}{\Gamma(\rho')} \left\{ |I(\rho, b) - I(\rho', b)| + |I(\rho, b') - I(\rho', b')| \right\}. \] (1.10.10)

The inequality (1.10.10) shows that in order to prove (1.10.7) we need in following statements.
\[
\lim_{|\rho - \rho'| \to 0} |\Gamma(\rho') - \Gamma(\rho)| = 0 \text{ uniformly on } \rho \in [\bar{\rho}, \rho] \text{ and } \rho' \in [\bar{\rho}, \rho]. \] (1.10.11)
\[
\lim_{|\rho - \rho'| \to 0} |I(\rho, b) - I(\rho', b)| = 0 \text{ uniformly on } \rho \in [\bar{\rho}, \rho], \rho' \in [\bar{\rho}, \rho], b \in [b, \bar{b}]. \] (1.10.12)
\[
\lim_{|b - b'| \to 0} |I(\rho, b) - I(\rho, b')| = 0 \text{ uniformly on } \rho \in [\bar{\rho}, \rho], b \in [b, \bar{b}], b' \in [b, \bar{b}]. \] (1.10.13)

### 1.10.3 Proof of Theorem 1.3

Let us prove (1.10.11). Without loss of generality we assume that $\rho' > \rho$.

For $\varepsilon > 0$ there is a number $\delta \in (0, \frac{1}{3})$ such that $\int_{\delta}^{+\infty} t^{\rho - 1} e^{-t} dt < \frac{\varepsilon}{2}$, and if $|\rho' - \rho| < \delta$, then $\int_{0}^{\delta} t^{\rho - 1} e^{-t} (t^{\rho' - \rho} - 1) dt < \frac{\varepsilon}{2}$.

Putting $|\rho' - \rho| < \delta$ and using these inequalities we may estimate the difference
\[
0 \leq \Gamma(\rho') - \Gamma(\rho) \leq \int_{0}^{\delta} t^{\rho - 1} e^{-t} (t^{\rho' - \rho} - 1) dt + \int_{\delta}^{+\infty} t^{\rho - 1} e^{-t} dt + \int_{\delta}^{+\infty} t^{\rho - 1} e^{-t} dt \leq \int_{0}^{\delta} t^{\rho - 1} e^{-t} (t^{\rho' - \rho} - 1) dt + 2 \cdot \int_{\delta}^{+\infty} t^{\rho - 1} e^{-t} dt < \varepsilon.
\]
Thus, for \( \rho \in [\rho', \bar{\rho}] \), \( \rho' \in [\rho, \bar{\rho}] \) we have \( 0 \leq \lim_{\rho' - \rho \to 0} |\Gamma(\rho') - \Gamma(\rho)| < \varepsilon \) for every \( \varepsilon < 0 \).
Tending here \( \varepsilon \downarrow 0 \) we obtain (1.10.11).

Let us prove (1.10.12). We assume that \( \rho' > \rho \).

For \( \varepsilon > 0 \) there is a number \( \delta \in (0, \frac{1}{3}) \) such that

\[
\left( \int_0^\delta + \int_{1-\delta}^1 \right) \left( \ln \frac{1}{1-x} \right)^{\rho-1} \frac{x^{b-1}}{1-x} dx < \frac{\varepsilon}{3},
\tag{1.10.14}
\]

and if \( |\rho' - \rho| < \delta \), then \( \int_0^1 (\ln \frac{1}{2})^{\rho-1} \frac{x^{b-1}}{1-x} ((\ln \frac{1}{2})^{\rho'-\rho} - 1) dx < \frac{\varepsilon}{3} \).

Putting \( |\rho' - \rho| < \delta \) and using these inequalities we may estimate the difference:

\[
0 \leq I(\rho', b) - I(\rho, b) \leq \int_{\delta}^{1-\delta} (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} ((\ln \frac{1}{x})^{\rho'-\rho} - 1) dx +
\]

\[
+ \left( \int_0^\delta + \int_{1-\delta}^1 \right) (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} dx + \left( \int_0^\delta + \int_{1-\delta}^1 \right) (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} dx \leq
\]

\[
\leq \int_\delta^{1-\delta} (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} ((\ln \frac{1}{x})^{\rho'-\rho} - 1) dx + 2 \left( \int_0^\delta + \int_{1-\delta}^1 \right) (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} dx \leq \varepsilon.
\]

Thus, for \( \rho \in [\rho, \bar{\rho}] \), \( \rho' \in [\rho, \bar{\rho}] \) and \( b \in [b, \bar{b}] \) we have

\[
0 \leq \lim_{\rho' - \rho \to 0} |I(\rho, b) - I(\rho', b)| < \varepsilon \text{ for every } \varepsilon > 0.
\]
Tending \( \varepsilon \downarrow 0 \) we obtain (1.10.12).

Let us prove (1.10.13). Without loss of generality we assume that \( b' > b \).

For \( \varepsilon > 0 \) there is a number \( \delta \in (0, \frac{1}{3}) \) such that (1.10.15) holds and if \( |b' - b| < \delta \), then

\[
\int_{\delta}^{1-\delta} (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} (x^{b'-b} - 1) dx < \frac{\varepsilon}{3}.
\tag{1.10.15}
\]

Putting \( |b' - b| < \delta \) and using inequalities (1.10.14) and (1.10.15) we may estimate the difference:

\[
0 \leq I(\rho, b') - I(\rho, b) \leq \int_{\delta}^{1-\delta} (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} (x^{b'-b} - 1) dx +
\]

\[
+ \left( \int_0^\delta + \int_{1-\delta}^1 \right) (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} dx + \left( \int_0^\delta + \int_{1-\delta}^1 \right) (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} dx \leq
\]

\[
\leq \int_{\delta}^{1-\delta} (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} (x^{b'-b} - 1) dx + 2 \left( \int_0^\delta + \int_{1-\delta}^1 \right) (\ln \frac{1}{x})^{\rho-1} \frac{x^{b-1}}{1-x} dx \leq \varepsilon.
\]

Thus, for \( \rho \in [\rho, \bar{\rho}] \), \( b \in [b, \bar{b}] \), \( b' \in [b, \bar{b}] \) we have

\[
0 \leq \lim_{b' - b \to 0} |I(\rho, b) - I(\rho, b')| = 0 \text{ for every } \varepsilon > 0.
\]
Tending \( \varepsilon \downarrow 0 \) we obtain (1.10.13). Theorem 1.3 is proved.
1.10.4 Supplement to Theorem 1.3

We are going to show that Theorem 1.3 takes place for all family of distributions (1.2.6)-(1.2.7) with range of parameters (1.10.1). It means that fixing constants \( \rho, \bar{\rho}, b, \bar{b} \) we conserve inequalities \( 1 < \rho \leq \bar{\rho} < +\infty \) and replace inequalities \( 0 < b \leq \bar{b} < +\infty \) by \( -1 < \bar{b} \leq \bar{b} < +\infty \).

Two cases shall be considered separately: \( -1 < \bar{b} \leq \bar{b} < +\infty \) and \( -\varepsilon < \bar{b} \leq \bar{b} < \varepsilon \), where \( \varepsilon \in (0, 1) \) is taken "small enough".

Case \( -1 < \bar{b} \leq \bar{b} < +\infty \). In order to establish Theorem 1.3 we have to prove that uniformly on \( \rho, \rho', b, b' \)

\[
\lim_{|\rho - \rho'| + |b - b'| \to 0} \left| \frac{1}{(n - |b|)^\rho} - \frac{1}{(n - |b'|)^\rho'} \right| = 0, \quad n = 1, 2, \ldots, \tag{1.10.16}
\]

and

\[
\lim_{|\rho - \rho'| + |b - b'| \to 0} \left| \frac{J(\rho, b)}{\Gamma(\rho)} - \frac{J(\rho', b')}{\Gamma(\rho')} \right| = 0, \tag{1.10.17}
\]

where, due to the second expression (1.7.8), \( J(\rho, b) = \int_0^1 (\ln \frac{1}{x})^{\rho - 1} \frac{b|b|}{1 - x} \, dx \).

The limit relationship is already proved for \( n = 2, 3, \ldots \) because we may make a replacement \( 1 - |b| = c \) and \( 1 - |b'| = c' \) and reduce (1.10.16) to (1.10.6). The remaining case is \( n = 1 \), i.e.

\[
0 \leq \left| \frac{1}{(1 - |b|)^\rho} - \frac{1}{(1 - |b'|)^\rho'} \right| = \frac{1}{c^\rho} - \frac{1}{(c')^\rho} \leq \frac{1}{(1 - |b|)^2} \cdot |(c')^\rho - c^\rho| \leq \frac{(1 - |b|)^\rho}{(1 - |b|)^2} \cdot \left( \frac{c'}{c} \right)^\rho - 1,
\]

where without loss of generality we assume \( 1 - |b'| \geq 1 - |b| \), or \( |b| \geq |b'| \).

Now, the conclusion (1.10.16) for \( n = 1 \) uniformly on \( b \) and \( b' \) is obvious.

Next, let us write down (compare to (1.10.9))

\[
\left| \frac{1}{(n - |b|)^\rho} - \frac{1}{(n - |b'|)^\rho'} \right| \leq \frac{1}{(\Gamma(\rho))^2} \left\{ \Gamma(\rho)|J(\rho, b) - J(\rho', b)| + |J(\rho', b) - J(\rho', b')| \right\} + \frac{J(\rho, b)}{(\Gamma(\rho))^2} |\Gamma(\rho) - \Gamma(\rho)|.
\]

From this point the estimations are quite similar to case considered in Theorem 1.3.

1.11 Stability of Waring Distributions

In the present Section the general stability statement is proved for the family of Waring Distributions (1.9.11)-(1.9.12) with constraints (1.9.13).

1.11.1 The Result and Its Discussion

For Waring Distribution \( \{\hat{p}_n(\rho, q)\} \) denote by \( \hat{F}(x; \rho, q) \) corresponding distribution function and denote \( \delta(\rho, q; \rho', q') = \sup_{0 \leq x \leq +\infty} |\hat{F}(x; \rho, q) - \hat{F}(x; \rho', q')| \). Let the constants \( \bar{\rho}, \bar{q}, \bar{q} \) be fixed and satisfy inequalities \( 1 < \bar{\rho} \leq +\infty, \ 0 < \bar{q} \leq +\infty \).
The following statement takes place.

**Theorem 1.4** Uniformly on \( \rho \in [\rho, \bar{\rho}], \rho' \in [\rho, \bar{\rho}], q \in [q, \bar{q}], q' \in [q, \bar{q}] \) the limit exists

\[
\lim_{|\rho - \rho'| + |q - q'| \to 0} \hat{\delta}(\rho, q; \rho', q') = 0. \tag{1.11.1}
\]

The particular cases of Theorem 1.4, namely, uniform convergences of type (1.11.1) but only by each parameter separately have been proved in [16].

Easily seen that for the family of Waring Distributions the conditions 1)-3) hold. Here we have, due to (1.9.11)-(1.9.12),

\[
g_n(\rho, q) = \prod_{K=1}^{n} (1 - \frac{\rho}{q + K}), \quad n = 0, 1, 2, \ldots, \tag{1.11.2}
\]

\[
g(\rho, q) = \frac{q}{\rho - 1}, \tag{1.11.3}
\]

where \( \prod_{K=1}^{0} = 1 \). In order to establish Theorem 1.4, due to consideration in previous Section, we have to prove that under conditions of (1.11.1)'s fulfillment uniformly on \( \rho, \rho', q, q' \)

\[
\lim_{|\rho - \rho'| + |q - q'| \to 0} \left| \prod_{K=1}^{n} (1 - \frac{\rho}{q + K}) - \prod_{K=1}^{n} (1 - \frac{\rho'}{q' + K}) \right| = 0, \quad n = 0, 1, 2, \ldots, \tag{1.11.4}
\]

\[
\lim_{|\rho - \rho'| + |q - q'| \to 0} \left| \frac{q}{\rho - 1} - \frac{q'}{\rho' - 1} \right| = 0, \tag{1.11.5}
\]

where (1.11.2)-(1.11.3) are used.

### 1.11.2 Proof of Theorem 1.4

It is easy to verify (1.11.5). Indeed, in conditions of Theorem 1.4 we have

\[
0 \leq \left| \frac{q}{\rho - 1} - \frac{q'}{\rho' - 1} \right| = \frac{1}{(\rho - 1)(\rho' - 1)} \cdot (|\rho' - 1|q - q' + q'(\rho' - \rho)) \leq \frac{1}{(\rho - 1)(\rho' - 1)} \cdot ((\rho' - 1)|q - q'| + q'|\rho' - \rho|) \leq \frac{|q - q'|}{\rho - 1} + \frac{q'|\rho' - \rho|}{(\rho - 1)(\rho' - 1)} \leq \frac{q'}{\rho' - 1},
\]

which implies (1.11.5).

Next, in conditions of Theorem 1.4 for \( n = 1, 2, \ldots \) let us estimate the difference

\[
0 \leq \left| \prod_{K=1}^{n} (1 - \frac{\rho}{q + K}) - \prod_{K=1}^{n} (1 - \frac{\rho'}{q' + K}) \right| \leq \sum_{m=1}^{n} \left| \prod_{K=1}^{m-1} (1 - \frac{\rho'}{q' + K})(\prod_{K=m}^{n} (1 - \frac{\rho}{q + K})) - \prod_{K=1}^{m} (1 - \frac{\rho'}{q' + K})(\prod_{K=m+1}^{n} (1 - \frac{\rho}{q + K})) \right| = \]

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\begin{equation}
\sum_{m=1}^{n} \left( \prod_{K=1}^{m-1} (1 - \frac{\rho'}{q' + K}) \right) \left( \prod_{K=m+1}^{n} (1 - \frac{\rho}{q + K}) \right) \left( \prod_{K=1}^{n} (1 - \frac{\rho}{q + m}) \right) \leq \sum_{m=1}^{n} \left( \prod_{K=1}^{n} (1 - \frac{\rho}{q + K}) \right) \left| \frac{\rho'}{q' + m} - \frac{\rho}{q + m} \right|.
\end{equation} 

Due to (1.11.6) and (1.11.4) the problem is reduced to the estimation for \( n = 1, 2, \cdots \) (in conditions of Theorem 1.4) of the difference

\[ 0 \leq \left| \frac{\rho}{q + n} - \frac{\rho'}{q' + n} \right| = \frac{1}{(q + n)(q' + n)} \left( (q' + n)(\rho - \rho') + \rho'(q' - q) \right) \leq \frac{|\rho - \rho'|}{q + n} + \frac{\rho'|q' - q|}{(q + n)(q' + n)} \leq \frac{|\rho - \rho'|}{q} + \bar{\rho}|q' - q| \frac{1}{(q')^2}, \]

which implies (1.11.4). Theorem 1.4 is proved.
Chapter 2

Frequency Distributions Generated by Stable Densities

2.1 From Scale-Invariant to Semi-Group Property

2.1.1 Self-Organized Systems

One of the important regularities of many large-scale biomolecular systems is their self-organization. The conception of the self-organization appeared in the Phase Transition System Theory where, very often, systems spontaneously self-organized themselves in fractals. The similar situation one may observe in networks, in particular, biomolecular networks. Here very often, we see the reproduction of properties of the network on previous fractals during the process of network’s enlargement.

We already mentioned in Section 1.1 that the Power Law being scale-invariant allows to extrapolate statistical properties of any part of complex biomolecular system to the whole system. That is why the Power Law is widely used in self-organized complex systems. Let us more precisely introduce self-organized systems. In self-organized systems knowing the local frequency distributions in two successive non-intersected intervals (fractals) we must be able to obtain the frequency distribution in the united interval (in the union of these intervals). It implies that we can extrapolate the frequency distribution in whole system. The second regularity consists in following. The frequency distribution must be of the same type in united interval as it is in each interval. These intervals (fractals) may be chosen with approximately equal lengths in the way, which allows to postulate either the independence or some type of "weak" dependence between the numbers of events’ occurrences on each fractal. These random numbers are characterized by local frequency distributions on fractals. In self-organized systems of such type for densities of continuous analogs of events’ occurrence numbers’ distributions, in-
Instead of scale-invariance the semi-group property has to take place. In contradiction to scale-invariance property, where the operation of multiplication is used, the semi-group property is based on operation of convolution.

The semi-group property implies that the convolution of densities (or distribution functions) of the same type equals to density (or distribution function) of exactly this type. Such semi-group property is intrinsic for normal, Cauchy’s, Levy’s distribution functions and for many other very useful ones. Notice that the semi-group property holds, for instance, for the four-parametric family of Stable Laws that is very important in Probability Theory. It explains our interest to this family.

Later we are going to substantiate that for self-organized large-scale biomolecular systems, in particular, for growing biomolecular networks of the type described above the conception of regular variation and the semi-group property for empirical frequency distributions’ continuous analogs are closely connected and supplement each other from the point of view of Probability Theory.

It is just the time to mention that Stable Laws not only satisfy semi-group property, but also have regularly varying at infinity tails.

Any distribution function $F$ defined on $R^1 = (-\infty, +\infty)$ has two tails $F(-x)$ (the left one) and $1 - F(x)$ (the right one), where $x \in R^+.$

2.1.2 Semi-Group Property: The Definition

In order to formulate the semi-group property for distribution functions mathematically we need in notion of convolution (see, for instance, [29]).

Let $\xi_1$ and $\xi_2$ be two independent random variables defined on the same probability space. Many new variables on the same probability space can be constructed as functions of $\xi_1$ and $\xi_2,$ for instance $\xi = \xi_1 + \xi_2.$

Denote by $F, F_1, F_2$ the distribution functions of random variables $\xi, \xi_1, \xi_2$ respectively. Then for any $x \in R^1$ the formula

$$F(x) = \int_{-\infty}^{+\infty} F_1(x-y)dF_2(y) = \int_{-\infty}^{+\infty} F_2(x-y)dF_1(y) \quad (2.1.1)$$

is a result of total probability formula. In (2.1.1) we deal with Stieltjes Integral. The operation given by (2.1.1) shall be written as $F = F_1 \ast F_2 = F_2 \ast F_1,$ where $\ast$ denotes the sign of convolution. Thus, due to (2.1.1) the convolution is a symmetric operation.

If one of presented random variables $\xi_1$ or $\xi_2$ has a density, say $f_1$ or $f_2$ respectively, then (2.1.1) may be rewritten in terms of Riemann Integral

$$F(x) = \int_{-\infty}^{+\infty} F_2(x-y) \cdot f_1(y)dy, \quad x \in R^1, \quad (2.1.2)$$
or
\[ F(x) = \int_{-\infty}^{+\infty} F_1(x-y)f_2(y)dy, \quad x \in R^1, \]  
(2.1.3)

respectively. In this case the random variable \( \xi \) also has a density, say \( f \), and
\[ f(x) = \int_{-\infty}^{+\infty} f_1(x-y)dF_2(y), \quad x \in R^1, \]  
(2.1.4)
or
\[ f(x) = \int_{-\infty}^{+\infty} f_2(x-y)dF_1(y), \quad x \in R^1, \]  
(2.1.5)
respectively. If both random variables \( \xi_1 \) and \( \xi_2 \) have densities, then
\[ f(x) = \int_{-\infty}^{+\infty} f_1(x-y)f_2(y)dy = \int_{-\infty}^{+\infty} f_2(x-y)f_1(y)dy, \quad x \in R^1, \]  
(2.1.6)
which shall be written in the form \( f = f_1 * f_2 = f_2 * f_1 \).

If \( \xi_1 \geq 0 \) and \( \xi_2 \geq 0 \) (with probability 1), then their distribution functions and densities (if exist) are concentrated on \([0, +\infty)\) and the convolution (2.1.1) is reduced to
\[ F(x) = \int_{0-}^{x} F_1(x-y)dF_2(y) = \int_{0-}^{x} F_2(x-y)dF_1(y), \quad x \in [0, +\infty). \]

Similar changes have to be done to formulas (2.1.2)-(2.1.6).

Let \( \{F_\alpha(x)\} \) be a one-parametric family of distribution functions and \( f_\alpha \) be the density of \( F_\alpha \). So, we have one-parametric family of densities \( \{f_\alpha(x)\} \).

**Definition 8.** We say that for family \( \{f_\alpha(x)\} \) the semi-group property holds if this family is closed under convolution, i.e.
\[ f_{\alpha_1} * f_{\alpha_2} = f_{\alpha_1 + \alpha_2}. \]  
(2.1.7)

The analog of (2.1.7) may be written down for distribution functions.

Such semi-group property is intrinsic for many often used in practise distribution functions and their densities. For instance, the gamma-densities are concentrated on \([0, +\infty)\) and are defined by formula
\[ f_{\alpha,\nu}(x) = \frac{1}{\Gamma(\nu)} \alpha^{\nu} x^{\nu-1} e^{-\alpha x}, \quad \nu \in R^+, \quad x \in R^+ \]  
(2.1.8)
and \( f_{\alpha,\nu}(0) = 0 \), where \( \Gamma(\cdot) \) is Euler’s Gamma Function.

Thus, the family of gamma-densities is a two-parametric family of densities. Here \( \alpha \) is a trivial scale parameter, but the parameter \( \nu \) is essential. That is why let us fix parameter \( \alpha \). In
In this case we get one-parametric family, which satisfies following semi-group property (see, [23], II, 2):

\[ f_{\alpha, \nu_1} * f_{\alpha, \nu_2} = f_{\alpha, \nu_1 + \nu_2}. \]  

(2.1.9)

Indeed, by (2.1.4) (or (2.1.5)) the left-hand-side of (2.1.9) takes the form

\[
\frac{\alpha^{\nu_1+\nu_2}}{\Gamma(\nu_1) \cdot \Gamma(\nu_2)} e^{-\alpha x} \int_0^x (x - y)^{\nu_1-1} \cdot y^{\nu_2-1} dy.
\]

(2.1.10)

After substitution \( y = xt \) in the integral the expression (2.1.10) is transformed into the following one

\[
\frac{(\alpha x)^{\nu_1+\nu_2}}{\Gamma(\nu_1) \cdot \Gamma(\nu_2)} e^{-\alpha x} \int_0^1 (1 - t)^{\nu_1-1} \cdot t^{\nu_2-1} dt.
\]

(2.1.11)

The integral in (2.1.11) is a well-known Beta Function (see, (1.2.17)). Since \( B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \) (see, 8.384.1, p.950, [20]), therefore

\[
\int_0^1 (1 - t)^{\nu_1-1} \cdot t^{\nu_2-1} dt = \frac{\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1+\nu_2)}.
\]

Substituting the last equality into (2.1.11) we come to \( f_{\alpha, \nu_1 + \nu_2}(x) \). The convolution formula (2.1.9) is proved.

### 2.1.3 Normal Laws

The Normal Law is famous because of its numerous applications in astrophysics, chemistry, biology, social and political, engineering sciences, everywhere. It appears everywhere if we deal with cumulative effect of very many very small factors of the same kind. Let

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du, \quad x \in \mathbb{R}^1,
\]

(2.1.12)

be standard Normal Law. It has mean value \( E\xi = \int_{-\infty}^\infty x d\Phi(x) := a = 0 \), where \( E \) denotes the sign of mathematical expectation, and \( \xi \) is a standard normal random variable having distribution function \( \Phi \), and variance \( D\xi = E(\xi - E\xi)^2 = E\xi^2 = \int_{-\infty}^\infty x^2 d\Phi(x) := \sigma^2 = 1 \), where \( D \) denotes the sign of variance. The distribution function \( \Phi \) has density

\[
\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}^1.
\]

(2.1.13)

The graphs of functions \( \Phi \) and \( \phi \) are presented in Figure 4. (a) and (b), respectively.
4(a)
With the help of standard normal random variable we construct a family of random variables \( \{ \eta = \sigma \xi + a, \sigma \in R^+, a \in R^1 \} \). The distribution function \( \Phi_{a,\sigma} \) of random variable \( \eta \) equals to
\[
\Phi_{a,\sigma}(x) = P(\eta < x) = P(\xi < \frac{x-a}{\sigma}) = \Phi\left(\frac{x-a}{\sigma}\right), \quad x \in R^1.
\]
Due to (2.1.14), we obtain a family of Normal Laws: \( \{ \Phi_{a,\sigma} : \sigma \in R^+, a \in R^1 \} \) of one type with \( \Phi \) (see, Definition 3.). Here \( \Phi_{a,\sigma} \) has mean value \( a \) and variance \( \sigma^2 \). Indeed, \( E\eta = E(\sigma \cdot \xi + a) = \sigma \cdot E\xi + a = a, \) \( D\eta = D(\sigma \xi + a) = D(\sigma \cdot \xi) = \sigma^2 \cdot D\xi = \sigma^2 \).

We may write \( \Phi_{a,\nu}(x) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{+\infty} e^{-\frac{(u-a)^2}{2\sigma^2}} du, \quad x \in R^1, \) by using (2.1.14).

The following semi-group property for normal distribution function is well-known [23]:
\[
\Phi_{a_1,1} * \Phi_{a_2,1} = \Phi_{a_1+a_2,1} \quad \text{for } a_1 \in R^1, \ a_2 \in R^1.
\]
Even a more general property takes place: for \( a_i \in R^1, \ \sigma_i^2 \in R^+ \), \( i = 1, 2 \),
\[
\Phi_{a_1,\sigma_1} * \Phi_{a_2,\sigma_2} = \Phi_{a_1+a_2, \sqrt{\sigma_1^2+\sigma_2^2}}. \quad (2.1.15)
\]
The standard way of proving the equalities of type (2.1.15) is the Method of Characteristic Functions [23], [29]. In case \( \Phi_{a,\sigma} \) we have

\[
\varphi_{a,\sigma}(t) := \int_{-\infty}^{+\infty} e^{itx} d\Phi_{a,\sigma}(x) = e^{iat-(\sigma^2/2)}, \quad t \in R^1, \quad i = \sqrt{-1}.
\]  

(2.1.16)

Taking into account that the characteristic function of the sum of independent random variables is a product of their characteristic functions we conclude: (2.1.15) is equivalent to the equality: \( \varphi_{a_1,\sigma_1}(t) \cdot \varphi_{a_2,\sigma_2}(t) = \varphi_{a_1+a_2,\sqrt{\sigma_1^2+\sigma_2^2}}(t), \quad t \in R^1 \), which, due to (2.1.15), is true.

How to evaluate directly the left-hand-side of (2.1.15) in terms of distribution functions and prove (2.1.15) one may find out in [29].

2.2 Densities which Possess the Semi-Group Property

The semi-group property has been discovered by us as an alternative to scale-invariance property, in order to figure out new frequency distributions. For constructing new frequency distributions by operation of dediscretization we need densities which possess the semi-group property.

The examples presented in Section 2.1 have one defect. They do not vary regularly at infinity as it requires by one of the important statistical facts for frequency distributions. Indeed, due to (2.1.8), the gamma-densities decrease and tend to zero with exponential speed as the argument goes to +\( \infty \). The density of Standard Normal Law (2.1.12) by its form (2.1.13), tends to zero even with more quick than it does the exponential speed.

In the next Chapter we’ll see that the Standard Normal Law may serve for other aim - to be an approximation of frequency distribution in growing biomolecular network.

In this Section we present two more realistic examples of densities which possess the semi-group property and lay claim to be a base for new frequency distributions building. These are densities of Levy’s and Cauchy’s distribution functions. Both distribution functions belong to the family of Stable Laws. For them the establishment of their semi-group property may be done by method of Characteristic Functions which has been applied for the Normal Law. But we prefer more complicate, direct methods, which are new and methodologically interesting.

2.2.1 Levy’s Law

Being concentrated on \( R^+ \) Levy’s Law

\[
F(x) = 2 \cdot (1 - \Phi(1/\sqrt{x})), \quad x \in R^+,
\]  

(2.2.1)
where $\Phi$ is a Standard Normal Law, has density

$$\frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{1}{2x}\right) \text{ for } x \in \mathbb{R}^+, \quad \text{and 0 for } x \leq 0.$$ 

This function under the name Holtsmark distribution was known before to astronomers, but not to mathematicians.

Similarly to the case of Normal Law, with the help of distribution function (2.2.1) a family $\{F_{a,\sigma} : \sigma \in \mathbb{R}^+, a \in \mathbb{R}^1\}$ of distribution functions of the type $F_{a,\sigma}(x) = F\left(\frac{x-a}{\sigma}\right)$ being of one type with $F$ is constructed. Denote by $f_{a,\sigma}$ the density of $F_{a,\sigma}$ and put $f_\sigma = f_{0,\sigma}$.

Our purpose is to show that the following semi-group property holds:

$$f_\sigma_1(x) * f_\sigma_2(x) = f_{\sigma_1 + \sigma_2}(x), \quad x \in \mathbb{R}^+. \quad (2.2.2)$$

In order to establish (2.2.2), first of all, we need to evaluate some integrals. Let $a \in \mathbb{R}^+, b \in \mathbb{R}^+$ are some constants. Let us evaluate the following integral

$$I(a, b) = \int_{0}^{+\infty} \exp(-ax^2 - \frac{b}{x^2})dx. \quad (2.2.3)$$

One method of its evaluation is presented in [24], p.689 and p.681. We suggest another method. Let us make in (2.2.3) a replacement of the variable of integration $x = \sqrt{\frac{b}{a}} \cdot t$. Then we have

$$I(a, b) = \sqrt{\frac{b}{a}} \cdot \int_{0}^{+\infty} \exp(-\sqrt{ab} \cdot (t^2 + \frac{1}{t^2}))dt = \sqrt{\frac{b}{a}} \cdot \left(\int_{0}^{1} e^{-2\sqrt{ab} \cdot \left(t - \frac{1}{t}\right)^2}dt + \int_{1}^{+\infty} e^{-2\sqrt{ab} \cdot \left(t - \frac{1}{t}\right)^2}dt\right). \quad (2.2.4)$$

In the first integral at the right-hand-side of (2.2.4) let us make replacement $t = (1/y)$, and combine both integrals of the right-hand-side. It gives

$$I(a, b) = \sqrt{\frac{b}{a}} e^{-2\sqrt{ab}} \int_{1}^{+\infty} (1 + \frac{1}{y^2}) \exp(-\sqrt{ab} \cdot (y - \frac{1}{y})^2)dy = \sqrt{\frac{b}{a}} e^{-2\sqrt{ab}} \int_{1}^{+\infty} \exp(-\sqrt{ab} \cdot (y - \frac{1}{y})^2)dy = \sqrt{\frac{b}{a}} e^{-2\sqrt{ab}} \int_{0}^{\infty} e^{-\sqrt{ab} \cdot x^2} dx. $$

Thus,

$$I(a, b) = \frac{1}{a} \cdot e^{-2\sqrt{ab}} \cdot \int_{0}^{\infty} e^{-x^2} dx. \quad (2.2.5)$$

Taking into account the equality $\int_{-\infty}^{+\infty} e^{-x^2/2}dx = \sqrt{2\pi}$, which follows from the equality $\Phi(+\infty) = 1$, and the case that at the right-hand-side of (2.2.5) the function under integral is even, we conclude

$$I(a, b) = \sqrt{\frac{\pi}{a}} \frac{1}{2} e^{-2\sqrt{ab}}. \quad (2.2.6)$$
Let us evaluate the integral
\[ J(a, b) = 2 \cdot \int_0^{+\infty} \frac{x^2 + 1}{x^2} \exp(-ax^2 - \frac{b}{x^2}) \, dx. \] (2.2.7)

Since
\[ J(a, b) = 2 \cdot \left\{ I(a, b) + \int_0^{+\infty} \frac{1}{x^2} \exp(-ax^2 - \frac{b}{x^2}) \, dx \right\}, \] (2.2.8)
therefore making replacement of variable \( x = \frac{1}{y} \) in the second term of the right-hand-side of (2.2.8) we obtain \( J(a, b) = a \cdot \{ I(a, b) + I(b, a) \} \). Due to the last equality, with the help of (2.2.6) we have
\[ J(a, b) = \sqrt{\pi} \left( \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \right) e^{-2\sqrt{ab}} = J(b, a). \] (2.2.9)

Now we are ready to check out the semi-group property (2.2.2). We deal with distribution functions \( (F_0, \sigma) := F_\sigma(x) = 2 \cdot (1 - \Phi(\sigma/\sqrt{x})), \, x \in R^+, \sigma \in R^+ \), and with their densities
\[ f_\sigma(x) = \begin{cases} \frac{\sigma}{\sqrt{2\pi}} \frac{1}{\sqrt{x^3}} e^{-(\sigma^2/2x)} & \text{for } x \in R^+, \\ 0 & \text{for } x \leq 0. \end{cases} \] (2.2.10)
Due to (2.2.2) and (2.2.10), we must verify the validity of the equality
\[ K(x, \sigma_1, \sigma_2) := \frac{\sigma_1 + \sigma_2}{2\pi} \int_0^{+\infty} \frac{1}{x^2} \exp(-\frac{\sigma_1^2}{2x}) \, dx = \frac{\sigma_1 \cdot \sigma_2}{2\pi} \int_0^{+\infty} \frac{1}{(x-y)^2 y^3} \exp\left(-\frac{\sigma_1^2}{2(x-y)} - \frac{\sigma_2^2}{2y}\right) \, dy \] (2.2.11)
for any \( x \in R^+, \sigma_1 \in R^+, \sigma_2 \in R^+ \).

Let us make replacement of integration variable \( y = \frac{1}{x^2 + 1} \) at the right-hand-side of the equality (2.2.11), where we put \( t > 0 \). Then we obtain
\[ K(x, \sigma_1, \sigma_2) = \frac{\sigma_1 \cdot \sigma_2}{\pi x^2} \exp\left(-\frac{\sigma_1^2 + \sigma_2^2}{2x}\right) \cdot \int_0^{+\infty} \frac{t^2 + 1}{t^2} \cdot \exp\left(-\frac{1}{2x} \frac{\sigma_1^2}{t^2} + \frac{\sigma_2^2}{2t^2}\right) \, dt. \] (2.2.12)
Comparing the right-hand-side of equality (2.2.12) with the form of \( J(a, b) \) defined by (2.2.7) we conclude that for \( x \in R^+, \sigma_1 \in R^+, \sigma_2 \in R^+ \)
\[ K(x, \sigma_1, \sigma_2) = \frac{\sigma_1 \cdot \sigma_2}{2\pi \cdot x^2} \exp\left(-\frac{\sigma_1^2 + \sigma_2^2}{2x}\right) \cdot J\left(\frac{\sigma_1^2}{2x}, \frac{\sigma_2^2}{2x}\right). \] (2.2.13)
Substituting into (2.2.13) the expression of \( J\left(\frac{\sigma_1^2}{2x}, \frac{\sigma_2^2}{2x}\right) \) taken from (2.2.9) we come to the expression at the left-hand-side of (2.2.12).

The semi-group property (2.2.2) for Levy’s Law is proved.
2.2.2 Cauchy’s Laws

The Standard Cauchy’s Law takes the form

\[ F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan x, \quad x \in \mathbb{R}^1, \tag{2.2.14} \]

with density \( \frac{1}{\pi(1+x^2)} \) for \( x \in \mathbb{R}^1 \). The distribution function (2.2.14) generates the family \( \{ F_{a,\sigma} : \sigma \in \mathbb{R}^+, a \in \mathbb{R}^1 \} \) of distribution functions of the type \( F_{a,\sigma}(x) = F\left(\frac{x-a}{\sigma}\right) \) being of one type with \( F \).

Denote by \( f_{a,\sigma} \) the density of distribution function \( F_{a,\sigma} \) and put \( f_\sigma = f_{0,\sigma} \). Our purpose is to show that the semi-group property (2.2.2) takes place for Cauchy’s Law.

In order to establish (2.2.2), first of all, we need to evaluate the following integral for \( x \in \mathbb{R}^+ \)

\[ I(x, t, s) = \int_{-\infty}^{+\infty} \frac{dy}{(t^2 + y^2)(s^2 + (x - y)^2)} \tag{2.2.15} \]

where \( t \in \mathbb{R}^+ \) and \( s \in \mathbb{R}^+ \) are some constants. The computation may be done with the help of long elementary method of expansion of the functions being under integral on elementary fractions. But it seems that the method of “contour integration” is more preferable. Denote for \( t \in \mathbb{R}^+ \) and \( s \in \mathbb{R}^+ \)

\[ f(z) = \frac{1}{(t^2 + z^2)(s^2 + (x - z)^2)} \]

where \( x \in \mathbb{R}^1 \) is fixed and \( z \) is an arbitrary complex number. The denominator of the function \( f(z) \) has following zeros \( z = it, \ z = -it, \ z = x + is, \ z = x - is, \) where \( i = \sqrt{-1} \), which are simple poles of \( f(z) \). Among them only \( z = it \) and \( z = x + is \) are located in positive quadrant of the complex plane. That is why

\[ I(x, t, s) = 2\pi i \left\{ \frac{\lim_{z \to it} f(z)}{t} + \frac{\lim_{z \to x+is} f(z)}{s} \right\} = \]

\[ = \pi \left\{ \frac{1}{t \cdot (s^2 + (x - it)^2)} + \frac{1}{s \cdot (t^2 + (x + is)^2)} \right\}, \]

where \( \text{Res} \) denotes the symbol of residue. Let us make expansion of multipliers

\[ I(x, t, s) = \pi \cdot \left\{ \frac{1}{t(s + ix + t)(s - ix - t)} + \frac{1}{s(t + ix - s)(t - ix + s)} \right\} = \]

\[ = \frac{\pi}{s - t - ix} \cdot \left\{ \frac{1}{st(x^2 + (s + t)^2)} + \frac{1}{st(x^2 + (s + t)^2)} \right\} = \frac{\pi(s + t)}{st(x^2 + (s + t)^2)}. \]

Thus, for \( s \in \mathbb{R}^+, t \in \mathbb{R}^+ \) and \( x \in \mathbb{R}^1 \) we get

\[ I(x, t, s) = \frac{\pi(s + t)}{st(x^2 + (s + t)^2)} = I(x, s, t). \tag{2.2.16} \]

Now we are ready to check out the semi-group property (2.2.2). We deal with one-parametric family of Cauchy’s Laws \( (F_{0,\sigma}(x) :=) F_\sigma(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x/\sigma), \) \( x \in \mathbb{R}^1, \sigma \in \mathbb{R}^+ \), and with their densities

\[ f_\sigma(x) = \frac{\sigma}{\pi \sigma^2 + x^2} \text{ for } x \in \mathbb{R}^1. \tag{2.2.17} \]
Due to (2.2.2) and (2.2.17), we must verify the validity of the equality
\[
\frac{1}{\pi^2} \int_{-\infty}^{+\infty} \frac{\sigma_2}{\sigma_2^2 + (x-y)^2} \cdot \frac{\sigma_1}{\sigma_1^2 + y^2} dy = \frac{1}{\pi} \frac{\sqrt{\sigma_1^2 + \sigma_2^2}}{\sigma_1^2 + \sigma_2^2} + (x-y)^2 (2.2.18)
\]
for any \( x \in R^1, \sigma_1 \in R^+, \sigma_2 \in R^+ \). Since the left-hand-side expression of the equality (2.2.18), due to (2.2.15), coincides with the expression \( \sigma_1 \sigma_2 \pi \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + \sigma_2^2}} I(x, \sigma_1, \sigma_2) \), therefore from (2.2.16) it follows (2.2.18). The semi-group property (2.2.2) for Cauchy’s Law is proved.

Finally, let us make the following remark.

Considered densities of Levy’s Laws and Cauchy’s Laws vary regularly at infinity.

Indeed, for the Levy’s Law, due to (2.2.10), their densities vary regularly at infinity with exponent \((-3/2)\) and exhibit constant slowly varying component being equal to \((\sigma/\sqrt{2\pi})\). The densities of Cauchy’s Laws being defined on \( R^1 \) are symmetric functions, so we may talk on their asymptotic behavior at \(+\infty\). The asymptotic behavior of these densities at \(-\infty\) is the same as at \(+\infty\). Due to (2.2.17), for the Cauchy’s Laws their densities vary regularly at infinity with exponent \((-2)\) and exhibit constant slowly varying component being equal to \((\sigma/\pi)\).

## 2.3 Stable Laws: Definition and Examples

We already mentioned that the Power Law plays an essential role in large-scale biomolecular systems because of its scale-invariance property. Next, we notice that this property very often might be replaced by semi-group property.

In this Section we introduce another, more powerful than the semi-group property and, obviously, more restrictable property, which extracts the well-known in Probability Theory family of Stable Laws.

### 2.3.1 The Family of Stable Laws

**Definition 9.** We say that distribution function \( S \) is stable if for any \( a_1 \in R^1, a_2 \in R^1, b_1 \in R^+, b_2 \in R^+ \) there are numbers \( a \in R^1, b \in R^+ \) such that
\[
S\left( \frac{x-a_1}{b_1} \right) * S\left( \frac{x-a_2}{b_2} \right) = S\left( \frac{x-a}{b} \right), \ x \in R^1,
\]
where * denotes the sign of convolution.

Let us consider gamma-distribution functions
\[
F_{\alpha, \nu}(x) = \begin{cases} \frac{1}{\Gamma(\nu)} \alpha^\nu \cdot \int_0^x u^{\nu-1} e^{-\alpha u} du, & x \in R^1, \\ 0, & x \leq 0, \end{cases}
\]

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with densities (2.1.8). \( F_{\alpha,\nu}(x) \) may be represented as
\[
F_{\alpha,\nu}(x) = \gamma(\nu, \alpha x) \Gamma(\nu),
\]
where \( \Gamma(\cdot) \) is the Euler’s Gamma Function and \( \gamma(\nu, x) = \int_0^x e^{-t} \cdot t^{\nu-1} dt \) – the incomplete Gamma Function, a well known in Mathematical Analysis.

Let us show that the gamma distribution function, which as we know satisfies the semi-group property, is not stable. Indeed, taking density (2.1.8) with \( \alpha = 1 \) and \( \nu \) is fixed let us check out (2.3.1) for \( a_1 = a_2 = 0, b_1 = b_2 = 1 \). Then (2.3.1), due to (2.1.8), in our case in terms of densities takes the form
\[
f_{1,2\nu}(x) = f_{1,\nu}(x) * f_{1,\nu}(x) = 1_b f_{1,\nu}(\frac{x-a}{b}), \quad x \in R^+.
\]
(2.3.2)

with some admissible constants \( a \in R^1, b \in R^+ \). Here also the semi-group property (2.1.9) for Gamma-densities was used. But the equality (2.3.2), i.e. \( f_{1,2\nu}(x) = 1_b f_{1,\nu}(\frac{x-a}{b}) \) for any \( a \) and cannot be true for all values \( x \in R^+ \).

In order to give another definition of Stable Laws let us introduce the notation \( \xi < d > \eta \) to indicate that random variables \( \xi \) and \( \eta \) are identically distributed. By this notation \( \eta < d > \alpha \xi + \beta \) with \( \alpha \in R^+, \beta \in R^1 \), means that distribution functions of \( \xi \) and \( \eta \) belong to the same class, and differ only by location parameters.

Throughout this Section \( \xi, \xi_1, \xi_2, \cdots \) denote independent identically distributed random variables, and \( S_n = \sum_{K=1}^n \xi_K \) for any integer \( n \geq 1 \).

Let us assume that the distribution function \( S \) of random variables \( \xi, \xi_1, \xi_2, \cdots \) is not concentrated at zero, i.e. \( P(\xi = 0) \neq 1 \).

Now the following definition is equivalent to Definition 8 (see, VI.7, [29])

**Definition 10.** We say that \( S \) is stable (in a broad sense) if for each integer \( n \geq 1 \) there are constants \( \sigma_n \in R^+ \) and \( a_n \in R^1 \) such that
\[
S_n < d > \sigma_n \xi + a_n.
\]
(2.3.3)

Let us supplement this definition. We say that \( S \) is stable in a strict sense if (2.3.3) holds for any integer \( n \geq 1 \) with \( a_n = 0 \).

Having Definition 10 we need also Criteria for \( S \) to be stable.

**Criterion 2.1** \( S \) is stable if (2.3.3) holds for \( n = 2 \) and \( n = 3 \).

This criterion belongs to Levy P. (see, VI.13, task1, p.215, [29]).

But if (2.3.3) holds only for \( n = 2 \) it is not enough for \( S \) to be stable (see, example XVII, (3f), p.538, [29]).

2.3.2 Normal Laws

At once the problem of non-emptiness of the family of Stable Laws arises.
Even here Normal Laws play an exceptional role.
Let us show that if \( \xi, \xi_1, \xi_2, \cdots \) have distribution function \( \Phi_{\alpha,\sigma}, \alpha \in \mathbb{R}^1, \sigma \in \mathbb{R}^+ \), then
\[
S_n < d > n^{1/2} \cdot \xi + (n - n^{1/2}) \cdot a \quad \text{for any integer } n \geq 1. \tag{2.3.4}
\]
It means that two-parametric family of Normal Laws is formed by Stable Laws with
\[
\sigma_n = n^{1/2} \quad \text{and} \quad a_n = (n - n^{1/2})a \quad \text{for any integer } n \geq 1 \tag{2.3.5}
\]
in Definition 10. Thus, the family of Stable Laws is not empty.

Let us prove (2.3.4). Due to (2.1.15), we have
\[
\Phi_{\alpha,\sigma} \ast \Phi_{\alpha,\sigma} \ast \cdots \ast \Phi_{\alpha,\sigma} = \Phi_{\frac{\alpha}{\sqrt{n}},\sigma \sqrt{n}} \quad \text{for any integer } n \geq 1,
\]
which means that
\[
S_n < d > \eta_n, \tag{2.3.6}
\]
where \( \eta_n \) is normal distributed random variable with mean value \( na \) and variance \( \sigma^2 \cdot n \). Since linear transformation of normal random variable, as it is well-known [29], is normal distributed, therefore
\[
\xi = \frac{\eta_n - na}{n^{1/2}} + a \tag{2.3.7}
\]
is a normal random variable. This variable has distribution function \( \Phi_{\alpha,\sigma} \) because
\[
E \xi = E\left(\frac{\eta_n - na}{n^{1/2}}\right) + a = \frac{1}{n^{1/2}}(E\eta_n - na) + a = a,
\]
\[
D \xi = D\left(\frac{\eta_n - na}{n^{1/2}}\right) = D\left(\frac{\eta_n}{n^{1/2}}\right) = \frac{1}{n} D\eta_n = \frac{\sigma^2}{n} = \sigma^2.
\]
Since, due to (2.3.7), \( \eta_n = n^{1/2} \xi + (n - n^{1/2})a \), so, from (2.3.6) it follows (2.3.4).

The exceptional role of Normal Laws inside the family of Stable Laws consists in following.

Among Stable Laws only Normal Laws have finite variance. \( \tag{2.3.8} \)

In order to prove statement (2.3.8) we may establish that if random variables \( \xi, \xi_1, \xi_2, \cdots \) have finite variance and for any integer \( n \geq 1 \) satisfy equality (2.3.3) with some \( \sigma_n \in \mathbb{R}^+ \) and \( a_n \in \mathbb{R}^1 \), then they are normal distributed and (2.3.5) holds, where
\[
a = E \xi \quad \text{and} \quad \sigma^2 = D \xi. \tag{2.3.9}
\]

First of all, let us show that if \( \sigma^2 < +\infty \) in (2.3.9), then (2.3.5) holds.
Indeed, from (2.3.3) we obtain
\[
n \cdot a = ES_n = E(\sigma_n \cdot \xi + a_n) = \sigma_n \cdot E \xi + a_n \quad \text{and} \quad a_n = \sigma_n - a + a_n,
\]
\[
n \cdot \sigma^2 = DS_n = D(\sigma_n \cdot \xi + a_n) = D(\sigma_n \xi) = \sigma_n^2 \cdot D \xi = \sigma_n^2 \cdot \sigma^2.
\]
From the second equality because of \(0 < \sigma^2 < +\infty\) we have \(\sigma_n = n^{1/2}\), which, due to the first equality, gives \(a_n = (n - n^{1/2})\).

Thus, in the case of finite variance (2.3.3) coincides with (2.3.4).

Let us now prove that the random variable \(\xi\) in (2.3.4) has distribution function \(\Phi_{a,\sigma}\). By Central Limit Theorem [29] in case of finite variance, the limit exists

\[
\lim_{n \to +\infty} P\left(\frac{S_n - na}{\sigma \cdot \sqrt{n}} < x\right) = \Phi(x)
\]

uniformly on \(x \in R^1\). Since, due to (2.3.4), \(\frac{S_n - na}{\sigma \cdot \sqrt{n}} := \frac{\xi}{\sigma} - \frac{a}{\sigma}\) (doesn’t depend on \(n\)) for any integer \(n \geq 1\), so, \(P\left(\frac{\xi}{\sigma} - \frac{a}{\sigma} < x\right) = \Phi(x)\), or \(P(\xi < y) = \Phi\left(\frac{y - a}{\sigma}\right) = \Phi_{a,\sigma}(y), y \in R^1\).

It remains to mention that \(\xi\) and \(\xi_1, \xi_2, \cdots\) are identically distributed.

The statement (2.3.8) is proved.

### 2.3.3 Further Examples of Stable Laws

**Example 1.** Let \(\xi, \xi_1, \xi_2, \cdots\) have density (2.2.10) of Levy’s Law with fixed \(\sigma\).

Due to (2.2.2), we may write down

\[
f_\sigma \ast f_\sigma \ast \cdots \ast f_\sigma = f_{n\sigma}\text{ for any integer } n \geq 1,
\]

which means that for \(x \in R^+\) and \(n = 1, 2, \cdots\)

\[
P(S_n < x) = \int_0^x f_\sigma \ast f_\sigma \ast \cdots \ast f_\sigma(u)du = \int_0^x f_{n\sigma}(u)du =
\]

\[
\frac{\sigma n}{2\pi} \int_0^x \frac{1}{\sqrt{u^3}} \exp\left(-\frac{n^2\sigma^2}{2u}\right)du,
\]

where (2.2.10) was used. At the right-hand-side of (2.3.12) let us make replacement of integration variable \(\frac{u^2}{v^2} = v\). Then, for \(n = 1, 2, \cdots\) we obtain

\[
P(S_n < x) = \frac{\sigma}{2\pi} \int_0^{x/n^2} \frac{1}{\sqrt{v^3}} e^{-\sigma^2/2v}dv = \int_0^{x/n^2} f_\sigma(v)dv = P(\xi < \frac{x}{n^2}) = P(n^2 \cdot \xi < x).
\]

Thus, we obtain

\[
S_n < d > n^2 \cdot \xi \text{ for any integer } n \geq 1.
\]

Due to (2.3.13), in general case of Levy’s Law we must replace \(\xi, \xi_1, \xi_2, \cdots\) in (2.3.13) by random variables \(\xi - a, \xi_1 - a, \xi_2 - a, \cdots\) with arbitrary \(a \in R^1\), respectively. It leads to the equality

\[
S_n < d > n^2 \cdot \xi + (n - n^2) \cdot a.
\]

So, Definition 10 takes place in this case with

\[
\sigma_n = n^2 \quad \text{and} \quad a_n = (n - n^2) \cdot a \text{ for any integer } n \geq 1.
\]
Example 2. Let $\xi, \xi_1, \xi_2, \cdots$ have density (2.2.17)

$$\frac{1}{\pi \sigma^2 + x^2} \quad \text{for } x \in \mathbb{R}^1$$

of Cauchy’s Law with fixed $\sigma$.

Due to (2.2.2), we come to (2.3.11), which, similarly to (2.3.12), may be written for $x \in \mathbb{R}$ and $n = 1, 2, \cdots$ as follows

$$P(S_n < x) = \int_{-\infty}^{x} f_{n\sigma}(u) du = \frac{1}{\pi} \int_{-\infty}^{x} \frac{n\sigma}{n^2\sigma^2 + u^2} du,$$  \hspace{1cm} (2.3.17)

where (2.3.16) was used. At the right-hand-side of (2.3.17) let us make replacement of integration variable $u_n = v$. Then, for $n = 1, 2, \cdots$ we obtain

$$P(S_n < x) = \frac{1}{\pi} \int_{-\infty}^{x/n} \frac{\sigma}{\sigma^2 + v^2} dv = \int_{-\infty}^{x} f_{\sigma}(v) dv = P(\xi < \frac{x}{n}) = P(n\xi < x).$$

Thus, we obtain

$$S_n < d > n\xi \quad \text{for any integer } n \geq 1.$$  \hspace{1cm} (2.3.18)

### 2.3.4 First Essential Parameter of Stable Laws

The relationships (2.3.4), (2.3.14) and (2.3.18) for Normal, Levy’s and Cauchy’s Laws, respectively, show that these distributions are stable. For them in Definition 10 (see, (2.3.3)) we have to put $\sigma_n = n^{1/2}$, $\sigma_n = n^2$, $\sigma_n = n$, $n = 1, 2, \cdots$, respectively. It is amazing that also in general case of Stable Law necessarily

$$\sigma_n = n^{1/\alpha} \quad \text{with } \alpha \in (0, 2].$$  \hspace{1cm} (2.3.19)

So, the values $\alpha = 2, \alpha = (1/2), \alpha = 1$ characterize in (2.3.3) Normal, Levy’s, Cauchy’s Laws, respectively, inside the family of Stable Laws.

Thus, $\alpha$ may serve as a parameter of Stable Laws.

**Definition 11.** The constant $\alpha \in (0, 2]$ in (2.3.3) taking part in presentation (2.3.19) of $\{\sigma_n\}$ is called the exponent of Stable Law $R$.

Two concluding remarks.

(a) Excluding Normal Laws any Stable Law has exponent $\alpha \in (0, 2)$ and infinite variance  \hspace{1cm} (2.3.20)

(b) Due to Definitions 9 – 10:

All distribution functions belonging to the same class are or not stable simultaneously.

If they are stable, then they have the same exponent.
Let us prove the statement (2.3.21).

We have to prove that if $\xi$ is stable with exponent $\alpha$, i.e. its distribution function, say $S_\alpha$, is stable, then the random variable $\eta = \sigma \xi + a$ with arbitrary $\sigma \in \mathbb{R}^+$ and $a \in \mathbb{R}^1$ is stable with the same exponent $\alpha$. Due to (2.3.3) and (2.3.19) we have

$$S_n < d > n^{1/\alpha} \cdot \xi + a_n \text{ for any integer } n \geq 1. \quad (2.3.22)$$

Now,

$$P\left( \sum_{K=1}^{n} (\sigma \xi_n + a) < x \right) = P(S_n < \frac{x - na}{\sigma}) =$$

$$= P(n^{1/\alpha} \cdot \xi + a_n < \frac{x - na}{\sigma}) = P(n^{1/\alpha} \cdot \eta + a'_n < x),$$

where the constant $a'_n$ takes the form $a'_n = \sigma \cdot a_n + na - n^{1/\alpha} \cdot a$.

The statement (2.3.21) is proved.

Because of the statement (2.3.21) now we are familiar with already three parameters of Stable Laws. Here they are: exponent, shifting parameter, scale factor.

2.4 Subfamilies of Stable Laws

In the present Section we give general information on the development of Theory of Stable Laws, introduce a fourth parameter of four-parametric family of Stable Laws (an asymmetry), and describe two important subfamilies.

2.4.1 Information

The Theory of Stable Laws is related with the name of P.Levy, who in 1924 figured out the characteristic functions of all strictly stable distribution functions.

Before the Fourier Transform of Symmetric Stable Laws have been pointed by A. Cauchy, but it was not clear that they correspond to distribution functions.

Later, when infinitely divisible distributions were discovered and various Canonical Representations for them were obtained (P. Levy, A. Khintchine, A. Komolgorov, B. Gnedenko), the Theory of Stable Laws has been built once more from this point of view (every Stable Law is infinitely divisible). The new approach also was based on characteristic functions and was due to P. Levy and A. Khintchine [30]-[31].

Different Canonical Representations of characteristic functions of Stable Laws have been proposed based on various parametrizations. Such a diversity arises in order to give simple formulations of properties of Stable Laws and leads to many confusions.

The definition easily implies the existence of density for any Stable Law (see, VI.I, p.167, Lemma 1, [29]). Unfortunately, it seems impossible to express stable densities
in a *closed* form, but series expansions were given independently by W. Feller and H. Bergström (see, XVII, 6, [23]). The *closed* form of *stable densities* is found only for *some* values of exponent $\alpha$ of *Stable Laws*. All existing examples were considered in Section 2.2. It is clear that it is complicate to deal with *stable densities* given in the form of series.

Anyway, Canonical Representations become a *source* for obtaining many facts on *Stable Laws*.

Let us present some of them which are necessary for deriving the *fourth* parameter of *Stable Laws*. (a) *Any Stable Law* $S$ *has a regularly varying at infinity sum of tails*

$$1 - S(x) + S(-x), \ x \in R^+$$

(2.4.1)

where $1 - S(x)$ and $S(-x)$ are right and left tails. (b) *The function* (2.4.1) *exhibits constant slowly varying component.*

There are two situations here. In the first one (a) and (b) hold for both tails. In the second either $1 - S(x) = o(S(-x))$ or $S(-x) = o(1 - S(x))$ as $x \to +\infty$. Due to (a),

$$1 - S(x) + S(-x) > 0 \text{ for all } x \in R^+.$$ 

(2.4.2)

Here the case of Normal Laws is excluded.

A further impulse for the Theory of *Stable Laws’ development has been done after establishment of the following *amazing* fact.

*The class of limit distributions for* (centralized and normed) 
*sums of independent identically distributed random variables* 
*coincides with the set of Stable Laws!* 

This aspect of the Theory shall be exploited in the next *Chapter.*

### 2.4.2 The Asymmetry

The set of $\{S^*\}$ of *Stable Laws* is a *four-parametric* family [23], [32]. We are familiar with three parameters: exponent, shifting parameter, scale factor. The fourth one is called *asymmetry*. Other three parameters being fixed the *asymmetry* characterizes the *skewness* of a *Stable Law*.

**Definition 12.** For a *Stable Law* $S$ the limit

$$\lim_{x \to +\infty} \frac{1 - S(x) - S(-x)}{1 - S(x) + S(-x)} := \beta \in [-1, 1]$$

(2.4.3)

is called *asymmetry*. 

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(2.4.2) makes it possible to consider the ratio in (2.4.3).

Statements (a), (b) imply: (2.4.3) is equivalent to the existence of limit

\[ \lim_{x \to +\infty} \frac{S(-x)}{1 - S(x)} = \frac{1 - \beta}{1 + \beta}. \]

That is why Definition 12 is correct.

Having two essential parameters of \( \{S^*\} \) we may improve (2.3.21)

All \( S \in \{S^*\} \) with same exponent \( \alpha \) and symmetry \( \beta \) are of the same type. Conversely, \( S \) and \( S' \) from \( \{S^*\} \) with different vectors \((\alpha, \beta)\) are not of the same type. (2.4.4)

Canonical Representations of Stable Laws in terms of characteristic functions are various [23], [32]. If we take for each values of exponent \( \alpha \in (0, 2) \) and asymmetry \( \beta \in [-1, 1] \) one representative \( S_{\alpha, \beta} \in \{S^*\} \), then

\[ \{S^*\} = \bigcup_{(\alpha, \beta)} \left\{ S_{\alpha, \beta}\left(\frac{x - a}{\sigma}\right) : \sigma \in R^+, a \in R^1 \right\}. \]

Choosing different representatives we get different Canonical Representations.

So, we derive characteristic function for any representative, and by operations of shifting and sealing obtain the Canonical Representation. The concrete expressions of various Canonical Representations one may find in VII, 4, [23], or Introduction, [32].

### 2.4.3 Important Subfamilies

Two subfamilies of \( \{S^*\} \) are interesting for us.

**Definition 13.** \( S \in \{S^*\} \) with asymmetry \( \beta = 1 \) is called a Right-side Stable Law.

Denote the set of Right-side Stable Laws by \( \{S^*_+\} \).

By (2.4.3), for \( S \in \{S^*_+\} \) the maximal value of asymmetry \( \beta \) equals to 1. \( S \) has this value iff

\[ \lim_{x \to +\infty} \frac{S(-x)}{1 - S(x)} = 0. \]

Thus, \( \{S^*_+\} \) is characterized by maximal skewness.

Density of any Levy’s Law (2.2.10) is an example of Right-side Stable Law’s density. It is concentrated on \( R^+ \).

Due to Theorem 3.1, p.43, [33], in particular, Right-side representative \( S_\alpha \) possess a convergent two-side Laplace-Stieljes Transform

\[ \rho_\alpha(s) = \int_{-\infty}^{\infty} e^{-sx}dS_\alpha(x) = \begin{cases} \exp(-s^\alpha) & \text{for } 0 < \alpha < 2, s \in R^+, \\ \exp(-s + s \cdot \log s) & \text{for } \alpha = 1, s \in R^+. \end{cases} \] (2.4.5)

With the help of scaling factor from (2.4.5) we create two-parametric family given by Laplace-Stieltjes Transforms

\[ \left\{ \rho_\alpha(C^{1/\alpha} \cdot s) : 0 < \alpha < 2, C \in R^+ \right\}. \] (2.4.6)
The second subfamily is a two-parametric family \( \{S^*_0\} \) of Symmetric Stable Laws with the characteristic functions

\[
\psi_\alpha(t) = \exp(-C \cdot |t|^\alpha), \quad t \in R^1,
\]

where \( C \in R^+ \) denotes a scaling factor and \( \alpha \in (0,2] \) an exponent. This fact one may find in first monograph by P. Levy (1925).

Some explanations. For symmetric \( S \) its characteristic function is real. Indeed,

\[
\int_{-\infty}^{\infty} e^{itx} dS_0(x) = \left( \int_{-\infty}^{0} + \int_{0}^{\infty} \right) e^{itx} dS_0(x) = \int_{0}^{\infty} e^{-itx} dS_0(-x) + \\
+ \int_{0}^{\infty} e^{itx} dS_0(x) = \int_{0}^{\infty} (e^{itx} + e^{-itx}) dS_0(x) = 2 \cdot \int_{0}^{\infty} \cos(tx) dS_0(x), \quad i = \sqrt{-1}.
\]

\( \{S^*_0\} \) is no more than two-parametric family. Indeed, for \( S \in \{S^*_0\}, \beta = 0 \) and for any \( \alpha \) scaling factor \( C \) among all \( S \in \{S^*_0\} \) of such a type only one value of shifting parameter leads to Symmetric Stable Law.

(2.4.7) with \( \alpha = 1 \) corresponds to family of Cauchy’s Laws with densities (2.2.17).

Sometimes one-side Laplace-Stieltjes Transform of \( S \in \{S^*_0\} \) may be useful. For Symmetric \( S_\alpha \in \{S^*_0\} \) with exponent \( \alpha \neq 1 \) taken as a representative we may write down (see, Theorem 2.6.2, p.137, [33])

\[
\rho_\alpha^+(s) = \int_{0}^{\infty} e^{-sz} dS_\alpha(x) = \frac{1}{\pi} \int_{0}^{\infty} e^{-(su)^\alpha} \frac{du}{u^2 + 1}.
\]

(2.4.8)

\( S_\alpha \) is called also Standard Symmetric Stable Law.

With the help of scaling factor from (2.4.8) we create two-parametric family of Symmetric Stable Laws given by one-side Laplace-Stieltjes Transforms

\[
\left\{ \rho_\alpha^+(C^{1/\alpha} \cdot s) : 0 < \alpha \leq 2, C \in R^+ \right\}.
\]

(2.4.9)

Finally, from the family of Right-side Stable Laws we extract a subfamily with \( 0 < \alpha < 1 \). This family is important because only Stable Laws with exponent \( \alpha \in (0,1) \) and asymmetry \( \beta = 1 \) are concentrated on \( R^+ \).

### 2.5 Stable Densities

In the present Section we discuss the known series expansions of stable densities and extract particular cases which correspond to Right-side and Symmetric Stable Laws. With the help of scaling the extracted stable densities are enlarged into two-parametric families of stable densities. The graphs of densities from important families are drown and different properties are formulated.
2.5.1 Series Expansion

Any stable density \( s(x; \alpha, \beta, a, \sigma) \) depends on following parameters: exponent \( \alpha \), asymmetry \( \beta \), shifting parameter \( a \), scale factor \( \sigma \). This notation is conditional depending on chosen parametrization, i.e. depending on chosen representatives with exponent \( \alpha \) and asymmetry \( \beta \). These representatives are \( s(x; \alpha, \beta) = s(x; \alpha, \beta, 0, 1) \) in terms of density. Usually, they are called Standard stable densities.

Standard stable densities, as we see, form two-parametric family depend on chosen form of parametrization. For the family of Standard stable densities in literature series expansions are known \([23],[29],[32]\). The following series expansions are presented in 2.4, p.109-110, [32].

For \( 0 < \alpha < 1 \), ”permutable” \( \beta \), and \( x \in \mathbb{R}^+ \)

\[
\begin{align*}
  s(x; \alpha, \beta) &= \frac{1}{\pi} \sum_{n \geq 1} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} \cdot \frac{1}{x^{n\alpha+1}} \cdot \sin \frac{\pi n\alpha(1 + \beta)}{2}, \\
  & \quad \text{(2.5.1)}
\end{align*}
\]

For \( 1 < \alpha < 2 \), ”permutable” \( \beta \), and \( x \in \mathbb{R}^1 \)

\[
\begin{align*}
  s(x; \alpha, \beta) &= \frac{1}{\pi} \sum_{n \geq 1} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} x^{n-1} \sin \frac{\pi n \cdot (1 + \beta)}{2}, \\
  & \quad \text{(2.5.2)}
\end{align*}
\]

The case \( \alpha = 1 \) also is presented, but we are not interested in it.

Easily seen that the value \( \beta = -1 \) is not permutable in (2.5.1), and the values \( \beta = \pm 1 \) are not permutable in (2.5.2). So, a priori, from (2.5.2) is not possible to get expansions for Right-side stable densities.

(2.5.1) and (2.5.2) were given independently by W. Feller [34], H. Bergstrom [35], Chao Chung-Jen [36]. Expansions for \( s(x; \alpha, 0) \) were obtained by A. Wintner [37].

From (2.5.1) and (2.5.2) we obtain: for \( 0 < \alpha < 1 \) and \( x \in \mathbb{R}^+ \)

\[
\begin{align*}
  s(x; \alpha, 0) &= \frac{1}{\pi} \sum_{n \geq 1} (-1)^{n-1} \cdot \frac{\Gamma(n\alpha + 1)}{n!} \frac{1}{x^{n\alpha+1}} \cdot \sin \frac{\pi n\alpha}{2}, \\
  & \quad \text{(2.5.3)}
\end{align*}
\]

and for \( 1 < \alpha < 2 \) and \( x \in \mathbb{R}^1 \)

\[
\begin{align*}
  s(x; \alpha, 0) &= \frac{1}{\pi} \sum_{n \geq 1} (-1)^{n-1} \cdot \frac{\Gamma(n\alpha + 1)}{n!} x^{n-1} \cdot \sin \frac{\pi n}{2} = \\
  &= \frac{1}{\pi} \sum_{n \geq 1} (-1)^{2m} \cdot \frac{\Gamma(2m-1)}{(2m-1)!} x^{2m-2} \cdot \sin \frac{\pi n}{2} = \frac{1}{\pi} \sum_{n \geq 1} (-1)^{m-1} \cdot \frac{\Gamma(2m-1)}{(2m-1)!} x^{2m-2}, \\
  & \quad \text{(2.5.4)}
\end{align*}
\]

Expansions for \( s(x; \alpha, 1), 0 < \alpha < 1 \), were pointed by H. Pollard [38]. By (2.5.1), for \( 0 < \alpha < 1 \) and \( x \in \mathbb{R}^+ \)

\[
\begin{align*}
  s(x; \alpha, 1) &= \frac{1}{\pi} \sum_{n \geq 1} (-1)^{n-1} \cdot \frac{\Gamma(n\alpha + 1)}{n!} \frac{1}{x^{n\alpha+1}} \cdot \sin(\pi n\alpha), \\
  & \quad \text{(2.5.5)}
\end{align*}
\]

In XVII.6, [29] characteristic functions were considered

\[
\Psi_{\alpha}(t) = \exp(-|t|^\alpha \exp(\pm i\pi\gamma/2)), \quad i = \sqrt{-1}, 0 < \alpha < 2, \alpha \neq 1, \gamma \in \mathbb{R}^1, \\
\]
where in ± the upper sign prevails for \( t \in R^+ \), the lower for \( (-t) \in R^+ \). It was proved that for \( \Psi_\alpha \) to be stable it is necessary and sufficient:

\[
|\gamma| \leq \begin{cases} 
\alpha & \text{if } 0 < \alpha < 1, \\
2 - \alpha & \text{if } 1 < \alpha < 2. 
\end{cases}
\] (2.5.7)

Here, obviously, \( \alpha \) is the exponent of Stable Law, but the meaning of parameter \( \gamma \) is not clear and was not explained in [29]. Anyway, it is connected with asymmetry. Indeed, characteristic function (2.5.6) for permutable by (2.5.7) value \( \gamma \) is real, i.e. corresponds to Symmetric Stable Law. For other permutable by (2.5.7) values of \( \gamma \) (2.5.6) corresponds to non-symmetric Stable Laws. So, \( \gamma \) is not a scale factor. It is not a shifting parameter either, because in (2.5.6) we do not see a multiplier \( \exp(it\gamma) \).

In [29] for \( \Psi_\alpha(t) \) the corresponding densities are derived. They take the following forms:

\[
s(-x; \alpha, \gamma) = \tilde{s}(x; \alpha, -\gamma)
\] (2.5.8)

and: for \( 0 < \alpha < 1 \) and \( x \in R^+ \)

\[
\tilde{s}(x; \alpha, \gamma) = \frac{1}{\pi x} \sum_{n \geq 1} \frac{\Gamma(n\alpha + 1)}{n!} (-x^{-\alpha})^n \cdot \sin \frac{\pi n(\gamma - \alpha)}{2};
\] (2.5.9)

for \( 1 < \alpha < 2 \) and \( x \in R^+ \)

\[
\tilde{s}(x; \alpha, \gamma) = \frac{1}{\pi x} \sum_{n \geq 1} \frac{\Gamma(n\alpha + 1)}{n!} (-x)^n \cdot \sin \frac{\pi n(\gamma - \alpha)}{2\alpha}.
\] (2.5.10)

(2.5.9) and (2.5.10) imply: for \( 0 < \alpha < 1 \) and \( x \in R^+ \)

\[
\tilde{s}(x; \alpha, 0) = s(x; \alpha, 0),
\] (2.5.11)

where \( s(x; \alpha, 0) \) is given by (2.5.3); for \( 1 < \alpha < 2 \) and \( x \in R^+ \) the equality (2.5.11) takes place, where \( s(x; \alpha, 0) \) is given by (2.5.4).

### 2.5.2 Shapes of Stable Densities

The graphs of stable densities \( g(x; \alpha, \beta) \) with \( \alpha = 0.25; 0.5; 0.75 \) and different \( \beta \) are drawn in Figure 5.
Figure 5.

The graphs of stable densities $g(x; \alpha; \beta)$ with $\alpha = 1, 2, 1, 5, \ldots$ and different $\beta$ are drawn in Figure 6.
Figure 6.

All stable densities (and their distribution functions) are unimodal in the sense of A. Khintchine [31].

Definition 14. The distribution function $F$ is unimodal if there is at least one value $x = a$ such that $F$ is downward convex in $(-\infty, a)$ and upward convex in $(a, +\infty)$.

Here downward (upward) convexity is meant in a broad sense, i.e.

$$F\left(\frac{x_1 + x_2}{2}\right) \leq (\geq) F(x_1) + F(x_2).$$
The unimodality of Symmetric Stable Laws has been proved by A. Wintner [39]. Non-direct proof of unimodality of all Stable Laws as a consequence of general statement has been done by M. Yamazato [40].

2.5.3 Properties of Stable Densities

Let us consider following two-parametric families of stable densities:

\[ \left\{ \hat{f}_{\alpha,\sigma}(x) = \sigma^{-1/\alpha} s(x \cdot \sigma^{-1/\alpha}; \alpha, 1) : 0 < \alpha < 1, \sigma \in R^+ \right\}, \quad (2.5.12) \]

where \( s(x; \alpha, 1) \) is given by series expansion (2.5.5);

\[ \left\{ f_{\alpha,\sigma}(x) = \sigma^{-1/\alpha} s(x \cdot \sigma^{-1/\alpha}; \alpha, 0) : 0 < \alpha \leq 2, \sigma \in R^+ \right\}, \quad (2.5.13) \]

where \( s(x; \alpha, 0) \) is given by series expansion (2.5.3) for \( 0 < \alpha < 1 \), (2.5.4) for \( 1 < \alpha < 2 \) and \( s(x; 1, 0) = \frac{1}{\pi} \frac{1}{1+x^2} \), \( s(x; 2, 0) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \). There is one-to-one correspondence between families (2.4.5) (but here \( 0 < \alpha < 1 \)) and (2.5.12), (2.4.9) and (2.5.13), i.e. one-to-one correspondence between sets of constants \( C \) and \( \sigma \), which in both cases present scale factor.

Remind that in (2.5.4) and (2.4.9) we deal with one-side Laplace-Stieltjes Transform, only in the first case Stable Law is concentrated on \( R^+ \) and in the second one on \( R_1 \).

From the general case considered in 2.7, p.173, [32] we extract following properties.

(a) \( \hat{f}_{\alpha,\sigma}(x) > 0 \) for \( x \in R^+ \) and \( \hat{f}_{\alpha,\sigma}(x) = 0 \) for \( x \leq 0 \); \( f_{\alpha,\sigma}(x) > 0 \) for any \( x \in R^1 \).

(b) The graphs of \( \hat{f}_{\alpha,\sigma} \) and \( f_{\alpha,\sigma} \) are downward/upward convex and are unimodal with only one mode.

(c) \( \hat{f}_{\alpha,\sigma}(x) \approx \text{const} \cdot \frac{1}{x^{\alpha+1}} \) and \( f_{\alpha,\sigma}(x) \approx \text{const} \cdot \frac{1}{x^{\alpha+1}} \) as \( x \to +\infty \), where the case \( \alpha = 2 \) is excluded.

(d) The graphs of \( \hat{f}_{\alpha,\sigma_1} \) and \( \hat{f}_{\alpha,\sigma_2} \) for \( \sigma_1 \neq \sigma_2 \) intersect each other only once on \( R^+ \). The graphs of \( f_{\alpha,\sigma_1} \) and \( f_{\alpha,\sigma_2} \) for \( \sigma_1 \neq \sigma_2 \) intersect each other in two points \( x_1 \) and \( x_2 \), where \( x_1 \in R^+ \), \( x_2 = -x_1 \).

The property (d) is proved by M.Kanter [41].

2.6 Continuous Analogs of Frequency Distributions

In this Section we build two-parametric families of infinite differentiable densities based on families of Stable Laws which were extracted in Section 2.5. We’ll show that they posses known statistical facts on frequency distributions in biomolecular systems. That is why these families may serve as continuous analogs of desired frequency distributions which we want to obtain.
2.6.1 Discussion

The *scale-invariance* property of Power Law becomes an initial point for applications of this distribution in large-scale biomolecular systems. By using the properties of Power Law, which are essential for known statistical facts’ fulfillment, its generalization - Pareto distribution has been suggested as a suitable approximation for empirical frequency distributions. Note that the Pareto distribution doesn’t posses the *scale-invariance* property.

In this Chapter we proceed by the same scheme. Namely, we substantiated that in many large-scale biomolecular systems the *scale-invariance* property is possible to replace by *semi-group* property of quite different nature which leads to a wide class of distribution functions. Next, we reduced the obtained class by introducing the notion of *Stable Law*. The reason may be explained as follows. The right tails of distribution functions, which posses the *semi-group property*, not always vary regularly at infinity. The requirement of regular variation, as we understood in Chapter 1, is one of known statistical fact’s mathematical formulation. *Stable Laws* excepting *Normal Laws* have regularly varying sum of tails. In Section 2.5 we extracted from family of stable densities two-parametric subfamilies which vary regularly at infinity, i.e. their distribution functions have regularly varying right tails.

In our way of constructing new frequency distributions based on these subfamilies several problems arise.

The first one says that, for instance, the subfamilies extracted from stable densities are concentrated on $R^1$. We talk about symmetric stable densities. So, we need for these densities a transformation procedure that shall lead to densities concentrated on $R^+$, because such a property is observed for empirical frequency distributions.

As we’ll see in below, this procedure requires the knowledge of the value initial *Stable Law* at point 0. This value is represented for *Stable Laws* with the help of asymptotical expansions of stable densities when the argument tends to zero (see, 2.5, [32]). But they are complicate and evaluation of this value is embarrassing. This problem is possible to solve with the help of *Tauberian Theorems* for corresponding *Integral Transforms*. We are able to do that for *Right-side Stable Laws* with exponent $\alpha \in (0, 1)$, but not for $\alpha \in (1, 2)$. There is no problem for *Symmetric Stable Laws*. For them the mentioned value is equal to $(1/2)$.

After transformation procedure we get distribution functions concentrated on $R^+$. Then the *Laplace-Stieltjes Transform* is more preferable for them. In order to build the first ones, it requires to have one-side *Laplace-Stieltjes Transforms* for *Stable Laws* which generate these distribution functions. First of all, it seems to be complicate to evaluate the desired transforms for distribution functions generated by *Right-side Stable Laws* with exponent $\alpha \in (1, 2)$. But it is an easy task for *Right-side Stable Laws* with
exponent \( \alpha \in (0, 1) \) and for symmetric Stable Laws. This is the second problem we have to deal with.

From the point of view of stable densities it is important to mention that the series expansions (2.5.2) and (2.5.10) give noting for the Right-side stable densities with exponent \( \alpha \in (1, 2) \). But for other considering families we have series expansions (2.5.3)-(2.5.5). This is the third problem we have to deal with.

From results of the Theory of Stable Laws it follows that any Stable Law has regularly varying sum of tails. In cases being considered here (Right-side and Symmetric Stable Laws) the Stable Laws have regularly varying right tails which exhibit constant slowly varying components (compare to statement (c) in Section 2.5). The fourth problem is related with the constant slowly varying components evaluation for our future considerations.

The following asymptotic expansion is known (see, Corollary 2, 2.5, p.115, [32]). For \( 1 < \alpha < 2 \) and \( \beta \neq -1 \)

\[
s(x; \alpha, \beta) \approx \frac{\alpha}{\pi} x^{-1} \cdot \sum_{K \geq 1} \frac{\Gamma(\alpha K)}{\Gamma(K)} \cdot \frac{1}{x^K} \cdot \sin \frac{\pi K \cdot (2 - \alpha)(1 + \beta)}{2}, \quad x \to +\infty. \tag{2.6.1}
\]

Two corollaries of (2.6.1) are of interest for us. Here they are:

\[
s(x; \alpha, 0) \approx -\frac{\alpha}{\pi} x \cdot \sum_{K \geq 1} \frac{\Gamma(\alpha K)}{\Gamma(K)} \frac{1}{x^K} \sin \left(\frac{\pi K \alpha}{2}\right), \quad x \to +\infty; \tag{2.6.2}
\]

\[
1 - S(x; \alpha, 0) = \int_{x}^{+\infty} s(u; \alpha, 0) du \approx \frac{1}{\pi} \sum_{K \geq 1} \frac{\Gamma(\alpha K)}{\Gamma(K + 1)} \frac{1}{x^K} \sin \left(\frac{\pi K \alpha}{2}\right), \quad x \to +\infty. \tag{2.6.3}
\]

It is possible to prove that \( s(x; \alpha, 1) \) and \( s(x; \alpha, 0) \) vary regularly at infinity with exponent \( \rho = -(\alpha + 1)^{-1} \) (it is not easy to do), but it seems to be embarrassing to evaluate the constant slowly varying component from these expansions.

It seems to be more easy to solve this problem with the help of Tauberian Theorems for Laplace-Stieltjes Transform. In this case after transformation procedure we also do not possess the Laplace-Stieltjes Transform for distribution functions generated by Right-side Stable Laws with exponent \( \alpha \in (1, 2) \).

All these technical difficulties force us to refuse to use Right-side Stable Laws with exponent \( \alpha \in (1, 2) \) for constructing new empirical frequency distributions.

The fifth problem says that the expressions of table densities from the extracted families in terms of convergent series expansions are complex. With respect to this problem two kind of arguments are appropriate.

We have to mention that after series of publications by B. Mandelbrot in 1956-1967 [42]-[51] the stage of intensive penetration of Stable Laws into Economics, Biology, Engineering Sciences begins. It is the result of following circumstances.

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Stable Laws are natural generalization of Normal Laws. They appear in the same conditions when we have a cumulative effect of many small similar factors. The only difference consists in a "large dispense" of cumulative effects values from the mean value. Such a cumulative effect takes place for various phenomena in mentioned scientific fields. By the opinion of acknowledged specialist in Probability, Statistics and Mathematical Analysis, V. A. Zolotarev, many facts point out that the functions \( s(x; \alpha, \beta) \) by the richness of their analytic properties are adequate to be extracted into independent class and admit for them "citizen’s rights" in the Theory of Special Functions (see, 2.10, p.200, [32]). By our opinion, at least for particular values \( \beta = 1 \) and \( \beta = 0 \) the functions \( s(x; \alpha, \beta) \) already have to be considered as special functions of Mathematical Analysis and Probability Theory, and corresponding tables for them have to be composed. Then, the problem of complexity partly could be solved. This is the first kind of arguments.

The second kind of arguments consists in following. Here are several Stable Laws whose densities are presented in a closed form: normal, Cauchy, Levy. For some values of \( \alpha \) there are expression for Stable Laws in term of enough simple form integrals (see, Section 2.7). Each of them generates one-parametric family of Stable Laws with the help of scaling. These one-parametric families may generate empirical frequency distributions for biomolecular needs.

### 2.6.2 The Slice of Random Variable

After discussion in 2.6.1 three one-parametric families of stable densities

\[
\begin{align*}
\{s(x; \alpha, 1) : 0 < \alpha < 1\}, \\
\{s(x; \alpha, 0) : 0 < \alpha < 1\}, \\
\{s(x; \alpha, 0) : 1 < \alpha < 2\}
\end{align*}
\]

remain as pretenders to generate desired frequency distributions. But the families (2.6.5) and (2.6.6) are concentrated on \( R^1 \). In order to transform these families we need in following

**Definition 15.** Let the random variable \( \xi \) have continuous distribution function \( F \), which is not completely concentrated on \( R^1 \setminus R^+ \). Then, for random variable \( \xi \) any positive random variable \( \hat{\xi} \) with distribution function

\[
\hat{F}(x) = \begin{cases} 
F(x) - F(0) & \text{if } x \in R^+, \\
\frac{1-F(0)}{1-F(x)} & \text{if } x \in R^1 \setminus R^+
\end{cases}
\]

is called the slice of \( \xi \).

Easily seen that \( \hat{F}(x) = P(\xi < x/\xi \geq 0), \ x \in R^+ \), where \( P(A/B) \) is the conditional probability of event \( A \) under condition of \( B \)'s occurrence.
If distribution function $F$ has density, say $f$, in $R^+$, then $\hat{F}$ also has a density in $R^+$

$$\hat{f}(x) = \frac{f(x)}{1 - F(0)}. \quad (2.6.7)$$

Given random variable $\xi$ there exist various constructions of the slice $\hat{\xi}$ for $\xi$. The most known is $\hat{\xi} = \hat{F}^{-1}(\hat{F}(\xi)) = F^{-1}(\frac{\hat{F}(\xi) + c}{1 + c})$, where $F^{-1}$ and $\hat{F}^{-1}$ are reverse functions for $F$ and $\hat{F}$, respectively, and $c = \frac{F(0)}{1 - F(0)} < +\infty$.

We may give statistical interpretation of the notion of slice. Namely, let $(\xi_1, \xi_2, \cdots, \xi_n)$, $n > 1$, be an independent selection on random variable $\xi$ ($\xi_1, \xi_2, \cdots, \xi_n$ are identically distributed). By choosing among $\xi_1, \xi_2, \cdots, \xi_n$ only positive ones, we come to the independent selection of random variable $\hat{\xi}$: $(\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_\nu})$ formed by random number of observations $\nu$ having binomial distributions, where $1 \leq \nu \leq n$, $1 \leq i_1 < i_2 < \cdots < i_\nu \leq n$.

There are many useful properties of random variable $\hat{\xi}$. For instance, let $\{\xi_n\}$ be a sequence of independent random variables. Then, there exists a sequence $\{\hat{\xi}_n\}$ of slices of $\{\xi_n\}$, that are also formed of independent random variables.

Now, let us apply the notion of slice to one-parametric families (2.6.5) and (2.6.6). Since they are formed by symmetric densities, i.e. $s(x; \alpha, 0) = s(-x; \alpha, 0)$, $x \in R^+$, (see, also Figures 5. and 6.), therefore for their distribution functions, say $S(x; \alpha, 0), x \in R^1$, the equality holds $S(0; \alpha, 0) = \frac{1}{2}$. That is why, due to (2.6.7), the slices of symmetric random variables with densities $s(x; \alpha, 0)$ have distribution functions with densities

$$\hat{s}(x; \alpha, 0) = 2 \cdot s(x; \alpha, 0), \quad x \in R^+. \quad (2.6.8)$$

Thus, from (2.6.4)-(2.6.6) we obtain three one-parametric families: (2.6.4) and

$$\{\hat{s}(x; \alpha, 0) : 0 < \alpha < 1\}, \quad (2.6.9)$$

$$\{\hat{s}(x; \alpha, 0) : 1 < \alpha < 2\}, \quad (2.6.10)$$

which are concentrated on $[0, +\infty)$.

**2.6.3 The Properties of Extracted Families**

The graphs of the first family (2.4.6) for values $\alpha: 0, 0.25; 0, 5; 0, 75$ are presented in Figure 5. As we see in Figures 5 and 6, the densities of Symmetric Stable Laws around 0 become "very large" as $\alpha \to 0$. That is why for simplicity with the help of Figures 5 and 6 the graphs of densities $\hat{s}(x; \alpha, 0)$ are drawn in Figure 7 starting from the value $\alpha = 0.5$ for values 0, 0.5; 0, 75; 1, 25. For larger values of $\alpha$ the graphs are "very close" to each other.
It is clear that the Right-side Stable Law with exponent $\alpha \in (1, 2)$ has more skewness (to the right), than distribution function with density $\hat{s}(x; \alpha, 0)$, where $\alpha$ is the same as mentioned exponent. So, Right-side Stable Laws with exponent $\alpha \in (1, 2)$ could be better pretender to generate suitable empirical frequency distributions than Symmetric Stable Laws with exponent $\alpha \in (1, 2)$, if we’ll be able to overcome all arising problems being described in 2.6.1. Next, comparing the graphs of functions $s(x; \alpha, 1)$ and $\hat{s}(x; \alpha, 0)$ with the same exponent $\alpha \in (0, 1)$ we see that the skewness of $\hat{s}(x; \alpha, 0)$ is comparatively small, which implies the following conclusion.

The best pretenders to generate empirical frequency distributions are the families (2.6.4) and (2.6.10) of densities.

Let us consider two-parametric families of densities concentrated on $\mathbb{R}^+$. Namely, (2.5.12) and the following one

$$\{ \hat{f}_{\alpha, \sigma}(x) = 2\sigma^{-1/\alpha} \cdot \hat{s}(x \cdot \sigma^{-1/\alpha}; \alpha, 0) : 1 < \alpha < 2, \sigma \in \mathbb{R}^+ \}. \quad (2.6.12)$$

From properties (a)-(b) being formulated in Section 2.5 we combine the following conclusions

**Theorem 2.1** (a) The graphs of $\hat{f}_{\alpha, \sigma}$ are downward/upward convex, and are unimodal with only one mode, say $m$, where $0 < m < +\infty$ for $0 < \alpha < 1$ and $m = 0$ for $1 < \alpha < 2$. 

Figure 7.
\( f_{\alpha, \sigma}(x) \approx \text{const} \cdot \frac{1}{x^{\alpha+1}}, \quad x \to +\infty. \)

It means that the density \( f_{\alpha, \sigma}(x) \) varies regularly at infinity and exhibits constant slowly varying component. The exponent of regular variation, say \((-\rho)\), of \( f_{\alpha, \sigma} \) and the exponent of stable density \( \alpha \) which generates \( f_{\alpha, \sigma} \) satisfy equality \( \rho = \alpha + 1 \).

(c) The graphs of functions \( f_{\alpha, \sigma_1}(x) \) and \( f_{\alpha, \sigma_2}(x) \) for any different \( \sigma_1 \in \mathbb{R}^+ \) and \( \sigma_2 \in \mathbb{R}^+ (\sigma_1 \neq \sigma_2) \) intersect each other only once.

More deep consideration of families (2.6.12) is waiting us in Section 2.7.

2.7 Regular Variation of Continuous Analogs

In the present Section a deeper analysis of regular variation of the tails of two-parametric families (2.5.12) and (2.6.12) shall be done. We improve the correspondence of densities and their Laplace-Stieltjes Transforms. The constant slowly varying components of densities from mentioned families shall be evaluated.

Our difficulties are a result of different parametrizations being used in order to derive separately the families of densities and the families of Laplace-Stieltjes Transforms.

2.7.1 The Case \( 0 < \alpha < 1 \)

In case \( 0 < \alpha < 1 \) the Right-side Stable Laws, in particular, the standard ones \( S_\alpha(x) \) are concentrated on \( \mathbb{R}^+ \). The two-side Laplace-Stieltjes Transforms of standard Stable Laws for \( 0 < \alpha < 2 \) are given by formula (2.4.5). Formula (2.4.5) in our particular case \( 0 < \alpha < 1 \) leads to the usual, i.e. one-side, Laplace-Stieltjes Transform

\[
\rho_\alpha(s) = \int_0^\infty e^{-sx}dS_\alpha(x) = \exp(-s^\alpha), \quad 0 \leq s < +\infty, 0 < \alpha < 1.
\]  
(2.7.1)

We already know that only Right-side Stable Laws with parameter \( \alpha \in (0, 1) \) are concentrated on \( \mathbb{R}^+ \).

By Theorem 1, XVIII.6, p.424, [29], for \( S_\alpha(x) \) given by (2.7.1) the limit exists

\[
\lim_{x \to +\infty} x^\alpha \cdot (1 - S_\alpha(x)) = \frac{1}{\Gamma(1 - \alpha)}, \quad 0 < \alpha < 1,
\]  
(2.7.2)

where \( \Gamma(\cdot) \) denotes the Euler's Gamma Function. The asymptotic relation (2.7.2) says that the function \( 1 - S_\alpha(x) \) varies regularly at infinity with exponent \((-\rho)\), where the number \( \rho \) coincides with the exponent of the same Stable Law \( S_\alpha \), i.e. with the number \( \alpha \). Moreover, \( 1 - S_\alpha(x) \) exhibits constant slowly varying component \( L = (1/\Gamma(1 - \alpha)) \).

The limit relation (2.7.2) is obtained in [29] with the help of Tauberian Theorem for Laplace-Stieltjes Transform.
In order to get the characteristic function of standard Stable Law \( S_\alpha(x) \), \( 0 < \alpha < 1 \), we must replace the argument \( s \) in (2.7.2) by \((-it)\), where \( i = \sqrt{-1} \), \( t \in \mathbb{R} \). Then for \( t > 0 \) we get

\[
\psi_\alpha(t) = \int_0^{+\infty} e^{i tx} dS_\alpha(x) = \exp(-it^\alpha) = \exp(-(\cos(-\pi/2) + i \sin(-\pi/2))^{\alpha} \cdot t^\alpha) = \exp(-t^\alpha e^{-i\pi\alpha/2}).
\]

Similarly, for \( t < 0 \) we obtain \( \psi_\alpha(-t) = \exp(-(-t)^\alpha e^{i\pi\alpha/2}) \). Combining these two expressions we come to the equality

\[
\psi_\alpha(t) = \exp(-|t|^\alpha \cdot e^{\pm i\pi\alpha/2}),
\]

where in \( \pm \) the upper sign prevails for \( t < 0 \), the lower for \( t > 0 \).

Indeed, the representation (2.7.3) coincides with the representation of standard Stable Law’s characteristic function in case of exponent \( \alpha \in (0, 1) \) and asymmetry \( \beta = 1 \) (see, Theorem B3, p.22, [32]). It is just the case when corresponding density \( s(x; \alpha, 1) \) exhibits the series expansion (2.5.5). It means that, due to (2.7.1), for \( 0 \leq s < +\infty, 0 < \alpha < 1 \)

\[
\int_0^{+\infty} e^{-sx} \cdot s(x; \alpha, 1) dx = \exp(-s^\alpha).
\]

Taking into account (2.7.4), for following two-parametric family (see, (2.5.12))

\[
\left\{ \hat{f}_{\alpha,\sigma}(x) := \sigma^{-1/\alpha} \cdot s(\sigma^{-1/\alpha} \cdot x; \alpha, 1) : 0 < \alpha < 1, \sigma \in \mathbb{R}^+ \right\}
\]

of densities concentrated on \( \mathbb{R}^+ \) we have

\[
\rho_{\alpha,\sigma}(s) := \int_0^{+\infty} e^{-sx} \cdot \hat{f}_{\alpha,\sigma}(x) dx = \sigma^{-1/\alpha} \cdot \int_0^{+\infty} e^{-sx} \cdot s(\sigma^{-1/\alpha} \cdot x; \alpha, 1) dx = \int_0^{+\infty} e^{-s\sigma^{1/\alpha} \cdot y} \cdot s(y; \alpha, 1) dy = \exp(-\sigma \cdot s^\alpha),
\]

where (2.7.4) is used. Finally we have to notice that for some values of \( \alpha \) in case \( 0 < \alpha < 1 \) it is possible to evaluate \( s(x; \alpha, 1) \) presenting it in the form of some improper integral.

The case of density of Levy’s Law (2.2.1) is included because in this case we have exponent \( \alpha = (1/2) \). Only we have to mention that in our notations this density takes a slightly different form than we are familiar to: \( s(x; 1/2, 1) = 1/2\sqrt{\pi} \cdot \frac{1}{x^{1/2}} \exp(-\frac{1}{4x}), \ x \in \mathbb{R}^+ \).

Further, in cases \( \alpha = (1/3) \) and \( \alpha = (1/4) \) the following equalities hold (see, 3.4, p.236, [32])

\[
\begin{align*}
s(x; 1/3, 1) &= \frac{1}{2\pi} x^{-3/2} \cdot \int_0^{+\infty} \exp(-\frac{1}{3\sqrt{3x}}(y^3 + \frac{1}{y^3})) dy \\
s(x; 1/4, 1) &= \frac{1}{2\pi} x^{-1/3} \cdot \int_0^{+\infty} \exp(-\frac{1}{4x^{-1/3}} \cdot (y^4 + \frac{1}{y^4})) dy.
\end{align*}
\]
2.7.2 The Case $1 < \alpha < 2$

Standard *Symmetric Stable Laws* $S_\alpha(x)$ with exponents $\alpha \in (1, 2)$, as we know, are concentrated on $R^1$. Their one-side Laplace Stieltjes Transform is given by formula (2.4.8)

$$\rho_\alpha^+(s) = \int_0^{+\infty} e^{-sx}dS_\alpha(x) = \frac{1}{\pi} \int_0^{+\infty} e^{-(su)^\alpha} \frac{du}{1+u^2}, \quad 0 \leq s < +\infty, \ 1 < \alpha < 2. \quad (2.7.7)$$

The generalization of formula (2.7.7) for all standard *Symmetric Stable Laws* originally has been obtained by V. Zolotarev [52].

According to notations of monograph [32], density of a standard *Stable Law* with exponent $\alpha \in (1, 2)$ and asymmetry $\beta$ exhibits series expansion (2.5.2). Being simplified for symmetric density it takes the form (2.5.4) for given $\alpha$, i.e.

$$s(x; \alpha, 0) = \frac{1}{\pi} \sum_{n \geq 1} (-1)^n \Gamma((2m-1)/\alpha + 1) x^{2m-1}, \ 1 < \alpha < 2, \ x \in R^1. \quad (2.7.8)$$

This form of symmetric densities, as it was shown in Section 2.5, corresponds to following characteristic functions (see, (2.5.6) with $\gamma = 0$)

$$\psi_\alpha(t) = \exp(-|t|^\alpha), \ 1 < \alpha < 2, \ t \in R^1. \quad (2.7.9)$$

By Lemma 2, XVII.4, p.541, [29] and Theorem 3, XVII.5, p.547, [29], for distribution function $G$ having the characteristic function

$$\int_{-\infty}^{+\infty} e^{itx}dG(x) = \exp \lambda(t), \ i\sqrt{-1}, \ t \in R^1, \quad (2.7.10)$$

where for $1 < \alpha < 2$ we put

$$\lambda(t) = -B_\alpha \cdot |t|^\alpha, \ B_\alpha = \frac{\Gamma(2-\alpha)}{\alpha-1} \cos \frac{\pi \alpha}{2}, \quad (2.7.11)$$

the following equivalency takes place

$$1 - G(x) \approx \frac{2 - \alpha}{2\alpha} \frac{1}{x^\alpha}, \ x \to +\infty. \quad (2.7.12)$$

Here the result taken from [29] is simplified in particular case of *Symmetric Stable Law* by putting in mentioned result $p = q = (1/2)$ and taking constant $C = 1$.

Now, let us make trivial calculations. Due to (2.7.9)-(2.7.11),

$$\exp \lambda(t/B_\alpha^{1/\alpha}) = \exp(-|t|^\alpha) = \int_{-\infty}^{+\infty} \exp\left(\frac{ix}{B_\alpha^{1/\alpha}}\right)dG(x) = \int_{-\infty}^{+\infty} e^{ix/y} dG(B_\alpha^{1/\alpha} \cdot y) \quad (2.7.13)$$

It means that for standard *Stable Law* $S_\alpha(x)$ the following equality holds

$$S_\alpha(x) = G(B_\alpha^{1/\alpha} \cdot x), \ 1 < \alpha < 2, \ x \in R^1.$$
where the constant $B_{\alpha}$ is defined in (2.7.11). Next, from formulas (2.7.11)-(2.7.13) we obtain
\[ 1 - S_{\alpha}(x) \approx \frac{2 - \alpha}{2\alpha} \cdot \Gamma(1 - \alpha) \cos(\pi\alpha/2) \cdot \frac{1}{x^\alpha}, \quad x \to +\infty, \]
and taking into account the following well-known relationships for the Gamma Function
\[ \Gamma(x+1) = x \cdot \Gamma(x) \quad \text{and} \quad \Gamma(1-x) \cdot \Gamma(x) = \frac{\pi}{\sin(\pi x)}, \]
we come to the equivalency
\[ 1 - S_{\alpha}(x) \approx \frac{2 - \alpha}{2\alpha} \cdot \frac{1}{x^\alpha} = (2 - \alpha) = \frac{2 - \alpha}{\pi \alpha} \cdot \sin\frac{\pi\alpha}{2} \cdot \frac{1}{x^\alpha}, \quad x \to +\infty, 1 < \alpha < 2. \] (2.7.14)

It is necessary to mention that the coefficient of $\frac{1}{x^\alpha}$ at the right-hand-side in (2.7.14) is positive because $\sin\frac{\pi\alpha}{2} < 0$ for $\alpha \in (1, 2)$.

During our calculations we deal with function $\Gamma(x)$, where argument $x$ takes also negative values. That is why above the generalization of Gamma Function
\[ \Gamma(z) = -\frac{1}{2i \cdot \sin(\pi z)} \int e^{(-t)^z} e^{-t} dt, \quad i = \sqrt{-1}, \]
where $z$ is not an integer, was used (see, for instance, 8.310.2, p.933, [20]).

The contour C is shown in the drawing.

$\Gamma(z)$ is a fractional analytic function of argument $z$ with simple poles at the points $z = n$, $n = 0, 1, 2, \ldots$, to which correspond the residues $(-1)^{n} \cdot \frac{\Gamma(z)}{z!}$ satisfies the equality $\Gamma(1) = 1$.

The asymptotic formula (2.7.14) says that $1 - S_{\alpha}(x)$ varies regularly at infinity with exponent $(-\rho)$, where $\rho$ coincides with the exponent of Symmetric Stable Law $S_{\alpha}$.

Moreover, the function $1 - S_{\alpha}(x)$ exhibits constant slowly varying component
\[ L = -\frac{2 - \alpha}{\pi \alpha} \cdot \sin\frac{\pi\alpha}{2}. \] (2.7.15)

Thus, according to (2.7.7) for stable densities $s(x; \alpha, 0)$ with $1 < \alpha < 2$ we have
\[ \int_{0}^{\infty} e^{-sx} \cdot s(x; \alpha, 0) dx = \frac{1}{\pi} \int_{0}^{\infty} e^{-(su)\alpha} \cdot \frac{du}{1 + u^2}. \] (2.7.16)

Taking into account (2.6.12) and (2.7.16), for two-parametric family
\[ \left\{ \hat{f}_{\alpha, \sigma}(x) := \sigma^{-1/\alpha} \cdot \hat{s}(\sigma^{-1/\alpha} \cdot x; \alpha, 0) : 1 < \alpha < 2, \sigma \in R^{+} \right\} \] (2.7.17)

of densities concentrated on $R^{+}$ we have
\[ \rho_{\alpha, \sigma}(x) := \int_{0}^{\infty} e^{-sx} \hat{f}_{\alpha, \sigma}(x) dx = \frac{2}{\pi} \int_{0}^{\infty} e^{-\sigma(su)\alpha} \cdot \frac{du}{1 + u^2}. \] (2.7.18)
where (2.6.8) was used. The distribution function which corresponds to density \( \tilde{f}_{\alpha,1}(x) \) has, obviously, only right tail. This tail, due to (2.7.14), varies regularly at infinity with exponent \((-\alpha)\) and exhibits constant slowly varying component being twice more than the function \( 1 - S_{\alpha}(x) \) has, i.e., due to (2.7.15), \( L = -2^{\frac{2-\alpha}{\alpha}} \pi^{\frac{\alpha}{2}} \sin\frac{\pi \alpha}{2} \).

### 2.7.3 The Case \( \alpha = 1 \)

In this case we prefer to deal with the prevalent form (familiar for us) of density of Gausshy’s Law and use the same notations as above, i.e.

\[
s(x; 1, 0) = \frac{1}{\pi(1 + x^2)}, \quad x \in \mathbb{R},
\]

which, obviously, is symmetric. The form given for standard stable density with exponent \( \alpha = 1 \) is slightly different

\[
g(x; 1, 0) = 2 \cdot s(x; 1, 0) = 2 \cdot \frac{1}{\pi} \left( 1 + x^2 \right)^{-1}.
\]

The one-side Laplace Transform for density (2.7.19) (it is amazing!) has the same form as for standard symmetric stable densities given by formula

\[
\rho_1^+(s) = \int_0^\infty e^{-sx} \cdot s(x; 1, 0) dx = \frac{1}{\pi} \int_0^\infty e^{-su} du \frac{1}{1 + u^2}, \quad 0 < s < +\infty.
\]

It is of interest that \( \rho_1^+(s) \) given by the last integral in (2.7.20) may be also expressed in terms of following Special Functions of Mathematical Analysis: sine integral and cosine integral. (\( \text{si}(x) \) and \( \text{ci}(x) \)). Remind that

\[
\text{si}(x) = - \int_x^{+\infty} \frac{\sin t}{t} dt = - \frac{\pi}{2} + \int_0^x \frac{\sin t}{t} dt
\]

and

\[
\text{ci}(x) = - \int_x^{+\infty} \frac{\cos t}{t} dt = \gamma + \ln x + \int_0^x \frac{\cos t - 1}{t} dt.
\]

(see, 8.230.1. and 2., p.928, [20]), respectively, where \( \gamma = 0, 5772156649015325 \cdots \) is the famous Euler’s constant. Then, due to 3.354, p.313, [20], we have

\[
\rho_1^+(s) = \text{ci}(s) \cdot \sin s - \text{si}(s) \cdot \cos s, \quad 0 < s < +\infty.
\]

Thus, we may talk on one-parametric family of densities

\[
\left\{ \tilde{f}_{1,\sigma}(x) := \sigma^{-1} \cdot \tilde{s}(\sigma x; 1, 0) : \sigma \in \mathbb{R}^+ \right\}
\]

concentrated on \( \mathbb{R}^+ \) and generated by density of the Cauchy’s Law (2.7.19) with the help of formula \( \tilde{s}(x; 1, 0) = 2 \cdot s(x; 1, 0), \quad x \in \mathbb{R}^+ \). Due to (2.7.20)-(2.7.22), we get two representations for the Laplace Transform \( \rho_{1,\sigma}(s), \sigma \in \mathbb{R}^+, 0 < s < +\infty:\n
\[
\rho_{1,\sigma}(s) := \int_0^\infty e^{-sx} \tilde{f}_{1,\sigma}(x) dx = \frac{2}{\pi} \int_0^\infty e^{-su} du \frac{1}{1 + u^2} =
\]

\[
= \text{ci}(\sigma \cdot s) \sin(\sigma \cdot s) - \text{si}(\sigma \cdot s) \cos(\sigma \cdot s).
\]
The distribution function which corresponds to density \( \hat{f}_{1,1}(x) \) has only right tail. This tail, due to Theorem 1.(i), VIII.9, p.273, [29] and to the form of density (2.7.19), varies regularly at infinity with exponent \((-1\)). It exhibits constant slowly varying component \( L = (2/\pi) \).

Thus, for introduced families of continuous analogs of desired frequency distributions which have to be constructed we find out the explicit values of their regular variations’ constant slowly varying components.

We’ll continue to collect the properties of continuous analogs, which, in particular, substantiate the fulfillment of known statistical facts for empirical frequency distributions, before the desired frequency distributions shall be derived.

Finally, we have to point out that in Theory of Stable Laws there are many asymptotic expansions for stable densities \( s(x; \alpha, \beta) \) as \( x \to +\infty \), or \( x \to 0 \). But they are complex and in such series expansions the terms change the sign infinitely often. Even the problem of the constant slowly varying component evaluation is not easy. The Method of Tauberian Theorems based on Integral Transforms, which leads to the results of this Section, is more preferable.

### 2.8 Integral Representations of Stable Laws

A powerful tool for investigation of asymptotic properties of Stable Laws is Integral Representation, which naturally is based on conversion formula for Canonical Representations of Stable Laws, because nothing else but Canonical Representations for Stable Laws initially are known. Further, series expansions of functions under integral in Integral Representation and their analysis lead to asymptotic expansion for Stable Laws.

In this Section we briefly give information on this topic in order to describe more deeply the introduced families of continuous analogs.

#### 2.8.1 Integral Representation (see, Chapter 2, [32])

Let \( S(x; \alpha, \beta) \) be a distribution function of the standard stable density \( s(x; \alpha, \beta) \) with asymmetry \( \beta \in [-1, 1] \) and exponent \( \alpha \in (0, 2] \).

For \( \alpha \neq 1, \beta \in [-1, 1], x \in R^1 \setminus \{0\} \) let us denote

\[
K(\alpha) = \alpha - 1 + \text{sign} (1 - \alpha), \quad \theta = \beta \cdot \frac{K(\alpha)}{\alpha} \text{sign} x, \quad (2.8.1)
\]

\[
\text{sign} x = \begin{cases} 
1 & \text{if } x \in R^+, \\
-1 & \text{if } x \in (-\infty, 0),
\end{cases}
\]

and for \( y \in [-\theta, 1] \)

\[
U_\alpha(y, \theta) = \left( \frac{\sin(\pi \alpha \cdot (y + 1))}{\cos(\pi y)} \right)^{1/\alpha} \cdot \frac{\cos(\pi ((\alpha - 1)y + \alpha \theta))}{\cos(\pi y)}. \quad (2.8.2)
\]

We are interested in only values \( x \in R^+ \) for \( S(x; \alpha, \beta) \). Then, for \( x \in R^+ \)

\[
1 - S(x; \alpha, \beta) = \frac{1}{4}(1 + \theta)(1 + \text{sign} (1 - \alpha)) - \frac{\text{sign} (1 - \alpha)}{2} \int_{-\theta}^{1} \exp(-x \pi/4 U_\alpha(y, \theta))dy. \quad (2.8.3)
\]
Formula (2.8.3) presents the Integral Representation of Stable Law $S(x; \alpha, \beta)$.

Consider the case

$$0 < \alpha < 1, \quad \beta = 1.$$  \hfill (2.8.4)

Then, for $x \in R^+$ from (2.8.1)-(2.8.2) we conclude

$$S(x; \alpha, 1) = \frac{1}{2} \int_{-1}^{1} \exp \left( - \frac{1}{x^{\alpha/(1-\alpha)}} U_{\alpha}(y) \right) dy,$$  \hfill (2.8.5)

where for $y \in [-1, 1]$

$$U_{\alpha}(y) = \left( \frac{\sin(\frac{\pi}{2} \cdot (y + 1))}{\cos(\frac{\pi}{2} y)} \right)^{\alpha-1} \frac{\cos(\frac{\pi}{2} ((\alpha - 1)y + \alpha))}{\cos(\frac{\pi}{2} y)}.$$  \hfill (2.8.6)

Now, making the variable’s replacement $\psi = \frac{\pi}{2} y$ in integral (2.8.5) and using the equalities in (2.8.6)

$$\cos((\alpha - 1)\psi + \frac{\pi}{2}) = \cos((\frac{\pi}{2} - (1 - \alpha)(\psi + \frac{\pi}{2})) = \sin((1 - \alpha)(\psi + \frac{\pi}{2})),
\cos \psi = \sin(\psi + \frac{\pi}{2}),
$$

we come to another representation for $S(x; \alpha, 1)$. New variable’s replacement $\phi = \psi + \frac{\pi}{2}$ leads to the following Integral Representation

$$S(x; \alpha, 1) = \frac{1}{\pi} \int_{0}^{\pi} \exp \left( - \frac{1}{x^{\alpha/(1-\alpha)}} U_{\alpha}(\phi) \right) d\phi,$$  \hfill (2.8.7)

where for $\phi \in [0, \pi]$

$$U_{\alpha}(\phi) = \left( \frac{\sin(\alpha \phi)}{\sin \phi} \right)^{\alpha-1} \frac{\cos(\frac{\pi}{2} (\alpha - 1)\phi)}{\cos(\frac{\pi}{2} \phi)}.$$  \hfill (2.8.8)

Consider the case

$$1 < \alpha \leq 2, \quad \beta = 0.$$  \hfill (2.8.9)

Then, from (2.8.1)-(2.8.2) for $x \in R^+$ we conclude

$$S(x; \alpha, 0) = 1 - \frac{1}{2} \int_{0}^{1} \exp(-x^{\alpha/(\alpha-1)} \cdot V_{\alpha}(y)) dy,$$  \hfill (2.8.10)

where for $y \in [0, 1]$

$$V_{\alpha}(y) = \left( \frac{\cos(\frac{\pi}{2} y)}{\sin(\frac{\pi}{2} \alpha y)} \right)^{\alpha-1} \frac{\cos(\frac{\pi}{2} (\alpha - 1)y)}{\cos(\frac{\pi}{2} y)}.$$  \hfill (2.8.11)

Now, making the variable’s replacement $\psi = \frac{\pi}{2} y$ in integral (2.8.10) we come to the following Integral Representation

$$S(x; \alpha, 0) = 1 - \frac{1}{\pi} \int_{0}^{\pi/2} \exp(-x^{\alpha/(\alpha-1)} \cdot \bar{V}_{\alpha}(\phi)) d\phi,$$  \hfill (2.8.12)
where

\[ V_\alpha(\varphi) = \left( \frac{\cos \varphi}{\sin(\alpha \varphi)} \right)^\frac{\alpha - 1}{\alpha - 1} \cdot \frac{\cos((\alpha - 1)\varphi)}{\cos \varphi}. \] (2.8.13)

Let us make one remark. Representation of type (2.8.7)-(2.8.8) in case (2.8.3) has been obtained in 2.5, p.118-119, [32]. But it differs from (2.8.7)-(2.8.8), and, in our opinion, there is a misprint in [32].

Let us make one more remark. Integral Representations for densities and other derivatives of Stable Laws are easy to derive by differentiating corresponding Integral Representations of Stable Laws under the sign of integral.

2.8.2 Standard Normal Law

From the obtained Integral Representations two integral forms for the Normal Standard Law \( \Phi(x) \) is possible to extract.

(a) Let \( \alpha = (1/2), \beta = 1 \). Due to (2.8.8) \( U_{1/2}(\varphi) = (\sin(\varphi/2)/\sin \varphi)^2 = \frac{1}{4 \cos^2(\varphi/2)} = \frac{1}{1 + \cos \varphi}, \) and from (2.8.7) and the last equality we obtain

\[ S(x; \frac{1}{2}, 1) = \frac{1}{\pi} \int_0^\pi \exp\left(-\frac{1}{x(1 + \cos \varphi)}\right) d\varphi. \] (2.8.14)

The following Levy’s Law

\[ 2 \cdot (1 - \Phi(1/\sqrt{x})), \ x \in R^+, \] (2.8.15)

has a density \( \frac{1}{\sqrt{2\pi x}} \exp(-\frac{1}{2x}), \ x \in R^+ \) (see, VI.2, p.170, [29]). This density is "slightly” distinguished from the standard one in terms of [32]: \( g(x; \frac{1}{2}, 1) = \frac{1}{\sqrt{2\pi} x^{-3/2}} \exp(-\frac{1}{4x}), \ x \in R^+. \)

That is why putting in (2.8.15) \( 2x \) instead of \( x \) and equating after that (2.8.14) and (2.8.15) we conclude \( 1 - \Phi(1/\sqrt{2x}) = \frac{1}{\sqrt{x}} \int_0^\pi \exp\left(-\frac{1}{x(1 + \cos \varphi)}\right) d\varphi, \) or

\[ \Phi(x) = 1 - \frac{1}{2\pi} \int_0^\pi \exp(-\frac{2x^2}{1 + \cos \varphi}) d\varphi, \ x \in R^+. \]

(b) Let \( \alpha = 2, \beta = 0 \). From (2.8.13) we have \( V_2(\varphi) = (\cos \varphi/\sin(2x^2))^2 = (\frac{1}{\sin \varphi})^2, \) and from (2.8.12) and the last equality \( \Phi(x) = S(\frac{x}{\sqrt{2}}; 2, 0) = 1 - \frac{1}{\pi} \int_0^{\pi/2} \exp(-\frac{x^2}{2\sin^2 \varphi}) d\varphi. \)

Remind that the Standard Normal Stable Law in terms of [32], i.e. \( S(x; 2, 0) \), has variance equals to 2, but the variance of \( \Phi \) is equal to 1.

2.8.3 Integral Representation for Derivatives

As we already mentioned, Integral Representations for derivatives of Stable Laws is possible to obtain by differentiation of (2.8.3). But in 2.2, p.97, [32] another, essentially distinguish form of derivatives has been obtained. The comparatively simple expression, for instance in case of \( \frac{dg(x; \alpha, 1)}{dx}, \ 0 < \alpha < 1 \), arises. Namely,

\[ \frac{dg(x; \alpha, 1)}{dx} = \frac{1}{\pi} \frac{1}{x^{2(1-\alpha)}} \int_0^\pi b(\varphi) \exp\left(-\frac{1}{x^{\alpha/(1-\alpha)} a(\varphi)}\right) d\varphi, \] (2.8.16)
where for $0 \leq \varphi \leq \pi$

\[
\begin{align*}
a(\varphi) &= \frac{(\sin(\alpha \varphi))^{\alpha/(1-\alpha)} \cdot \sin((1-\alpha)\varphi)}{\sin \varphi}, \\
b(\varphi) &= \frac{1}{1-\alpha} \cdot \left( \frac{(\sin(\alpha \varphi))^{2/(1-\alpha)}}{\sin \varphi} \right) \cdot \left( \frac{\alpha \cdot \sin((2-\alpha)\varphi)}{\sin(\alpha \varphi)} - 1 \right).
\end{align*}
\]

The mentioned form in case $\alpha \neq 1$, $\beta = 1$ for the first time has been considered by I. Ibragimov and K. Chernin [53], and implies in this particular case the unimodality’s prove of Stable Laws.

Remind that in general case the unimodality of Stable Laws non-directly as a consequence of more general distribution functions’ property has been done by M. Yamazato [40].

The unimodality’s proof in this particular case is based on Integral Representation (2.8.16) and on the following properties of functions $a(\varphi)$ and $b(\varphi)$ (see, Lemma 2.7.5, 2.7, p.159, [32]).

On interval $(0, \pi)$: a) The function $a(\varphi)$ is positive and strictly increasing; b) The function $\omega(\varphi) := \frac{\alpha \cdot \sin((2-\alpha)\varphi)}{\sin(\alpha \varphi)} - 1$ strictly decreases.

Note that there is a relationship between functions $b(\varphi)$ and $\omega(\varphi)$. Namely,

\[b(\varphi) = \frac{1}{1-\alpha} \left( \frac{\sin(\alpha \varphi)}{\sin \varphi} \right)^{2/(1-\alpha)} \cdot \omega(\varphi).\]

Let us formulate, finally, the following statement (see, Corollary 2, 2.2, p.86, [32])

\[
s(0; \alpha, \beta) = \frac{1}{\pi} \Gamma \left( 1 + \frac{1}{\alpha} \right) \cos \left( \frac{\pi}{2} \beta \cdot \frac{K(\alpha)}{\alpha} \right). \tag{2.8.17}
\]

In case (2.8.4) the formula (2.8.17) implies $s(0, \alpha, 1) = 0$, $0 < \alpha < 1$, and in case (2.8.9) we have

\[
s(0, \alpha, 0) = \frac{1}{\pi} \Gamma \left( 1 + \frac{1}{\alpha} \right), \quad 1 < \alpha < 2. \tag{2.8.18}
\]

### 2.9 Integral Representations of Continuous Analogs

Let us return to two-parametric families of continuous analogs

\[
\left\{ \hat{f}_{\alpha, \sigma}(x) = \sigma^{-1/\alpha} \cdot s(\sigma^{-1/\alpha} x; \alpha, 1) : 0 < \alpha < 1, \sigma \in \mathbb{R}^+ \right\},
\]

and

\[
\left\{ \hat{f}_{\alpha, \sigma}(x) = 2\sigma^{-1/\alpha} \cdot s(\sigma^{-1/\alpha} x; \alpha, 0) : 1 < \alpha < 2, \sigma \in \mathbb{R}^+ \right\},
\]

for desired frequency distributions. These densities are concentrated on $[0, +\infty)$.

According to results of Section 2.8, for distribution functions of continuous analogs

\[
\hat{F}_{\alpha, \sigma}(x) = \int_{0}^{x} \hat{f}_{\alpha, \sigma}(u) du, \quad x \in \mathbb{R}^+, \quad 0 < \alpha < 2, \quad \alpha \neq 1, \quad \sigma \in \mathbb{R}^+,
\]

the following Integral representations hold. Namely, for $x \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}^+$
in case $0 < \alpha < 1$

$$\hat{F}_{\alpha,\sigma}(x) = \frac{1}{\pi} \int_0^\pi \exp\left(-\left(\frac{\sigma}{x/\alpha}\right)^{1/(1-\alpha)} \cdot \tilde{U}_\alpha(\varphi)\right) d\varphi,$$  \hspace{1cm} (2.9.1)

where $\tilde{U}_\alpha(\varphi)$ is given by formula (2.8.8), and

in case $1 < \alpha < 2$

$$\hat{F}_{\alpha,\sigma}(x) = 1 - \frac{2}{\pi} \int_0^{\pi/2} \exp\left(-\left(\frac{x/\alpha}{\sigma}\right)^{1/(\alpha-1)} \tilde{V}_\alpha(\varphi)\right) d\varphi,$$  \hspace{1cm} (2.9.2)

where $\tilde{V}_\alpha(\varphi)$ is given by formula (2.8.13).

In order to understand the impact of parameters $\sigma$ and $\alpha$ on distribution function $\hat{F}_{\alpha,\sigma}$ we need in series expansions by $\varphi$ the functions $\tilde{U}_\alpha(\varphi)$ and $\tilde{V}_\alpha(\varphi)$ under the integrals at the right-hand-sides of (2.9.1) and (2.9.2), respectively, because the forms (2.8.6) and (2.8.13) are not suitable for such analysis.

### 2.9.1 Series Expansions I

Below, in series expansions the Bernoulli numbers $B_n, n = 0, 1, 2, \ldots$, appear. The numbers $B_n, n = 0, 1, 2, \ldots$, by definition, represent coefficients of $(t^n/n!)$ in the expansion of the function $(e^t - 1)^{-1} = \sum_{n=0}^{\infty} B_n \cdot \frac{t^n}{n!}$ (see, 9.610.1, p.1076, [20]). There are several integral representations for numbers $B_n, n = 2K, K = 1, 2, \ldots$, and relations with different Special Functions in Theory of Special Functions. For instance, due to 9.611.1, p.1076, [20], $B_{2n} = (-1)^{n-1} \cdot 4^n \cdot \int_0^{+\infty} \frac{2^{n-1}x^n}{\exp(2\pi x) - 1} dx$, $n = 1, 2, \ldots$, which, in particular, proves that for $n = 1, 2, \ldots$

$$(-1)^{n-1} \cdot B_{2n} = |B_{2n}| \in \mathbb{R}^+.$$  \hspace{1cm} (2.9.3)

Mention also that $B_{2n+1} = 0$ for $n = 1, 2, \ldots$, and $B_0 = 1, B_1 = -(1/2)$. From (2.9.3) one may obtain

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad \ldots.$$  \hspace{1cm} (2.9.4)

Consider the case $0 < \alpha < 1$ (see, 2.5, p.119-120, [32]). It is possible to derive the series expansion for the function $\ln w_\alpha(\varphi), 0 < \alpha < 1, 0 \leq \varphi < \pi$, where

$$w_\alpha(\varphi) = \left(\frac{\sin(\alpha\varphi)}{\alpha \sin \varphi}\right)^{\alpha/(1-\alpha)} \cdot \frac{\sin((1-\alpha)\varphi)}{(1-\alpha) \sin \varphi},$$  \hspace{1cm} (2.9.5)

with the help of formula for $\ln \frac{\sin x}{x}$ from 1.518.1, p.46, [20]. Namely, for $0 < \alpha < 1$ and $0 \leq \varphi < \pi$

$$\ln w_\alpha(\varphi) = \sum_{n \geq 1} a_n(\alpha) \cdot \varphi^{2n},$$  \hspace{1cm} (2.9.6)
where
\[
a_n(\alpha) = \frac{2^{2n-1}|B_{2n}|}{n \cdot (2n)!} \left( \frac{\alpha}{1 - \alpha} (1 - \alpha^{2n}) + 1 - (1 - \alpha)^{2n} \right). \quad (2.9.7)
\]

Formulas (2.9.6)-(2.9.7) allow to find with the help of Bruno formula for derivatives of functions' superposition (see, p.42-49, J.Riordan "Introduction to Combinatorial Analysis", Moscow, Foreign Literature Press, 1963, in Russian) series expansion
\[
w_\alpha(\varphi) = \exp\left(\sum_{n \geq 1} a_n(\alpha) \cdot \varphi^{2n}\right) = 1 + \alpha \cdot \frac{\varphi^2}{2} + \sum_{n \geq 2} b_n(\alpha) \cdot \varphi^{2n}, \quad (2.9.8)
\]
where
\[
b_n(\alpha) = \sum_{i=1}^{n} \frac{1}{K_i!} (a_i(\alpha))^{K_i}, \quad n = 2, 3, \cdots. \quad (2.9.9)
\]
The summation in (2.9.9) is made over all integer solutions of the system
\[
K_1 + 2K_2 + \cdots + nK_n = n, \quad K_i \geq 0, \quad i = 1, 2, \cdots, n. \quad (2.9.10)
\]
Just the case \(n = 1\) in (2.9.10), by using \(B_2 = \frac{1}{6}\) (see, (2.9.4)), leads to term \((\alpha \frac{\varphi^2}{2})\) in (2.9.8).
Due to (2.9.5) and (2.8.8),
\[
\bar{U}_\alpha(\varphi) = (1 - \alpha) \cdot \alpha^{\alpha/(1 - \alpha)} \cdot w_\alpha(\varphi). \quad (2.9.11)
\]
Denote
\[
\xi_\alpha(x) = (1 - \alpha) \cdot \left(\frac{\alpha}{x}\right)^{\alpha/(1 - \alpha)}. \quad (2.9.12)
\]
Then, from (2.9.1), (2.9.11)-(2.9.12) for \(x \in R^+, \sigma \in R^+\) in case \(0 < \alpha < 1\) we conclude following:
\[
\hat{F}_{\alpha, \sigma}(x) = \frac{1}{\pi} \int_0^\pi \exp(-\sigma^{1/(1 - \alpha)} \cdot \xi_\alpha(x) \cdot w_\alpha(\varphi))d\varphi. \quad (2.9.13)
\]
Denote
\[
c_n(\alpha) = \frac{\alpha}{1 - \alpha} (1 - \alpha^{2n}) + 1 - (1 - \alpha)^{2n} =
\]
\[
= \alpha + \alpha^2 + \cdots + \alpha^{2n} + 1 - (1 - \alpha)^{2n} > 0 \text{ for } n = 1, 2, \cdots.
\]
We have for \(n = 1, 2, \cdots\)
\[
\frac{dc_n(\alpha)}{d\alpha} = 1 + 2\alpha + \cdots + 2n \cdot \alpha^{2n-1} + 2n \cdot (1 - \alpha)^{2n-1} > 0,
\]
\[
\frac{d^2c_n(\alpha)}{d\alpha^2} = 1 \cdot 2 + 2 \cdot 3\alpha + \cdots + (2n - 1) \cdot 2n \cdot \alpha^{2n-1} - (2n - 1) \cdot 2n (1 - \alpha)^{2n-2},
\]
so, there is \(\alpha_0 \in (0, 1)\) such that for \(n = 2, 3, \cdots\)
\[
\frac{d^2c_n(\alpha)}{d\alpha^2} < 0 \text{ for } \alpha \in (0, \alpha_0), \quad \frac{d^2c_n(\alpha)}{d\alpha^2} > 0 \text{ for } \alpha \in (\alpha_0, +\infty), \quad \frac{d^2c_n(\alpha)}{d\alpha^2}|_{\alpha = \alpha_0} = 0.
\]
Since \(a_n(\alpha) = \frac{2^{2n-1}|B_{2n}|}{n \cdot (2n)!} c_n(\alpha), \quad n = 1, 2, \cdots\), therefore it means that: \(a_n(\alpha) \in R^+, \quad a_n(\alpha)\) increases, is upward convex in \((0, \alpha_0)\) and downward convex in \((\alpha_0, +\infty)\). Note that here \(\alpha_0 = 89\)
\( \alpha_0(n) \) and \( \{ \alpha_0(n) \} \) decreases. Thus, due to (2.9.9), for \( n = 1, 2, \cdots \) the coefficient \( b_n(\alpha) \) are positive, increase, and are downward convex starting from some point \( \alpha_1 \in (0, 1) \), where \( \alpha_1 \) doesn’t depend on \( n \).

The final conclusion is possible for \( w_\alpha(\varphi) \): it is positive, increases and starting from some point is downward convex.

Since the function \( \xi_\alpha(x) \) with fixed \( x \) by \( \alpha \) is simple enough, therefore we are able to investigate the behavior of \( \xi_\alpha(x) \) by \( \alpha \), and as a result of this, investigate the behavior of \( \hat{F}_{\alpha,\sigma}(x) \) by parameter \( \alpha \). The structure of dependence of \( \hat{F}_{\alpha,\sigma}(x) \) from parameter \( \sigma \) is very simple. Since \( \xi_\alpha(x) > 0, w_\alpha(\varphi) > 0 \), therefore, due to (2.9.13): \( \hat{F}_{\alpha,\sigma}(x) \) decreases as \( \sigma \) increases, etc.

### 2.9.2 Series Expansions II

Consider the case \( 1 < \alpha < 2 \). We proceed similarly to the way being presented above in case \( 0 < \alpha < 1 \). Let us derive the series expansion for the function \( \ln w_\alpha(\varphi) \), \( 1 < \alpha < 2 \), \( 0 \leq \varphi < \pi/2 \), where

\[
\alpha_\alpha(\varphi) = (\frac{\alpha \varphi}{\sin(\alpha \varphi)})^{\alpha/(\alpha - 1)} \cdot (\cos(\varphi)) \cdot \cos((\alpha - 1) \varphi). \tag{2.9.14}
\]

By using the expansions 1.518.1 and 1.518.2, p.46, [20] for \( \ln(\sin x) \) and for \( \ln \cos x \), respectively, from (2.9.14) we come to the mentioned series expansion. Namely, for \( 1 < \alpha < 2 \) and \( 0 \leq \varphi < (\pi/2) \) we have the expansion (2.9.6) also in this case, where for \( n = 1, 2, \cdots \)

\[
a_n(\alpha) = \frac{2^{2n-1} \cdot |B_{2n}|}{n \cdot (2n)!} \cdot ((2^{2n-1} - 1) \cdot (\frac{1}{\alpha - 1} + (\alpha - 1)^2) - \frac{\alpha^{2n-1}}{\alpha - 1}). \tag{2.9.15}
\]

Formulas (2.9.6) (in this case) and (2.9.15) allow to find with the help of Bruno formula for derivatives of functions’ superposition series expansion for the function \( w_\alpha(\varphi) \) in the form

\[
w_\alpha(\varphi) = 1 + \sum_{n \geq 1} b_n(\alpha) \cdot \varphi^{2n}, \tag{2.9.16}
\]

where \( b_n(\alpha), n = 1, 2, \cdots, \) are defined by (2.9.9)-(2.9.10). In (2.9.16), in particular, due to (2.9.9)-(2.9.10) and (2.9.15), we have

\[
b_1(\alpha) = a_1(\alpha) = \frac{2 \cdot |B_2|}{2} \cdot (\frac{1}{\alpha - 1} + (\alpha - 1)^2 - \frac{\alpha}{1 - \alpha}) = \frac{1}{6} \cdot (1 + (\alpha - 1))^2,
\]

where (2.9.4) was used.

The peculiarity of this case consists in the form of sequence \( \{ a_n(\alpha) \} \): the functions \( a_n(\alpha), n = 1, 2, \cdots, \) now, are not always positive. They change (each of them) once their sign. So, the analysis in this case is more complex than in case \( 0 < \alpha < 1 \).

Finally, we may conclude that now for families of continuous analogs of desired frequency distributions we have:

1. **series expansions for densities and distribution functions**;
2. asymptotic expansions for densities and distribution functions for large values of argument;
3. Integral Representations for densities and distribution functions.
4. Laplace-Stieltjes Transforms.

2.10 Stability Problem for Continuous Analogs

In this Section for chosen families of continuous analogs Problems of stability by Parameters are formulated and discussed. We begin from the Simple Version on Continuity by Parameters based on Continuity Theorem for Laplace-Stieltjes Transform. Next, the General Stability Problem on uniform convergence by parameters is formulated and simplified.

2.10.1 The Simple Version

Below we continue the demonstration of Simple Version of Stability Problem’s study based on Continuity Theorems (see, Section 1.8). Since chosen families are formed by densities being concentrated on $[0, +\infty)$, therefore here we deal with Continuity Theorem for Laplace-Stieltjes Transform.

There are different formulations of such theorems. As a rule, they use a weak convergence notion for distribution functions. In our case corresponding distribution functions are infinite differentiable. Then, the weak convergence becomes even uniform convergence on $[0, +\infty)$.

**Definition 16.** Distribution function $F$ is defective if $F(+\infty) < 1$.

Let $\{F_n\}$ be a sequence of distribution functions concentrated on $[0, +\infty)$ and $\varphi_n(s) = \int_0^{+\infty} e^{-sx}dF_n(x)$ for $0 \leq s < +\infty$ and $n = 1, 2, \cdots$ be a Laplace-Stieltjes Transform of $F_n$.

Now, let us formulate the following (see, XIII.1, p.408, [29])

**Continuity Theorem** If

$$F_n \to F \text{ as } n \to +\infty, \quad (2.10.1)$$

where $F$ is possibly defective distribution function with Laplace-Stieltjes Transform $\varphi$, then $\varphi_n(s) \to \varphi(s)$ as $n \to +\infty$ for any $s \in \mathbb{R}^+$.

Conversely, if the sequence of Laplace-Stieltjes Transforms $\{\varphi_n(s)\}$ of distribution functions $\{F_n\}$ converges as $n \to +\infty$ for each $s \in \mathbb{R}^+$ to a limit $\varphi(s)$, then $\varphi$ is a Laplace-Stieltjes Transform for a possibly defective distribution function and (2.10.1) holds.

The limit $F$ in (2.10.1) is a proper distribution function iff $\varphi(s) \to 1$ as $s \downarrow 0$. 

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Due to (2.7.6) and (2.7.18), we have the following forms of families of continuous analogs in terms of their Laplace-Stieltjes Transforms:

\[
\rho_{\alpha,\sigma}(s) = \exp(-\sigma \cdot s^\alpha) : 0 \leq s < +\infty, 0 < \alpha < 1, \sigma \in R^+ \}
\]

and

\[
\left\{ \rho_{\alpha,\sigma}(s) = \frac{2}{\pi} \int_{0-}^{+\infty} e^{-\sigma(su)^\alpha} \frac{du}{1+u^2} : 0 \leq s < +\infty, 1 < \alpha < 2, \sigma \in R^+ \right\}.
\]

By the Continuity Theorem and (2.10.2)-(2.10.3), it follows that the families of distribution functions of continuous analogs

\[
\{ \hat{F}_{\alpha,\sigma}(x) : 0 < \alpha < 1, \sigma \in R^+ \}
\]

and

\[
\{ \hat{F}_{\alpha,\sigma}(x) : 1 < \alpha < 2, \sigma \in R^+ \}
\]

are continuous by both parameters \( \alpha \) and \( \sigma \).

This is the Simple Version of Stability Results by Parameters.

Let us make one remark.

In [32] among several Canonical Representations of four-parametric family of Stable Laws there is one, which is especially constructed by corresponding parametrization choice in such a way that the family of Stable Laws becomes continuous by all four parameters. This fact easily comes out from the form of mentioned Canonical Representation and from the Continuity Theorem for characteristic functions.

2.10.2 The General Version

Denote

\[
\delta(\alpha, \sigma; \alpha', \sigma') = \delta(\alpha', \sigma'; \alpha, \sigma) = \sup_{0 \leq x \leq +\infty} |\hat{F}_{\alpha,\sigma}(x) - \hat{F}_{\alpha',\sigma'}(x)|,
\]

where \( \sigma, \sigma' \in R^+ \) and either \( \alpha \in (0, 1), \alpha' \in (0, 1) \), or \( \alpha \in (1, 2), \alpha' \in (1, 2) \).

Let the constants \( \sigma \) and \( \sigma' \) be fixed and satisfy inequalities \( 0 < \sigma \leq \sigma' < +\infty \).

The constants \( \alpha \) and \( \alpha' \) are fixed too and for the family of distribution functions (2.10.4) defined by (2.10.2) satisfy inequality \( 0 < \alpha \leq \alpha' < 1 \). For the family of distribution functions (2.10.5) defined by (2.10.3) these constants satisfy inequality \( 1 < \alpha \leq \alpha' < 2 \).

The general Stability Problem for families (2.10.4)-(2.10.5) consists in discovering the conditions on parameters under which the functional \( \delta(\alpha, \sigma; \alpha', \sigma') \) converges uniformly by parameters to zero. It comes that no additional conditions are needed.
Theorem 2.2. Uniformly on \(\alpha \in [\alpha, \overline{\alpha}], \alpha' \in [\alpha, \overline{\alpha}], \sigma \in [\sigma, \overline{\sigma}], \sigma' \in [\sigma, \overline{\sigma}]\) the limit exists
\[
\lim_{|\alpha - \alpha' + |\sigma - \sigma'|| \to 0} \delta(\alpha, \sigma : \alpha', \sigma') = 0. \tag{2.10.7}
\]

The formulation of Theorem 2.2 is quite similar to formulation of Theorems 1.3 and 1.4 for Pareto and Waring Distributions, respectively. Their distinction consists only in forms of expressions of functional \(\delta\), which in each concrete case requires specific estimations.

Denote
\[
\delta_1(\alpha; \sigma, \sigma') = \sup_{0 \leq x \leq +\infty} |\hat{F}_{\alpha, \sigma}(x) - \hat{F}_{\alpha, \sigma'}(x)| \tag{2.10.8}
\]

\[
\delta_2(\sigma; \alpha, \alpha') = \sup_{0 \leq x \leq +\infty} |\hat{F}_{\sigma, \alpha}(x) - \hat{F}_{\sigma, \alpha'}(x)| \tag{2.10.9}
\]

According to the obvious inequality, which follows from (2.10.6) and (2.10.8)-(2.10.9), \(0 \leq \delta(\alpha, \sigma; \alpha', \sigma') \leq \delta_1(\alpha; \sigma, \sigma') + \delta_2(\sigma; \alpha, \alpha')\), we may formulate the following:

Remark 2.1. In order to prove general Stability statement it is enough to do it in two particular cases. Namely, uniformly on \(\alpha \in [\alpha, \overline{\alpha}], \alpha' \in [\alpha, \overline{\alpha}], \sigma \in [\sigma, \overline{\sigma}], \sigma' \in [\sigma, \overline{\sigma}]\) simultaneously the limits exist
\[
\lim_{|\sigma - \sigma'| \to 0} \delta_1(\alpha; \sigma, \sigma') = 0, \tag{2.10.10}
\]

and
\[
\lim_{|\alpha - \alpha'| \to 0} \delta_2(\sigma; \alpha, \alpha') = 0. \tag{2.10.11}
\]

2.10.3 Preliminary Estimations I

Lemma 2.1. Given \(\varepsilon \in (0, 1)\) in conditions of Theorem 2.2 for distribution functions of continuous analogs for there is a number \(x_0 \in R^+\) (\(x_0\) doesn’t depend on \(\alpha\) and \(\sigma\)) such that for all \(x \in [x_0, +\infty)\)
\[
1 - \hat{F}_{\alpha, \sigma}(x) < \frac{\varepsilon}{16}. \tag{2.10.12}
\]

Proof. For diversity the proof in cases \(0 < \alpha < 1\) and \(1 < \alpha < 2\) shall be done by different methods. Consider the case \(0 < \alpha < 1\). We deal with the following series expansion (see, Theorem 2.4.2, 2.4, p.108-109, [32] and (2.7.5))
\[
1 - \hat{F}_{\alpha, \sigma}(x) = \frac{1}{\pi \alpha} \sum_{n \geq 1} (-1)^{n-1} \cdot \frac{\Gamma(n\alpha + 1)}{n \cdot n!} \sin(\pi n\alpha) \cdot \frac{1}{x^{n\alpha}}, \tag{2.10.13}
\]

which may be obtained by integration term by term inside the sum in formula (2.5.3). Here \(\Gamma(x)\) is the Euler’s Gamma Function, for which, in particular, the following asymptotic formula holds (see, 8.327, p.937, [29])
\[
\Gamma(x) \sim x^{-(1/2)} \cdot e^{-x} \cdot \sqrt{2\pi}, \quad x \to +\infty. \tag{2.10.14}
\]
By (2.10.14), \( \frac{\Gamma(n\alpha+1)}{n!} \sim \pi^{1/2} (\pi\alpha)^{n(1-\alpha)} \cdot \alpha^n, n \to +\infty, \) which proves the convergence of series \( \frac{1}{\alpha} \sum_{n \geq 1} \Gamma(n\alpha+1)/(n! \alpha^n). \) That is why given \( \varepsilon \in (0, 1) \) there is an integer \( n_0 > 1 \) such that

\[
\frac{1}{\alpha} \sum_{n \geq n_0} \Gamma(n\alpha+1)/(n! \alpha^n) < \frac{\varepsilon}{16}.
\] (2.10.15)

But for any \( x \in (\max(1, (\sigma)^{1/\alpha}), +\infty) \) from (2.10.13) we have

\[
0 \leq 1 - \hat{F}_{\alpha,\sigma}(x) \leq \frac{1}{\alpha} \sum_{n \geq 1} \Gamma(n\alpha+1)/(n! \alpha^n) < \frac{1}{\alpha} \sum_{n \geq n_0} \Gamma(n\alpha+1)/(n! \alpha^n),
\]

which, due to (2.10.15), proves Lemma 2.1 in this case.

Consider the case \( 1 < \alpha < 2. \) We deal with Integral Representation (2.9.2), i.e.

\[
1 - \hat{F}_{\alpha,\sigma}(x) = \frac{2}{\alpha} \int_0^{\pi/2} \exp(-x^n/\sigma^{1/(\alpha-1)}) \cdot V_{\alpha}(\phi) d\phi,
\] (2.10.16)

where, due to (2.8.13), the function

\[
V_{\alpha}(\phi) = \left( \frac{\cos \phi}{\sin(\alpha \phi)} \right)^{\alpha/(\alpha-1)} \cdot \frac{\cos((\alpha-1)\phi)}{\cos \phi}
\] (2.10.17)

by \( \varphi \) in \( (0, \pi/2) \) is positive. Moreover, the function \( V_{\alpha}(\phi) \) defined by (2.10.17) for a fixed \( \varphi \in (0, \pi/2) \) decreases as \( \alpha \) increases.

Indeed, the functions \( \frac{1}{\sin(\alpha \phi)}, \cos((\alpha-1)\phi), \frac{\alpha}{\alpha-1} = \frac{1}{1-(1/\alpha)} \) decrease as \( \alpha \) increases. As a result of this, the function \( \left( \frac{1}{\sin(\alpha \phi)} \right)^{\alpha/(\alpha-1)} \) decreases as \( \alpha \) increases because the last function is more than one for \( 1 < \alpha < 2 \) and \( \varphi \in (0, \pi/2) \), which proves the statement.

Since the function \( V_{\alpha}(\varphi) \) is positive and decreases as \( \alpha \) increases, therefore from (2.10.16) for \( x \in (1, +\infty) \) we obtain

\[
0 \leq 1 - \hat{F}_{\alpha,\sigma}(x) \leq \frac{2}{\alpha} \int_0^{\pi/2} \exp(-x^n/\pi^{1/(\alpha-1)} \cdot \pi V_{\pi}(\phi)) d\phi \leq
\]

\[
\leq \begin{cases} 
\frac{2}{\alpha} \int_0^{\pi/2} \exp(-x/\pi^{1/(\alpha-1)} \cdot \pi V_{\pi}(\phi)) d\phi & \text{if } \pi \in (0, 1), \\
\frac{2}{\alpha} \int_0^{\pi/2} \exp(-x/\pi^{1/(\alpha-1)} \cdot \pi V_{\pi}(\phi)) d\phi & \text{if } \pi \in (1, +\infty).
\end{cases}
\]

In case \( \sigma \in (0, 1) \) solve the equation \((\hat{\sigma})^{1/(\alpha-1)} = (\sigma)^{1/(\alpha-1)}\) with unknown \( \hat{\sigma} \in R^+; \) i.e. \( \hat{\sigma} = (\sigma)^{(\alpha-1)/(\alpha-1)} \), and put \( \sigma_* = \hat{\sigma} \) if \( \sigma \in (0, 1) \) and \( \sigma_* = \sigma \) if \( \sigma \in (1, +\infty) \) with \( \sigma_* \in R^+ \).

Thus, for \( x \in (1, +\infty) \) we come to the equality

\[
1 - \hat{F}_{\alpha,\sigma}(x) \leq \frac{2}{\alpha} \int_0^{\pi/2} \exp(-x/\sigma_*^{1/(\alpha-1)} \cdot \pi V_{\pi}(\phi)) d\phi,
\]

or, due to (2.10.16),

\[
1 - \hat{F}_{\alpha,\sigma}(x) \leq 1 - \hat{F}_{\pi,\sigma_*}(x) \text{ for } x \in (1, +\infty).
\] (2.10.18)
Since the function \( 1 - \hat{F}_{\pi, \sigma}(x) \) varies regularly at infinity with exponent \(-\pi\), therefore for a given \( \varepsilon \in (0, 1) \) there is a number \( x_0 \in \mathbb{R}^+ \) such that \( 1 - \hat{F}_{\pi, \sigma}(x) < \frac{\varepsilon}{16} \) for all \( x \in (x_0, +\infty) \), which together with (2.10.18) under additional assumption \( x \in (1, +\infty) \) implies (2.10.12) in this case. Lemma 2.1 is proved.

### 2.10.4 Preliminary Estimations II

Denote

\[
\gamma(\alpha, \sigma; \alpha', \sigma') = |f_{\alpha, \sigma}(0) - f_{\alpha', \sigma'}(0)|,
\]

where \( f_{\alpha, \sigma} \) is a density of \( \hat{F}_{\alpha, \sigma} \). According to (2.8.18) in case \( 1 < \alpha < 2 \)

\[
f_{\alpha, \sigma}(0) = \sigma^{-1/\alpha} \cdot \frac{2}{\pi} \Gamma\left(1 + \frac{1}{\alpha}\right),
\]

and in case \( 0 < \alpha < 1 \), obviously, \( f_{\alpha, \sigma}(0) = 0 \).

**Lemma 2.2.** In conditions of Theorem 2.2 \( 0 \leq \lim_{|\alpha - \alpha'| + |\sigma - \sigma'| \to 0} \gamma(\alpha, \sigma; \alpha', \sigma') < \frac{\varepsilon}{16} \) for a given \( \varepsilon \in (0, 1) \) uniformly on \( \alpha \in [\alpha, \overline{\alpha}], \alpha' \in [\alpha, \overline{\alpha}], \sigma \in [\sigma, \overline{\sigma}], \sigma' \in [\sigma, \overline{\sigma}] \). This limit relationship implies that for \( |\alpha - \alpha'| + |\sigma - \sigma'| \) small enough

\[
\gamma(\alpha, \sigma; \alpha', \sigma') \leq \frac{\varepsilon}{8}.
\]

**Proof.** The case \( 0 < \alpha < 1 \) is obvious. Consider the case \( 1 < \alpha < 2 \). Due to (2.10.20), we have

\[
\frac{1}{2} \pi \gamma(\alpha, \sigma; \alpha', \sigma') = \frac{1}{(\sigma - \sigma')^{1/\alpha}} \cdot |(\sigma')^{1/\alpha} \cdot \Gamma\left(1 + \frac{1}{\alpha}\right) - \sigma^{1/\alpha} \cdot \Gamma\left(1 + \frac{1}{\alpha}\right)| \leq \frac{1}{2} A_1 \cdot |(\sigma')^{1/\alpha} - \sigma^{1/\alpha}| + A_2 \cdot |\Gamma\left(1 + \frac{1}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha'}\right)|,
\]

where \( A_1 = (\sigma)^{-2/\pi} \cdot \Gamma\left(1 + \frac{1}{\alpha}\right), \quad A_2 = (\sigma')^{1/2} \). Here the monotony of Gamma Function was used. The estimation of the Gamma Functions’ difference has been made in 1.10.3 of Section 1.10. Thus, we conclude: uniformly on \( \alpha \in [\alpha, \overline{\alpha}], \alpha' \in [\alpha, \overline{\alpha}] \)

\[
\lim_{|\alpha - \alpha'| \to 0} |\Gamma\left(1 + \frac{1}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha'}\right)| = 0.
\]

Without loss of generality let us assume that \( \sigma' \geq \sigma \). Then,

\[
0 \leq |(\sigma')^{1/\alpha} - \sigma^{1/\alpha}| = \sigma^{1/\alpha} \cdot ((\sigma')/\sigma)^{1/\alpha} - 1) \leq (\sigma)^{1/2} \cdot ((\sigma')/\sigma)^{1/2} - 1).
\]

Thus, uniformly on \( \alpha \in [\alpha, \overline{\alpha}], \sigma \in [\sigma, \overline{\sigma}], \sigma' \in [\sigma, \overline{\sigma}] \), we have

\[
\lim_{|\sigma - \sigma'| \to 0} |(\sigma')^{1/\alpha} - \sigma^{1/\alpha}| = 0.
\]

The limit relationships (2.10.23) and (2.10.24), and the inequality (2.10.22) imply the inequalities in Lemma 2.2. Lemma 2.2 is proved.
For \( \tau \in (0, 1) \) denote
\[
I_{\tau}(\alpha, \sigma) = \left| \int_{0^-}^{\tau} (\hat{f}_{\alpha, \sigma}(0) - \hat{f}_{\alpha, \sigma}(u)) du \right|.
\] (2.10.25)

**Lemma 2.3.**

1. There is a constant \( B \in \mathbb{R}^+ \) such that uniformly on \( \alpha \in [\underline{\alpha}, \overline{\alpha}], \sigma \in [\underline{\sigma}, \overline{\sigma}] \) for all \( x \in R^+ \) the inequality holds
\[
\left| \frac{d}{dx} \hat{f}_{\alpha, \sigma}(x) \right| \leq B.
\] (2.10.26)

2. For a given \( \varepsilon \in (0, 1) \) and any \( \tau \in (0, \varepsilon/(8B)) \) uniformly on \( \alpha \in [\underline{\alpha}, \overline{\alpha}], \sigma \in [\underline{\sigma}, \overline{\sigma}] \)
\[
I_{\tau}(\alpha, \sigma) < \frac{\varepsilon}{8}.
\] (2.10.27)

**Proof.** We need in the following general fact on standard stable densities’ derivative of order \( n \) (see, 2.4, p.106, [32]): for \( x \in \mathbb{R}^+ \)
\[
\left| \frac{d^n}{dx^n} s(x; \alpha, \beta) \right| \leq \frac{1}{\pi \alpha} \Gamma(\frac{n+1}{\alpha}) \cdot (\cos(\frac{\pi}{2} K(\alpha) \cdot \beta))^{-\frac{n+1}{\alpha}},
\] (2.10.28)

where \( K(\alpha) = \alpha - 1 + \text{sign}(1 - \alpha) \).

Here \( \alpha \) and \( \beta \) are exponent and a symmetry of the standard stable density, \( \Gamma(x) \) is the Euler’s Gamma Function. Applying (2.10.28)-(2.10.29) to our cases: 1) \( 0 < \alpha < 1, \beta = 1, n = 1 \), and 2) \( 1 < \alpha < 2, \beta = 0, n = 1 \), we obtain the following inequalities.

In case 1)
\[
\left| \frac{d}{dx} \hat{f}_{\alpha, 1}(x) \right| \leq \frac{1}{\pi \alpha} \Gamma(\frac{2}{\alpha}) \cdot \frac{1}{\cos(\frac{\pi \alpha}{2} \cdot 2/\alpha)}.
\] (2.10.30)

Since \( \Gamma(x) \) is an increasing function and \( \cos(\frac{\pi \alpha}{2}) \) decreases as \( \alpha \) increases (remind that \( \alpha \in (0, 1) \)), therefore
\[
\Gamma(\frac{2}{\alpha}) \leq \Gamma(\frac{2}{\alpha}), \quad (\cos(\frac{\pi \alpha}{2}))^{2/\alpha} \leq (\cos(\frac{\pi \alpha}{2}))^{-2/\alpha}.
\] (2.10.31)

The inequalities (2.10.30) and (2.10.31) imply
\[
\left| \frac{d}{dx} \hat{f}_{\alpha, 1}(x) \right| \leq \frac{1}{\pi \alpha} \cdot \Gamma(\frac{2}{\alpha}) \cdot \frac{1}{(\cos(\frac{\pi \alpha}{2}))^{2/\alpha}}.
\]

That is why
\[
\left| \frac{d}{dx} \hat{f}_{\alpha, \sigma}(x) \right| = \sigma^{-2/\alpha} \cdot \left| \frac{d}{dy} \hat{f}_{\alpha, \sigma}(y) \right| \leq (\sigma^{-2/\alpha} \cdot \frac{1}{\pi \alpha} \cdot \Gamma(\frac{2}{\alpha}) \cdot \frac{1}{(\cos(\frac{\pi \sigma}{2}))^{2/\alpha}}
\]
where \( y = \sigma^{-1/\alpha} \cdot x \), which implies (2.10.26) in this case.

In case 2) we have \( \left| \frac{d}{dx} \hat{f}_{\alpha, 1}(x) \right| \leq \frac{2}{\pi \alpha} \cdot \Gamma(\frac{2}{\alpha}) \leq \frac{2}{\pi \alpha} \Gamma(\frac{2}{\alpha}). \) Thus, continuing by the same way as above we conclude that (2.10.26) holds also in this case.

Now, by the Mean Value Theorem, from (2.10.25) with the help of (2.10.26) we come for both considering cases to the same type inequality
\[
I_{\tau}(\alpha, \sigma) = \left| \frac{d}{dx} \hat{f}_{\alpha, \sigma}(x) \right|_{x=\theta \tau}, \tau \leq \tau \leq B, \quad \text{where} \quad \theta = \theta_{\tau} \in (0, 1). \]

Thus, from the last inequality for \( \tau \in (0, \varepsilon/(8B)) \) we obtain (2.10.27). **Lemma 2.3** is proved.
2.11 Proof of General Stability Problem

In this Section we continue the preliminary estimations, and with the help of preliminary estimations being realized in the previous Section as well as in the present one the technical proof of the general Stability Problem for families of continuous analogs is completed.

2.11.1 Continuing the Preliminary Estimations

For a given integer \( N > 1 \) consider the following series:

In case \( 0 < \alpha < 1 \)

\[
\hat{f}_{\alpha,\sigma,N}(x) = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n\alpha+1)}{n!} \frac{\sigma^n}{x^{\alpha+1}} \sin(\pi n\alpha), \quad \tau \in (0,1), x \in [\tau, 1/\tau], \alpha \in [\alpha, \overline{\alpha}], \sigma \in [\sigma, \overline{\sigma}];
\]

(2.11.1)

In case \( 1 < \alpha < 2 \)

\[
\hat{f}_{\alpha,\sigma,N}(x) = \frac{2}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(\frac{2n\alpha-1}{\alpha})}{(2n-1)!} \frac{\sigma^{2n-2}}{x^{\alpha}} \sin(\pi \frac{n\alpha}{\alpha}) \quad \tau \in (0,1), x \in [\tau, 1/\tau], \alpha \in [\alpha, \overline{\alpha}], \sigma \in [\sigma, \overline{\sigma}].
\]

(2.11.2)

It is easy to see, that (2.11.1) and (2.11.2) represent the partial sums of series expansions for densities \( \hat{f}_{\alpha,\sigma}(x), 0 < \alpha < 1 \), and \( \hat{f}_{\alpha,\sigma}(x), 1 < \alpha < 2 \), respectively. For \( \tau \in (0,1), N > 1, \alpha \in [\alpha, \overline{\alpha}], \sigma \in [\sigma, \overline{\sigma}] \) denote

\[
J_{\alpha,\sigma,N}(\tau) = \int_{\tau}^{1/\tau} \left( \hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha,\sigma,N}(x) \right) dx = \int_{\tau}^{1/\tau} \left( \sum_{n>N} (-1)^{n-1} \frac{\Gamma(n\alpha+1)}{n!} \frac{\sigma^n}{x^{\alpha+1}} \sin(\pi n\alpha) \right) dx + \int_{\tau}^{1/\tau} \left( \sum_{n>N} (-1)^{n-1} \frac{\Gamma(\frac{2n\alpha-1}{\alpha})}{(2n-1)!} \frac{\sigma^{2n-2}}{x^{\alpha}} \sin(\pi \frac{n\alpha}{\alpha}) \right) dx
\]

(2.11.3)

Lemma 2.4.

1. The integrals at the right-hand-side of (2.11.3) exist.

2. For given \( \varepsilon \in (0,1), \tau \in (0,1) \) there is an integer \( N > 1 \) (\( N \) doesn’t depend on \( \alpha \) and \( \sigma \)) such that

\[
|J_{\alpha,\sigma,N}(\tau)| < \frac{\varepsilon}{8};
\]

(2.11.4)

**Proof.** Consider the case \( 0 < \alpha < 1 \). From (2.11.3) we have

\[
|J_{\alpha,\sigma,N}(\tau)| \leq \frac{1}{\pi} \int_{\tau}^{1/\tau} \left( \sum_{n>N} \frac{\Gamma(n\alpha+1)}{n!} \frac{\sigma^n}{x^{\alpha+1}} \sin(\pi n\alpha) \right) dx \leq \frac{1}{\pi} (-\tau + \frac{1}{\tau}) \sum_{n>N} \frac{\Gamma(n\alpha+1)}{n!} \frac{\sigma^n}{x^{\alpha+1}} \cdot \tau^-(n\alpha+1).
\]

(2.11.5)

According to (2.11.5), with the help of asymptotic formula (2.10.14), for \( N \) large enough we estimate \( |J_{\alpha,\sigma,N}(\tau)| \). Namely,

\[
|J_{\alpha,\sigma,N}(\tau)| < \frac{2}{\pi \tau} (-\tau + \frac{1}{\tau}) \cdot \frac{C_{\alpha,\sigma}}{\tau^{1/2}} \sum_{n>N} \left( \frac{c_{\alpha,\sigma}}{n^{1-\alpha}} \right)^n,
\]

(2.11.6)
where \( c_\tau(\alpha, \sigma) = \exp(1 - \alpha) \cdot \sigma \cdot \tau^{-\alpha} \). At the right-hand-side of (2.11.6) we have a convergent series, which proves the statement 1 of Lemma 2.4. The statement 2 is proved too because the last series doesn’t depend on \( \alpha \) and \( \sigma \).

Consider the case \( 1 < \alpha < 2 \). From (2.11.3) we have

\[
|J_{\alpha, \sigma, N}(\tau)| \leq \frac{2}{\pi} \int_{\tau}^{1/\tau} \left( \sum_{n>N} \frac{\Gamma(\frac{2n-1}{\alpha} + 1)}{(2n-1)!} \frac{x^{2n-1}}{\sigma^{(2n-1)/\alpha}} \right) dx \\
\leq \begin{cases} 
\frac{2}{\pi}(-\tau + \frac{1}{\tau}) & \text{if } \alpha \in (1, +\infty), \\
\frac{2}{\pi}(-\tau + \frac{1}{\tau}) & \text{if } \alpha \in (0, 1).
\end{cases}
\tag{2.11.7}
\]

A similar to case \( 0 < \alpha < 1 \) estimations of series at the right-hand-side of (2.11.7), with the help of asymptotic formula (2.10.14), imply the statements 1. and 2. of Lemma 2.4 in this case. Lemma 2.4 is proved.

### 2.11.2 Simplification of Stability Problem

Let us write down the following inequality

\[
\delta(\alpha, \sigma; \alpha', \sigma') \leq \int_{0-}^{+\infty} |\hat{f}_{\alpha, \sigma}(x) - \hat{f}_{\alpha', \sigma'}(x)| dx. \tag{2.11.8}
\]

In conditions of Theorem 2.2 choose for a given \( \varepsilon \in (0, 1) \) a number \( \tau \) satisfying restrictions \( \tau \in (\varepsilon/8B), \frac{1}{2} > x_0 \), where \( B \) is defined in Lemma 2.3 and \( x_0 \) is defined in Lemma 2.1. Then, for small enough \( |\alpha - \alpha'| + |\sigma - \sigma'| \) from Lemmas 2.1 and 2.3 we have the following inequalities.

1. \[
\int_{0-}^{\tau} |\hat{f}_{\alpha, \sigma}(x) - \hat{f}_{\alpha', \sigma'}(x)| dx \leq \int_{0-}^{\tau} |\hat{f}_{\alpha, \sigma}(0) - \hat{f}_{\alpha', \sigma'}(0)| dx + \\
\int_{0-}^{\tau} |\hat{f}_{\alpha, \sigma}(x) - \hat{f}_{\alpha, \sigma}(0)| dx + \int_{0-}^{\tau} |\hat{f}_{\alpha', \sigma'}(x) - \hat{f}_{\alpha', \sigma'}(0)| dx = \\
= I_r(\alpha, \sigma) + I_r(\alpha', \sigma') + \tau \cdot \gamma(\alpha, \sigma; \alpha', \sigma') < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{3\varepsilon}{8}, \tag{2.11.9}
\]

where the monotonity of \( \hat{f}_{\alpha, \sigma} \) around the origin (point zero) and (2.10.19), (2.10.25) were used.

2. \[
\int_{1/\tau}^{+\infty} |\hat{f}_{\alpha, \sigma}(x) - \hat{f}_{\alpha', \sigma'}(x)| dx \leq \int_{1/\tau}^{+\infty} \hat{f}_{\alpha, \sigma}(x) dx + \int_{1/\tau}^{+\infty} \hat{f}_{\alpha', \sigma'}(x) dx = \\
= (1 - \hat{F}_{\alpha, \sigma}(\frac{1}{\tau})) + (1 - \hat{F}_{\alpha', \sigma'}(\frac{1}{\tau})) < \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = \frac{\varepsilon}{8}. \tag{2.11.10}
\]

In accordance with (2.11.9) and (2.11.10), for given \( \varepsilon \in (0, 1) \) and already chosen \( \tau \) from (2.11.8) we obtain the following inequality

\[
\delta(\alpha, \sigma; \alpha', \sigma') \leq \frac{\varepsilon}{2} + \int_{\tau}^{1/\tau} |\hat{f}_{\alpha, \sigma}(x) - \hat{f}_{\alpha', \sigma'}(x)| dx. \tag{2.11.11}
\]

Note that below \( \varepsilon \) and \( \tau \) are fixed.
Now, let us choose an integer $N > 1$ such that for given $\varepsilon$ and $\tau$ (2.11.4) takes place, and fix

$N$. Then, by (2.11.11) and (2.11.4), for $|\alpha - \alpha'| + |\sigma - \sigma'|$ small enough we come to the following inequalities

$$
\delta(\alpha, \sigma; \alpha', \sigma') \leq \int_\tau^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma',N}(x)|\,dx + \frac{\varepsilon}{2} + \int_\tau^{1/\tau} (\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha,\sigma,N}(x))\,dx + 
\int_\tau^{1/\tau} (\hat{f}_{\alpha',\sigma'}(x) - \hat{f}_{\alpha',\sigma',N}(x))\,dx \leq \frac{3\varepsilon}{4} + \int_\tau^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma',N}(x)|\,dx. 
$$

(2.11.12)

It is clear that if we proceed as above in particular cases $\delta_1(\alpha, \sigma; \sigma')$ and $\delta_2(\sigma'; \alpha, \alpha')$ defined by (2.10.8) and (2.10.9), then we obtain the following analogs of (2.11.12)

$$
\delta_1(\alpha; \sigma, \sigma') \leq \frac{3\varepsilon}{4} + \int_\tau^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha,\sigma',N}(x)|\,dx, 
$$

(2.11.13)

and

$$
\delta_2(\sigma; \alpha, \alpha') \leq \frac{3\varepsilon}{4} + \int_\tau^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma',N}(x)|\,dx. 
$$

(2.11.14)

In (2.11.14) we take $\sigma$ instead of $\sigma'$, which changes nothing.

Because of Remark 2.1, if we prove that for given $\varepsilon, \tau, N$ and $|\alpha - \alpha'| + |\sigma - \sigma'|$ small enough in conditions of Theorem 2.2

$$
T_N^{(1)}(\tau) = \int_\tau^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha,\sigma',N}(x)|\,dx < \frac{\varepsilon}{4}, 
$$

(2.11.15)

and

$$
T_N^{(2)}(\tau) = \int_\tau^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma,N}(x)|\,dx < \frac{\varepsilon}{4}, 
$$

(2.11.16)

then (2.11.13), (2.11.15) and (2.11.14),(2.11.16) imply (2.10.10) and (2.10.11), respectively, which due to Remark 2.1 proves Theorem 2.2. Indeed, assuming that (2.11.15) and (2.11.16) take place, we may rewrite (2.11.13) and (2.11.14) in the forms

$$
0 \leq \delta_1(\alpha; \sigma, \sigma') < \varepsilon, 
$$

(2.11.17)

and

$$
0 \leq \delta_2(\sigma; \alpha, \alpha') < \varepsilon, 
$$

(2.11.18)

respectively, for $|\alpha - \alpha'| + |\sigma - \sigma'|$ small enough, Tending $|\alpha - \alpha'| + |\sigma - \sigma'| \to 0$ from (2.11.17) and (2.11.18) we obtain $0 \leq \lim_{|\sigma - \sigma'| \to 0} \delta_1(\alpha; \sigma, \sigma') < \varepsilon$, $0 \leq \lim_{|\alpha - \alpha'| \to 0} \delta_2(\sigma; \alpha, \alpha') < \varepsilon$ because $\varepsilon$ doesn’t depend on $\alpha, \alpha', \sigma, \sigma'$. Now, tending $\varepsilon \downarrow 0$ we prove Theorem 2.2.

**2.11.3 The Proof of Inequalities (2.11.15)-(2.11.16)**

By (2.11.1), (2.11.2) and (2.11.5), (2.11.6) we have

$$
T_N^{(i)}(\tau) = \sum_{n=1}^{N} T_n^{(i)}(\tau), \quad i = 1, 2. 
$$

(2.11.19)
Here in case 0 < α < 1, for n = 1, 2, \ldots, N we have

\[ t_n^{(1)}(\tau) = \frac{\Gamma(n\alpha + 1)}{n!} \left| \sin(\pi n\alpha) \right| \cdot |\sigma^n - (\sigma')^n| \cdot \int_{\tau}^{1/\tau} \frac{dx}{x^{n\alpha + 1}} \leq \frac{\Gamma(n\alpha + 1)}{\alpha \cdot n!} \cdot \left( \frac{1}{\tau^{n\alpha}} - \frac{1}{\tau^{n(\alpha + 1)}} \right) \cdot |\sigma - \sigma'|, \]

\[ t_n^{(2)}(\tau) = \frac{\sigma^n}{n!} \int_{\tau}^{1/\tau} \left( \frac{\Gamma(n\alpha + 1)}{x^{n\alpha + 1}} - \frac{\Gamma(n\alpha' + 1)}{x^{n\alpha' + 1}} \right) \sin(\pi n\alpha) \cdot \sin(\pi n\alpha') |dx. \]

Similarly, in case 1 < α < 2, for n = 1, 2, \ldots, N we have

\[ t_n^{(1)}(\tau) = \frac{\Gamma(2n - 1)}{2n - 1}! \cdot \frac{1}{\sigma^{(2n - 1)/\alpha}} - \frac{1}{(\sigma')^{(2n - 1)/\alpha}} \cdot \int_{\tau}^{1/\tau} x^{2n - 2} dx \leq \frac{\Gamma(2n - 1)}{2n - 1}! \cdot \left( \frac{1}{\tau^{2n - 1}} - \tau^{2n - 1} \right) \cdot |(\sigma'/\sigma)^{2n - 1} - 1|, \]

where without loss of generality we assume that \( \sigma' \geq \sigma \) and \( \alpha_* = \alpha \) if \( \sigma \in (0, 1) \), \( \alpha_* = \alpha \) if \( \sigma \in (1, +\infty) \),

\[ t_n^{(2)}(\tau) = \frac{1}{(2n - 1)(2n - 1)!} \cdot \frac{1}{(2n - 1)/\alpha}, \]

\[ \left( \frac{1}{\tau^{2n - 1}} - \tau^{2n - 1} \right) \cdot |(\sigma'/\sigma)^{2n - 1} - 1| (2.11.23) \]

We have to show that given \( \varepsilon, \tau, N \) fixed for \( |\alpha - \alpha'| + |\sigma - \sigma'| \) small enough

\[ |t_n^{(i)}(\tau)| < \frac{\varepsilon}{4 \cdot N}, \quad n = 1, 2, \ldots, N, \quad i = 1, 2. \]

(2.11.24)

Due to (2.11.20) and (2.11.22), it is clear that (2.11.24) is true for \( t_n^{(1)}(\tau) \) with \( n = 1, 2, \ldots, N \) in both cases 0 < \( \alpha < 1 \) and 1 < \( \alpha < 2 \).

The estimation of the Gamma Functions’ difference has been made in 1.10.3 of Section 1.10. Taking this into account, due to (2.11.23), it is clear that (2.11.24) is true for \( t_n^{(2)}(\tau) \) with \( n = 1, 2, \ldots, N \) in case 1 < \( \alpha < 2 \).

By the same reason, due to (2.11.21), using the standard estimation method developed in [16] and also illustrated in Chapter 1, we reduce the proof of (2.11.24) for \( t_n^{(2)}(\tau) \) with \( n = 1, 2, \ldots, N \) in case 0 < \( \alpha < 1 \) to the following statement: uniformly on \( \alpha \in [\underline{\alpha}, \overline{\alpha}], \alpha' \in [\underline{\alpha}, \overline{\alpha}] \) and \( n = 1, 2, \ldots, N \)

\[ \lim_{|\alpha - \alpha'| \to 0} |\sin(n\alpha) - \sin(n\alpha')| = 0. \]

(2.11.25)

Since

\[ |\sin(n\alpha) - \sin(n\alpha')| = 2 \cdot |\sin(n\alpha + \alpha')| \cdot |\cos(n\alpha - \alpha')| \leq \leq 2 \cdot |\sin(n\alpha - \alpha')| \leq \pi \cdot |\alpha - \alpha'| \text{ for } |\alpha - \alpha'| \text{ small enough,} \]

therefore \( 0 \leq \lim_{|\alpha - \alpha'| \to 0} |\sin(n\alpha) - \sin(n\alpha')| \leq \lim_{|\alpha - \alpha'| \to 0} \pi n \cdot |\alpha - \alpha'| = 0 \), which implies (2.11.25). Thus, the Stability Problem for families of continuous analogs in its General Version is completely solved.
Chapter 3

The Stable Approximation in Biomolecular Networks

3.1 Growing Biomolecular Networks

Based on huge datasets of a large-scale biomolecular systems several common statistical facts on frequency distribution has been discovered: skewness to the right, regular variation, some convexity properties, continuity by parameters, unimodality. Many statistical frequency distributions in Macroevolution Theory are proposed which as we already saw in Chapters 1 and 2, satisfy the above statistical facts. The discovering of new frequency distributions in Macroevolution Theory is in progress. The powerful tool for obtaining new frequency distributions remains the usage of stationary distributions of standard birth-death process with various forms of coefficients which exhibit regular variation at infinity. This direction of investigations shall be presented in the second part of this monograph.

The following problem is of special interest. We may use before known empirical frequency distributions as distributions of events’ frequency occurrence number also in growing biomolecular systems not only for ”fractals” but for whole systems too. In particular, for some class of growing biomolecular networks the Power Laws may be taken as the empirical frequency distribution. Because of its self-invariance property the Power Law allows to ”stick” together the frequency distributions on ”fractals” and get the frequency distribution for the whole system. Its generalization - Pareto Distributions already doesn’t possess this property. Such a ”glueling” being done by Power Law is given in terms of the frequency distributions fractal by fractal multiplication. But we have to mention that any growing biomolecular network has a ”purely” stochastic nature. It means that we must summarize the random numbers of events occurrences over fractals which in terms of distributions is determined by the operation of convolution. This observation is true for a large class of growing biomolecular networks. That is why in Chapter 2 we pointed
at Stable Laws which possess to ”stick” together the continuous analogs of desired to be constructed frequency distributions in fractals and get the continuous analog for the whole system with the help of convolution operation. It was based on definition of Stable Laws. For instance, already the Right-side stable densities which correspond to Stable Laws with exponent \( \alpha \in (0, 1) \) are concentrated on \( R^+ \) and satisfy all known statistical facts on empirical frequency distributions. Thus, they may be chosen as the continuous analog of frequency distributions for growing biomolecular networks. Here the form of frequency distributions in fractals and in the whole system is the same.

The class of growing biomolecular networks is possible to enlarge from the point of view of usage of Stable Laws, their applicability.

This is the main topic of the present Chapter and our discussion.

3.1.1 On The Class of Networks

The self-organization conception was transferred from the Phase Transition Systems Theory to biology phenomena [1]-[3]. Such systems spontaneously self-organized themselves in fractals. The main problem here consists in description of events in the vicinity of critical points. Generalizing such critical systems, S.Kauffman [1] applied the self-organization conception to large-scale biomolecular systems, mostly to growing biomolecular networks as on of their important regularities. Here the selection process cannot avoid the order exhibited by most members of the system. This regularity can be explained as follows. Knowing the statistical properties of any part of a growing biomolecular network this regularity allows to extrapolate statistical properties to the whole system. Rising by size over time the biomolecular network repeats its behavior in fractals.

Thus, in a growing biomolecular network knowing the local frequency distributions in two successive non-intersected fractals we must be able to obtain the frequency distribution in the union of these fractals. So, we can extrapolate the frequency distribution in the whole system. The following additional regularities are valid in some systems of such type. The successive fractals may be chosen of approximately equal lengths, the independence or some kind ”weak” dependence for random numbers of the occurrence of events in the sequence of successive fractals takes place, frequency distributions on fractals are the same.

In Chapter 2 has been considered the case when based on convolution operation we come to the frequency distribution of the whole growing biomolecular network being of the same type as in each fractal. This, very restrictable requirement leads to stable densities as continuous analogs of frequency distributions for such systems, not only in fractals but for the whole system too.

In this Chapter we get rid of the mentioned restrictable requirement and conserve all
other before presented regularities of growing biomolecular networks.

In this new situation the frequency distribution on fractals may satisfy not all known statistical facts but only some most important among them. These are: to be concentrated on \([0, +\infty)\); either to vary regularly at infinity, or to have finite variance. It means that the corresponding distribution function, having only the right tail, has either a ”heavy” tail of power type, or more easy tail which tends to zero with exponential rate as argument tends to \(+\infty\). It is just the situation being described in various physical, chemical, biological phenomena as ”chaotic mass movement”, when many small similar to each other factors are combined and create the phenomena. It means that we are in a situation when the methods of the Theory of Limit Theorems of Probability works.

Let us describe this situation in mathematical terms and formulate the arising problems.

3.1.2 The Problems (see, XVII.5, [23])

Denote by \(\xi_1, \xi_2, \cdots\) the random numbers of events’ occurrences on successive fractals with ”approximately” same lengths of growing biomolecular network. These fractals have not intersected each other and the end of \(n\)-th fractal is the beginning of \((n + 1)\)-st one. For any integer \(n \geq 1\) the frequency distribution of \(\xi_n\) must satisfy at least one above mentioned some most important statistical facts. For many of such type of networks the fractals are possible to choose in such a way that the sequence of random variables \(\{\xi_n\}\) forms a sequence of identically distributed and either independent, or, at least, ”weakly” dependent in some sense random variables.

Then the General Theory of Limit Theorems of Probability allows to estimate the distribution function of random variable \(S_n = \xi_1 + \xi_2 + \cdots + \xi_n, n = 1, 2, \cdots\), of events-occurrences’ number of network with \(n\) fractals, which is a growing network when \(n\) increases, as \(n \to +\infty\). In this situation the sequence \(\{S_n\}\), being correspondingly centered and normed, as \(n \to +\infty\) has a limit. The distribution function of this limit we recommend as desired approximation for the distribution function of centered and normed \(S_n\) if \(n\) is ”large enough”.

Since the problems for independent and weakly dependent random variables do not distinguish in their formulation and the results are the same, therefore below we deal with independent, identically distributed random variables

\[
\xi_1, \xi_2, \cdots
\]  

(3.1.1)

The following problem is the most important for sums of independent, identically distributed random variables (3.1.1).
Problem 1. Let the random variables (3.1.1) have distribution function \( R \) and for any integer \( n \geq 1 \) there are centring and norming constants
\[
A_n \in \mathbb{R}^1 \text{ and } B_n \in \mathbb{R}^+,
\]
respectively, such that in \( \mathbb{R}^1 \)
\[
\lim_{n \to +\infty} P(B_n^{-1} \cdot (S_n - A_n) < x) = S(x)
\]
in sense of weak convergence of distribution functions.

Definition 17. We say that a sequence of distribution functions \( \{F_n\} \) weakly converges to some distribution function \( F \) if
\[
\lim_{n \to +\infty} F_n(x) = F(x) \text{ for any continuity point of } F.
\]
Weak convergence shall be written in the form \( F_n \Rightarrow F, n \to +\infty \), where \( \Rightarrow \) denotes the sign of weak convergence.

The Problem 1 requires to describe the class, say \( \{S\} \), of all possible limit distribution functions of Scheme (3.1.2)-(3.1.3).

The solution to the Problem 1 is surprising:
\[
\{S\} = \{S^*\}.
\] (3.1.4)

Remind that, due to our notations, \( \{S^*\} \) is a four-parametric family of infinite differentiable distribution functions - Stable Laws (see, Section 2.4). Since limit distributions in Scheme 3.1.2-3.1.3 are continuous, therefore the convergence in (3.1.3) is uniform by \( x \in \mathbb{R}^+ \).

The solution to Problem 1 generates the following

Problem 2. Excluding the case of \( S(x) = \Phi(x) \) in (3.1.3) (i.e. the exponent of Stable Law equals to 2) find conditions on
\[
R(x) = P(\xi_n < x), \; n = 1, 2, \ldots, \; x \in \mathbb{R}^+,
\]
such that \( R \) belongs to the domain of attraction of Stable Law \( S_{\alpha, \beta} \) with exponent \( \alpha \in (0, 2) \) and asymmetry \( \beta \in [-1, 1] \).

Definition 18. The distribution function (3.1.5) belongs to the domain of attraction of a Stable Law \( S \), if there exist centring and norming constant (3.1.2) such that (3.1.3) holds.

The solution to Problem 2 is given by the following statements.

A distribution function \( R \) belongs to the domain of attraction of a Stable Law \( S \) with exponent \( \alpha \in (0, 2) \) if
\( R \) satisfies
\[
\lim_{x \to +\infty} \frac{1 - R(x)}{1 - R(x) + R(-x)} = p,
\]
\[
\lim_{x \to +\infty} \frac{R(-x)}{1 - R(x) + R(-x)} = q;
\] (3.1.6)
(b) The sum of the tails $1 - R(x) + R(-x)$ varies regularly at infinity with exponent $(-\alpha)$.

From (3.1.6) we conclude that the limit exists

$$\lim_{x \to +\infty} \frac{1 - R(x) - R(-x)}{1 - R(x) + R(-x)} = p - q, := \beta \in [-1, 1]. \quad (3.1.7)$$

Now we are able to supplement our statement.

(c) The corresponding limit Stable Law has asymmetry $\beta$ given by (3.1.7).

According to properties of empirical frequency distributions, we are interested in the case of non-negative random variables (3.1.1). It means that in our case in (3.1.6) $p = 1$, $q = 0$ and, due to the statement (c), limit distribution functions in Scheme (3.1.2)-(3.1.3) may be only Right-side Stable Laws. Thus, other problems concerning the Theory of Sums of Independent, Identically Distributed Random Variables have to be formulated in this particular case. It is the contents of next Section.

The important conclusion of this Section sounds as follows.

In growing biomolecular networks, in general, in order to apply stable approximation to the distribution function of events' occurrence number in the whole system the regular variation of frequency distribution in separate taken fractal with exponent $(-\rho)$, $\rho \in (1, +\infty)$, is necessary and sufficient.

### 3.2 Limit Right-side Stable Laws

In this Section we give additional explanations around Problems 1 and 2 in case when in Scheme (3.1.2)-(3.1.3) the limit distribution function is a Right-side Stable Law. Next, we talk on the moments' convergence/divergence of distribution function $R$ which belongs to the domain of attraction of Right-side Stable Law.

Information on moments allows to describe the solution to the problem of centring and norming constants' determination in Scheme (3.1.2)-(3.1.3).

### 3.2.1 Canonical Representation

We are familiar with Canonical Representation of standard Right-side Stable Laws with exponent $\alpha \in (0, 2)$ in terms of Laplace-Stieltjes Transform (here the asymmetry $\beta = 1$), which allows with the help of location parameters (shifting parameters and scale factor) to describe three-parametric family of Right-side Stable Laws (see, Section 2.4)
Now, we are going to deal with Canonical Representations of Stable Laws in terms of characteristic functions.

Due to conditions (a) and (b) in Section 3.1, the limit Stable Law is completely characterized by three parameters \( p, q \) and \( \alpha \).

By Lemma 2, XVII, 4, [28] and Theorem 3, XVII, 5, [28], if \( S \) is stable with the characteristic function \( \psi_{\alpha,C} \), where for \( t \in \mathbb{R} \)

\[
\ln \psi_{\alpha,C}(t) = \begin{cases} 
-|t|^\alpha \cdot C^\frac{(1-\alpha)}{\alpha} \cdot (\cos \frac{\pi\alpha}{2} \mp i(p-q) \sin \frac{\pi\alpha}{2}) & \text{for } 0 < \alpha < 1, \\
|t|^\alpha \cdot C^\frac{(2-\alpha)}{1-\alpha} \cdot (\cos \frac{\pi\alpha}{2} \mp i(p-q) \sin \frac{\pi\alpha}{2}) & \text{for } 1 < \alpha < 2, \\
-t \cdot C^{\frac{\pi}{2} \pm i \log |t|} & \text{for } \alpha = 1,
\end{cases}
\]

(3.2.1)

then

\[
\begin{align*}
\lim_{x \to +\infty} x^\alpha (1 - S(x)) &= Cp \cdot \frac{2-\alpha}{\alpha}, \\
\lim_{x \to -\infty} x^\alpha \cdot S(-x) &= Cq \cdot \frac{2-\alpha}{\alpha}.
\end{align*}
\]

(3.2.2)

In (3.2.1) in symbols ”\( \pm \)” and ”\( \mp \)” the upper, or lower sign prevails to \( t > 0 \), or \( t < 0 \), and \( i = \sqrt{-1} \).

(3.2.1) is one of known in literature Canonical Representations of Stable Laws in terms of characteristic functions. The parameter \( C \) is a scale factor. The difference \( p - q \) presents assymetry \( \beta \) (see, (3.1.7)). The parameters \( p \) and \( q \) coincide with \( p \) and \( q \) in (3.1.6), respectively.

If distribution function \( R \) belongs to the domain of attraction of a Stable Law with some value of parameter \( C \), then it belongs to the domain of attraction of the same Stable Law of type (3.2.1) with any value of \( C \).

Indeed, if we replace the value \( C \) of scale factor by some value \( C' \), then in (3.1.3) instead of \( \{B_n\} \) we take a sequence \( \{B_n \cdot (\frac{C}{C'})^{1/\alpha}\} \). Then the limit \( S(x) \) is replaced by \( S((\frac{C}{C'})^{1/\alpha} \cdot x) \) and, due to (3.2.1),

\[
\int_{-\infty}^{\infty} e^{itx} dS((\frac{C}{C'})^{1/\alpha} x) = \int_{-\infty}^{\infty} \exp(i \cdot ((\frac{C'}{C})^{1/\alpha} \cdot t \cdot y) dS(y) = \psi_{\alpha,C}((\frac{C'}{C})^{1/\alpha} \cdot t) = \psi_{\alpha,C'}(t),
\]

where \( p \) and \( q \) in (3.2.1) stay unchanged. If we denote \( S'(x) = S((\frac{C}{C'})^{1/\alpha} \cdot x) \), then

\[
\psi_{\alpha,C'}(t) = \int_{-\infty}^{\infty} e^{itx} dS'(x).
\]

Thus, for the same \( R \) we get in Scheme (3.1.2)-(3.1.3) two limits \( S(x) \) and \( S'(x) \).

The statement is proved.

Let us make one remark.

The Stable Law \( S \) belongs to the domain of attraction of any Stable Law with the same exponent and assymetry as \( S \) has.
Indeed, first of all, comparing (3.1.6) with \( R = S \) and (3.2.2), we conclude that the Stable Law \( S \) belongs to the domain of its attraction. Now, due to the previous statement, it belongs to the domain of attraction of Stable Law with the same exponent and any scale factor. The coincideness of asymmetry for both Stable Laws, i.e. \( p - q \), is obvious. The statement is proved.

The last statement implies that the Stable Law mentioned in solution to Problem 2 in form of conditions (a) and (b) is defined by (3.2.1)-(3.2.2).

3.2.2 Particular Case

Now, applying above said to the distribution function \( R(x) \) concentrated on \([0, +\infty)\) of an empirical frequency distribution we come to the following

Corollary 3.1 1. A distribution function \( R \) concentrated on \([0, +\infty)\) may belong to the domain of attraction only of the Right-side Stable Law.

2. A distribution function \( R \) concentrated on \([0, +\infty)\) belongs to the domain of attraction of Right-side Stable Law with exponent \( \alpha \in (0, 2) \) and characteristic function \( \psi_{\alpha, C}^+ \), where

\[
\ln \psi_{\alpha, C}^+(t) = \begin{cases} 
-|t|^\alpha \cdot C^{\Gamma(1-\alpha)} \exp(\mp \frac{\pi \alpha}{2}) & \text{for } 0 < \alpha < 1, \\
-|t|^\alpha \cdot C^{\Gamma(2-\alpha)} \exp(\mp \frac{\pi \alpha}{2}) & \text{for } 1 < \alpha < 2, \\
-t \cdot C\left(\frac{\pi}{2} \pm i \log |t|\right) & \text{for } \alpha = 1. 
\end{cases} 
\tag{3.2.3}
\]

iff \( 1 - R \) varies regularly at infinity with exponent \( -\alpha \).

Thus, with the additional restriction in Problem 1, namely, \( \{\xi_n\} \) is a sequence of non-negative random variables the family of limiting distribution, say \( S_+ \), coincides with the family of Right-side Stable Laws with two independent parameters: exponent \( \alpha \) and scale factor \( C \). Here asymmetry equals to one, and shifting parameter is fixed.

We may also fix scale factor \( C \) by putting \( C = 1 \) because if \( R \) belongs to the domain of attraction of Right-side Stable Law with exponent \( \alpha \) and scale factor \( C \) given by characteristic function of form (3.2.3), then the same is true for Right-side Stable Law with exponent \( \alpha \) and scale factor 1 in (3.2.3).

Thus, the approximation of \( R \) may be done with the help of only standard Right-side Stable Laws with characteristic functions \( \psi_{\alpha}^+(t) = \psi_{\alpha, 1}^+(t) \).

From this point we assume that \( \{\xi_n\} \) is a sequence of non-negative, independent, identically distributed random variables with common distribution function \( R_{\alpha}(x) \), \( x \in [0, +\infty) \), satisfying following condition

\[
1 - R_{\alpha}(x) = x^{-\alpha} L(x), \ x \in R^+, \ 0 < \alpha < 2, \tag{3.2.4}
\]

where \( L(x) \) varies slowly at infinity.

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Also we have to make one remark.

We say that Stable Law with characteristic function $\psi_{\alpha,i}(t)$ (see, (3.2.3)) is a standard Right-side Stable Law, say $\tilde{S}_\alpha$. For this distribution function, due to (3.2.2),

$$\lim_{x \to +\infty} x^\alpha \cdot (1 - \tilde{S}_\alpha(x)) = \frac{2 - \alpha}{\alpha}, \quad 0 < \alpha < 2.$$  \hspace{1cm} (3.2.5)

It is a result of parametrization being used in (3.2.3) and contradicts to definition of standard Right-side Stable Law with exponent $\alpha$ introduced in Section 2.4, say $S_\alpha$, which is a result of another parametrization.

Remind that, for instance, for $0 < \alpha < 1$

$$\lim_{x \to +\infty} x^\alpha(1 - S_\alpha(x)) = \frac{1}{\Gamma(1 - \alpha)}. \hspace{1cm} (3.2.6)$$

In this case easily verify that

$$\int_{-\infty}^{\infty} e^{itx} dS_\alpha(x) = \psi_{1,C'}(t), \quad t \in \mathbb{R}^1,$$  \hspace{1cm} (3.2.7)

where

$$C' = \left(\frac{2 - \alpha}{\alpha} \Gamma(1 - \alpha)\right)^{1/\alpha}. \hspace{1cm} (3.2.8)$$

Next, we need some information on moments $m_{\alpha,\nu} = \int_{-\infty}^{\infty} x^\nu dR_\alpha(x)$, $0 < \alpha < 2$, $\nu \in \mathbb{R}^+$, of order $\nu$ for regularly varying at infinity with exponent $(-\alpha)$ distribution function $R_\alpha(x) = P(\xi_n < x)$, $n = 1, 2, \cdots, x \in \mathbb{R}^+$.

In the Theory of Regularly Varying Functions it is well-known that

$$m_{\alpha,\nu} \begin{cases} < +\infty & \text{if } 0 < \nu < \alpha, \\ = +\infty & \text{if } \nu > \alpha. \end{cases} \hspace{1cm} (3.2.9)$$

In particular, for $1 < \alpha < 2$ (3.2.9) implies

$$0 < E\xi_1 < +\infty. \hspace{1cm} (3.2.10)$$

Now, we are ready to formulate the next problem and solution to it.

### 3.2.3 Centring and Norming Constants

**Problem 3** Let for the distribution function $R_\alpha(x) = P(\xi_n < x)$, $x \in [0, +\infty)$, $n = 1, 2, \cdots$, the equality (3.2.4) holds, i.e. $R_\alpha(x)$ belongs to the domain of attraction of standard Right-side Stable Law $\tilde{S}_\alpha(x)$ satisfying (3.2.3) with $C = 1$.

Find centring and norming constants $A_n$ and $B_n$, $n = 1, 2, \cdots$, respectively in Scheme (3.1.2)-(3.1.3).
In order to avoid technical difficulties in future we omit the case $\alpha = 1$.

Theory gives in our particular case following solution to Problem 3 (see, XVII.5, [23]). For $x \in R^+$ let us denote

$$\mu(x) = \int_{0}^{x} y^2 dR_\alpha(y).$$

(3.2.11)

The Solution:

1. The sequence $\{B_n\}$ satisfies the limit relation

$$\lim_{n \to +\infty} \frac{n \cdot \mu(B_n)}{B_n^2} = 1,$$

(3.2.12)

2. $A_n = \begin{cases} 0 & \text{for } 0 < \alpha < 1, \\ nE_1 & \text{for } 1 < \alpha < 2. \end{cases}$

(3.2.13)

Let us transform relation (3.2.12). If (3.2.4) holds, then (see, XVII, 5, [23])

$$\lim_{x \to +\infty} \frac{x^2(1 - R_\alpha(x))}{\mu(x)} = \frac{2 - \alpha}{\alpha},$$

or

$$\lim_{x \to +\infty} \frac{x^{2-\alpha} \cdot L(x)}{\mu(x)} = \frac{2 - \alpha}{\alpha}.$$  

(3.2.14)

Substituting $B_n$ into (3.2.14) instead of $x$ ($B_n \to +\infty$ as $n \to +\infty$) and applying (3.2.12) we get the following limit condition $\lim_{n \to +\infty} \frac{n^{-1/\alpha} \cdot B_n}{(L(B_n))^{1/\alpha}} = \left(\frac{\alpha}{2 - \alpha}\right)^{1/\alpha}$, or

$$\lim_{n \to +\infty} \frac{B_n}{n \cdot (L(B_n))^{1/\alpha}} = \left(\frac{\alpha}{2 - \alpha}\right)^{1/\alpha}.$$  

(3.2.15)

### 3.3 Domain of Normal Attraction

The notion of the domain of normal attraction of Stable Law has been introduced by B. Gnedenko. The delimitation of this domain originally poses a serious problem but after general consideration on this problem being presented in XVII, [23] it becomes a simple consequence of general results of Theory of Stable Laws.

We have to distinguish the domain of normal attraction of Stable Law and the domain of attraction of Normal Law.

In this Section necessary information on the domain of normal attraction of standard Right-side Stable Laws is given.
3.3.1 Introductory Words

The most interesting case for us appears when for the distribution function $R_\alpha(x)$, $\alpha \in (0, 2]$, $x \in [0, +\infty)$, of empirical frequency distribution its only tail (right tail) $1 - R_\alpha(x)$ in representation (3.2.4) exhibits constant slowly varying component. It means that for $L(x)$ in (3.2.4) the limit exists

$$L = \lim_{x \to +\infty} L(x) \in R^+.$$  

(3.3.1)

Let us return to Problem 3 for distribution function $R_\alpha$, because, due to (3.2.4) and conditions (a), (b) of Section 3.1, $R_\alpha$ belongs to the domain of attraction of Right-side Stable Laws with exponent $\alpha$. Then $R_\alpha$ belongs to the domain of attraction of standard Right-side Stable Law $S_\alpha$ with characteristic function $\psi_{\alpha,1}^+(t)$ given by (3.2.3).

According to solution to Problem 3 given by conditions (3.2.12)-(3.2.13), we have

$$B_n \approx \left(\frac{\alpha \cdot L}{2 - \alpha}\right)^{1/\alpha} \cdot n^{1/\alpha}, \ n \to +\infty.$$  

(3.3.2)

Indeed, here instead of (3.2.12) we deal with equivalent to (3.2.12) limit relation (3.2.15). Due to (3.3.1) and $B_n \to +\infty$ as $n \to +\infty$, we have $\lim_{n \to +\infty} L(B_n) = L$. That is why, $\frac{B_n}{(nL(B_n))^{1/\alpha}} \approx \frac{B_n}{(nL)^{1/\alpha}}$, $n \to +\infty$ and (3.2.15) is transformed into (3.3.2).

Now, in this case, due to solution to Problem 3 (see, also (3.2.4), (3.3.1) and (3.3.2)), we may formulate the following

**Limit Theorem 1** If for $0 < \alpha < 2$, $\alpha \neq 1$, for distribution function $R_\alpha(x) = P(\xi_n < x)$, $x \in [0, +\infty)$, $n = 1, 2, \cdots$ concentrated on $[0, +\infty)$

$$1 - R_\alpha(x) \approx L \cdot x^{-\alpha}, \ x \to +\infty, \ L \in R^+,$$  

(3.3.3)

then uniformly on $x \in R^+$ the limit exists

$$\lim_{n \to +\infty} P\left(\frac{S_n - A_n}{B_n} < x\right) = \tilde{S}_\alpha(x).$$  

(3.3.4)

Here the constants $A_n$, $n = 1, 2, \cdots$, are given by (3.2.13) and

$$B_n = \left(\frac{\alpha L}{1 - \alpha}\right)^{1/\alpha} \cdot n^{1/\alpha}.$$  

(3.3.5)

(Compare to (3.3.2)). The characteristic function $\psi_{\alpha,1}^+(t) = \int_{-\infty}^{+\infty} e^{itx} d\tilde{S}_\alpha(x)$, $t \in R^+$, is given by Canonical Representation (3.2.3).

The strong formal mathematical separation of such an important situation considered above shall be done in 3.3.3 of this Section.
3.3.2 Back to Canonical Representations

If we replace the sequence \( \{B_n\} \) of form (3.3.5) by the sequence \( \{B'_n\} \) in Limit Theorem 1, where

\[
B'_n = (\alpha \cdot L)^{1/\alpha} \cdot n^{1/\alpha}, \quad n = 1, 2, \ldots, \tag{3.3.6}
\]

then in (3.3.4) we come to the limit distribution function

\[
\tilde{S}_\alpha((2 - \alpha)^{-1/\alpha} \cdot x), \quad x \in \mathbb{R}^1. \tag{3.3.7}
\]

For \( t \in \mathbb{R}^1 \) we have \( \int e^{itx}d\tilde{S}_\alpha((2 - \alpha)^{1/\alpha} \cdot x) = \psi_{\alpha, 1}((2 - \alpha)^{1/\alpha} \cdot t), \quad t \in \mathbb{R}^1 \), where, due to (3.3.3), and the equality \( \Gamma(y + 1) = y \cdot \Gamma(y) \), we obtain

\[
\psi_{\alpha, 1}((2 - \alpha)^{1/\alpha} \cdot t) = \begin{cases} 
-|t|^\alpha \frac{\Gamma(3 - \alpha)}{\Gamma(1 - \alpha)} \exp(\mp \frac{\pi t}{2}) & \text{for } 0 < \alpha < 1, \\
-|t|^\alpha \frac{\Gamma(3 - \alpha)}{\alpha - 1} \exp(\mp \frac{\pi t}{2}) & \text{for } 1 < \alpha < 2.
\end{cases} \tag{3.3.8}
\]

It is of interest to mention that the Limit Theorem (3.3.4) with the sequence \( \{B'_n\} \) instead of \( \{B_n\} \) has been proved with the help of direct computations by Method of Characteristic Functions. But for the limit distribution function (3.3.7) another Canonical Representation has been obtained. Namely, for our particular case, due to result from VIII, 9, [54],

\[
\psi_{\alpha, 1}((2 - \alpha)^{1/\alpha} \cdot t) = \int_0^{+\infty} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) x^{-\alpha - 1} dx. \tag{3.3.9}
\]

Thus, comparing (3.3.3) and (3.3.9) we get the following equality for \( t \in \mathbb{R}^1 \)

\[
\int_0^{+\infty} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) x^{-\alpha - 1} dx = \begin{cases} 
-|t|^\alpha \frac{\Gamma(3 - \alpha)}{\Gamma(1 - \alpha)} \exp(\mp \frac{\pi t}{2}) & \text{for } 0 < \alpha < 1, \\
-|t|^\alpha \frac{\Gamma(3 - \alpha)}{\alpha - 1} \exp(\mp \frac{\pi t}{2}) & \text{for } 1 < \alpha < 2.
\end{cases} \tag{3.3.10}
\]

In (3.3.8) and (3.3.10) the upper sign “−” (the lower sign “+”) prevails to \( t > 0 \) (to \( t < 0 \)).

From Theorem 3.1, p. 43, [34] we may describe the form of two-side Laplace-Stieltjes Transform of distribution function (3.3.7) taking into account (3.3.9).

\[
\int_{-\infty}^{+\infty} e^{-sx} d\tilde{S}_\alpha((2 - \alpha)^{1/\alpha} \cdot x) = \exp(-D \cdot \frac{s}{\alpha} - \Gamma(1 - \alpha) \cdot \frac{s^\alpha}{\alpha}), \quad 0 \leq s < +\infty, \tag{3.3.11}
\]

where

\[
A = \text{sign}(1 - \alpha) \cdot \int_0^{\infty} \frac{x^\alpha}{1 + x^2} dx, \quad 0 < \alpha < 2, \quad \alpha \neq 1. \tag{3.3.12}
\]

This consideration confirms once more the basement of misunderstanding arising very often in the Theory of Stable Laws on example of more simple situation in present Section. It is related to various Canonical representations of Stable Laws’ characteristic functions.
3.3.3 Domain of Normal Attraction

**Definition 19** A distribution function \( R_\alpha \) is said to belong to the domain of *normal attraction* of a *Stable Law* with *exponent* \( \alpha \in (0, 2] \) if it belongs to the domain of attraction of the *same Stable Law* with *norming constants* \( B_n = c \cdot n^{1/\alpha}, \) \( n = 1, 2, \ldots, \) \( c \in \mathbb{R}^+ \).

Due to consideration above in our case the following conclusion may be made.

*A distribution function \( R_\alpha \) concentrated on \([0, +\infty), \alpha \in (0, 2], \) belongs to the domain of normal attraction of standard Right − side Stable Law with exponent \( \alpha \) iff (3.3.3) holds.*

**Examples**

(a) If distribution function \( R \) has *finite* first two moments \( a := E\xi_1 < +\infty, \) \( E\xi_1^2 < +\infty, \) where \( R(x) = P(\xi_n < x), \) \( x \in \mathbb{R}^+, \) \( n = 1, 2, \ldots, \) and \( R \) is concentrated on \([0, +\infty), \) then the *Central Limit Theorem* for sequence of independent, identically distributed random variables \( \xi_1, \xi_2, \ldots \) takes place. Namely, *uniformly* on \( x \in \mathbb{R}^1 \) the limit exists (see, for instance, [29]).

\[
\lim_{n \to +\infty} P\left( \frac{S_n - n \cdot a}{\sigma \sqrt{n}} < x \right) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du,
\]

where \( \sigma = D\xi_1 = E\xi_1^2 - a^2. \)

It is clear that, due to *Central Limit Theorem*, if \( R \) has *finite* mean value and *finite* variance, then \( R \) belongs to the domain of normal attraction of standard (in usual sense) *Normal Law*.

It is important to know that *Normal Law*, in particular, the standard one may have domain of non-normal attraction in the sense of *Definition 19*.

(b) Due to *Definition 10* (Section 2.3, (2.3.3)) and (2.3.22) any Right-side Stable Law with *exponent* \( \alpha \in (0, 2], \) belongs to the domain of normal attraction of standard Right-side Stable Law with exponent \( \alpha \).

Let us formulate the corresponding result in case \( \alpha = 2 (\tilde{S}_\alpha = \Phi). \)

*A distribution function \( R \) being not concentrated at one point belongs to the domain of attraction of the Normal Law iff the function \( \mu(x) = \int_{0}^{x} u^2 dR(x) \) varies slowly at infinity. Here distribution function \( R \) is concentrated on \([0, +\infty). \)*
If $R$ has finite variance, then, obviously, (3.3.15) at infinity equals to finite constant, so, varies slowly at infinity. then we are in conditions of example (a) and $R$ belongs to the domain of normal attraction of $\Phi(x)$.

But if, for instance, $\mu(x) \approx \ln x$, $x \to +\infty$, then the variance of $R$ is infinite, $\mu(x)$ varies slowly at infinity. In this case $R$ doesn’t belong to the domain of normal attraction of $\Phi(x)$.

### 3.3.4 The Case $\alpha = 1$

Building two families of continuous analogs with parameters $\alpha \in (0, 1)$, $\sigma \in R^+$ and $\alpha \in (1, 2)$, $\sigma \in R^+$ for desired frequency distribution we omitted the case $\alpha = 1$. The reason is: in case $\alpha = 1$ the construction of one-parametric family of continuous analogs may be easily done with the help of density $\frac{1}{\pi \sigma} \cdot \frac{1}{1 + \sigma^2}$, $x \in R^1$, of Cauchy’s Law, which is symmetric. The continuous analogs of desired frequency distributions in case $\alpha = 1$ form one-parametric family \[
\{ \hat{f}_{1,\sigma}(x) = \sigma^{-1} \cdot \hat{f}_{1,1}(\sigma^{-1} \cdot x) : \sigma \in R^+ \} \]
of densities concentrated on $[0, +\infty)$, where
\[
\hat{f}_{1,1}(x) = \frac{2}{\pi^2} \cdot \frac{1}{1 + x^2}, \quad x \in [0, +\infty). \tag{3.3.16}
\]

The standard Right-side Stable Law with exponent $\alpha = 1$ may serve an approximation for frequency distribution in whole growing biomolecular network when frequency distribution in fractals varies regularly with exponent $(-1)$.

Below a substantiation for this case is done.

The Solution to Problem 3.

Let $R_1(x) = P(\xi_n < x)$, $x \in [0, +\infty)$, $n = 1, 2, \cdots$,  
\[
\mu(x) = \int_0^x y^2 dR_1(y). \tag{3.3.17}
\]

Then (see, XVIII, 5, [23]):

1. The sequence \(\{B_n\}\) in Scheme (3.1.2)-(3.1.3) satisfies (3.2.12), or due to (3.2.15),  
\[
\lim_{n \to +\infty} \frac{B_n}{n \cdot L(B_n)} = 1, \tag{3.3.18}
\]
where $L(x) \approx \frac{\mu(x)}{x}$, $x \to +\infty$.

2. The sequence \(\{A_n\}\) is formed with the help of $R_1$ and \(\{B_n\}\) as follows  
\[
A_n = \int_{0^-}^{+\infty} \sin(\frac{x}{B_n}) dR_1(x), \quad n = 1, 2, \cdots. \tag{3.3.19}
\]

Limit Theorem 2 If for distribution function $R_1$ concentrated on $[0, +\infty)$ (see, (3.3.17)) \(1 - R_1(x) \approx L \cdot x^{-1}$, $x \to +\infty$, $L \in R^+$, then uniformly on $x \in R^1$ the limit (3.3.4) with $\alpha = 1$ exists, where normalizing and centring constants \(\{A_n\}\) and \(\{B_n\}\) are given by (3.3.18) and (3.3.19), respectively.

The characteristic function $\phi_{1,1}^+(t) = \int_{-\infty}^{+\infty} e^{itx} dS_1(x)$, $i = \sqrt{-1}$, $t \in R^1$, is given by (3.2.3)
3.4 Method of Laplace Transforms: Convergence to Stable Laws I

In this Section a direct method of proving Limit Theorems for sums of independent, identically distributed random variables (being correspondingly centered and normed) to Right-side Stable Laws in case when distribution function of terms belongs to the domain of normal attraction of Right-side Stable Laws is presented and discussed.

3.4.1 Introductory Words

The properties of growing biomolecular networks automatically discover the class of Right-side stable densities as a natural material on which the creation of frequency distributions may base. In this way the regularities of frequency distributions are simple and clear but the form of densities is complex. How to explain to bioinformatics the ideas and methods of Theory on Stable Laws which form the mathematical basis of constructed above continuous analogs of empirical frequency distributions? In order to do that we have to give information on infinite divisible distribution functions and their Canonical Representations. After that we have to extract from these Canonical Representations corresponding representation for particular case of Stable Laws. This is a starting point for the Theory of Stable Laws and the approach which is traditional in Probability Theory. In the next more low level the Problems 1-3 have to be formulated and methods of their solutions be explained. This is also not an easy problem to make it clear for engineers. The only possible way is: to present results from general theory on properties of Stable Laws, their Canonical Representations, series expansions for densities, etc. in particular cases which can be explained and discussed without proofs. To formulate the Problems and Solutions to them. This is just the way we proceed.

Any known frequency distribution varies regularly at infinity and does exhibit constant slowly varying component. The same property does and has to posses any new constructed frequency distribution. It means that passing on stable approximation we are in situation when the frequency distribution belongs to the domain of normal attraction of standard Right-side Stable Law. These circumstances create a possibility to come down to the theory of more low level (in mathematical sense) proving Limit Theorems. Roughly speaking, this is a purpose of the present Section.

More explicitly, we are going to prove Limit Theorems on convergence to Right-side Stable Laws. Generally speaking, the method, which shall be used, is a variation of well-known Method of Characteristic Functions in terms of Laplace-Stieltjes Transforms. Remind that the Central Limit Theorem (convergence to the standard Normal Law) for centered and normed sums of independent, or weakly dependent random variables traditionally is established in Probability Theory by Method of Characteristic Functions. The Method of Characteristic Functions (in terms of Fourier-Stieltjes Transforms) has been developed in 8.9., 210-217, [54] by A. Borovkov.
in order to prove Limit Theorems to Stable Laws related with the domain of their normal attraction. Below we follow the idea of A.Borovkov for our particular case. But there is a distinction. We deal with sums of non-negative random variables which, obviously, reduces the family of limit distributions to Right-side Stable Laws in Scheme (3.1.2)-(3.1.3). Our approach consists in following. For non-negative random variables it is more preferable and clear for engineers to work with Laplace-Stieltjes Transform (or with Generating Functions in discrete case) instead of Characteristic Functions. Then we have real-valued functions instead of complex ones. In our particular case this approach is applicable because we already know that two-side Laplace Transform for Right-side stable densities exists.

Thus, in this case below the so-called Method of Laplace Transforms (by analogy with the Method of Characteristic Functions) is used.

### 3.4.2 Around The Method

Since we consider a case when distribution function \( R_n(x) = P(\xi_n < x), x \in [0, +\infty) \), \( n = 1, 2, \cdots \), in Scheme (3.1.2)-(3.1.3) belongs to the domain of normal attraction of a Right-side Stable Law with exponent \( \alpha \in (0, 2) \), therefore we have to assume that (3.3.3) holds. Without loss of generality we may put \( L = 1 \) in (3.3.3) because with the help of scale-factor we come to \( L \in R^+ \) in (3.3.3). Thus, let us assume that

\[
1 - R_n(x) \approx x^{-\alpha}, \quad x \to +\infty. \tag{3.4.1}
\]

Denote \( S_n = \xi_1 + \xi_2 + \cdots + \xi_n, \quad \zeta_n = \frac{S_n}{B_n} - A_n, \quad n = 1, 2, \cdots \), where

\[
B_n = (\alpha \cdot n)^{1/\alpha}, \quad n = 1, 2, \cdots, \tag{3.4.2}
\]

and the sequence \( \{A_n\} \) shall be chosen later. By (3.3.5), we already know that \( \{B_n\} \) of type (3.4.2) is a ”good” choice. A ”good” choice of \( \{A_n\} \) must imply the weak convergence

\[
P(\zeta_n < x) \Rightarrow S(x) \quad \text{as} \quad n \to +\infty, \tag{3.4.3}
\]

where \( S(x) \) is a Right-side Stable Law with exponent \( \alpha \) and Laplace-Stieltjes Transform

\[
\varphi(s) = \int_{-\infty}^{+\infty} e^{-sx}dS(x), \quad s \in R^+. \tag{3.4.4}
\]

For \( s \in R^+ \) and \( n = 1, 2, \cdots \) let us denote \( \xi = \xi_1 \) and \( \varphi_\xi(s) = Ee^{-s\xi}, \quad \varphi_\zeta(s) = Ee^{s\zeta}. \)

Then,

\[
\varphi_\zeta(s) = e^{sA_n} \cdot E \exp\left(-\left(\frac{s}{B_n}\right) \cdot \sum_{K=1}^{n} \xi_K\right) = e^{sA_n} \cdot (\varphi_\xi(\frac{s}{B_n}))^n.  
\]
The Method of Laplace Transforms means: if for any fixed \( s \in \mathbb{R}^+ \) we prove that
\[
\lim_{n \to +\infty} \varphi_{\zeta_n}(s) = \varphi(s) \quad \text{and} \quad \varphi(0) = 1,
\]
then, by Continuity Theorem for Laplace Transform, (3.4.3) holds and \( S(x) \) in (3.4.3) is a distribution function having \( \varphi(s) \) a Laplace-Stieltjes Transform (see, (3.4.4)). It is easier to deal with \( \ln \varphi_{\zeta_n}(s) \) and prove that
\[
\lim_{n \to +\infty} \ln \varphi_{\zeta_n}(s) = \ln \varphi(s), \quad s \in \mathbb{R}^+.
\] (3.4.5)

Similarly to traditional proof of Central Limit Theorem let us consider the equality
\[
\ln \varphi_{\zeta_n}(s) = s \cdot A_n - n \cdot (1 - \varphi_{\xi}(s/B_n)) + R_n, \quad n = 1, 2, \ldots. \tag{3.4.6}
\]

Since \( \ln \varphi_{\xi}(s/B_n) = \ln(1 - (1 - \varphi_{\xi}(s/B_n))) = -\sum_{m \geq 1} (1 - \varphi_{\xi}(s/B_n))^m \), therefore in (3.4.6) the first term of \( \ln \varphi_{\xi}(s/B_n) \)'s series expansion is extracted and the remained part is denoted as \( R_n \).

For fixed \( s \in \mathbb{R}^+ \) we have, due to (3.4.2), \( (s/B_n) \to 0 \) as \( n \to +\infty \). That is why
\[
0 > \frac{R_n}{n} = -\sum_{m \geq 2} \frac{1}{m-1} \cdot (1 - \varphi_{\xi}(\frac{s}{B_n}))^m > -\sum_{m \geq 2} (1 - \varphi_{\xi}(\frac{s}{B_n}))^m = -\varphi_{\xi}(\frac{s}{B_n})^2 \cdot \frac{1}{2} \cdot g_n(s),
\]
where \( g_n(s) < 2 \) for large \( n \). Now, for fixed \( s \in \mathbb{R}^+ \) and large \( n \) we get the inequality
\[
|R_n| < n \cdot (1 - \varphi_{\xi}(\frac{s}{B_n})). \tag{3.4.7}
\]

Due to (3.4.5)-(3.4.7), the standard idea is: to prove with the help of (3.4.7)
\[
\lim_{n \to +\infty} R_n = 0, \tag{3.4.8}
\]
and
\[
\lim_{n \to +\infty} (s \cdot A_n - n \cdot (1 - \varphi_{\xi}(\frac{s}{B_n}))) = \ln \varphi(s), \quad s \in \mathbb{R}^+. \tag{3.4.9}
\]

It is easy to establish (3.4.8). Indeed, for any \( \tau \in \mathbb{R}^+ \) the inequality \( 1 - \exp(-\tau) < \tau \) implies that
\[
1 - \exp(-\tau) < \tau^\nu \quad \text{for any} \quad \nu \in (0, 1). \tag{3.4.10}
\]

If \( 1 \leq \tau < +\infty \), then (3.4.10) is obvious. If \( 0 < \tau < 1 \), then \( \tau^\nu > \tau \) for any \( \nu \in (0, 1) \) and also we come to (3.4.10).

From the known property of regularly varying functions, the equivalency (3.4.1) implies:
\[
E\xi^\nu = \int_{0-}^{+\infty} x^\nu dR_\alpha(x) = \nu \int_{0-}^{+\infty} (1 - R_\alpha(x))x^{\nu-1}dx < +\infty \quad \text{for any} \quad \nu \in (0, \alpha).
\]
Thus, \(0 \leq 1 - \varphi_\xi(\tau) = E(1 - e^{-\tau \xi}) \leq \tau^{\nu} \cdot E\xi^\nu, \ \nu \in (0, 1)\) for any \(\tau \in R^+\). That is why for fixed \(s \in R^+, n = 1, 2, \ldots, \) and \(\nu \in (0, 2, 1)\) we get \(n \cdot (1 - \varphi_\xi(\frac{s}{B_n}))^2 \leq n \cdot (\frac{s}{B_n})^{2\nu} \cdot (E\xi^\nu)^2\), or, by (3.4.2), \(n \cdot (1 - \varphi_\xi(\frac{s}{B_n}))^2 \leq \text{const} \cdot n^{1-(2\nu/\alpha)} \to 0\) as \(n \to +\infty\), which proves (3.4.8).

The most complicate part in Limit Theorems’ establishment is the limit (3.4.9).

Here different choice of \(\{A_n\}\) may lead to different forms of \(\varphi(s)\).

### 3.4.3 The Results

Let us estimate the integrals \(I_n = \int_{0-}^{+\infty} \frac{x}{1+x^2} dR_\alpha(xB_n), \ n = 1, 2, \ldots\).

We want to show that \(I_n < +\infty\) for all \(n = 1, 2, \ldots\). Indeed, for \(x \in R^+\)
\[
0 \leq \int_{x}^{+\infty} \frac{u}{1 + u^2} dR_\alpha(uB_n) < \frac{1}{x}(1 - R_\alpha(xB_n)) \to 0 \text{ as } x \to +\infty.
\]

That is why we may put
\[
A_n = n \cdot \int_{0-}^{+\infty} \frac{x}{1 + x^2} dR_\alpha(xB_n), \ n = 1, 2, \ldots. \quad (3.4.11)
\]

**Theorem 3.1** For \(0 < \alpha < 2\) and \(\{A_n\}\) given by formula (3.4.11)
\[
P\left(\frac{S_n}{(\alpha n)^{1/\alpha}} - A_n < x\right) \Rightarrow \tilde{S}_\alpha(x). \quad (3.4.12)
\]

Here \(\tilde{S}_\alpha\) is a distribution function which depends only on parameter \(\alpha\) and the function \(\hat{\alpha}_\alpha(s) = \int_{-\infty}^{+\infty} e^{-sx} d\tilde{S}_\alpha(x), \ s \in R^+, \) is given by its logarithm
\[
\ln \hat{\alpha}_\alpha(s) = \int_{0-}^{+\infty} (e^{-sx} - 1 + \frac{sx}{1+x^2})x^{-1-\alpha} dx. \quad (3.4.13)
\]

If we consider the cases \(0 < \alpha < 1, \ \alpha = 1, \ 1 < \alpha < 2\) separately, then, in each case we may correspondingly choose the sequence \(\{A_n\}\) in order to simplify either the form of \(\{A_n\}\), or the form of limit distributions’ Laplace-Stieltjes Transform. Let us put
\[
\tilde{A}_n = \left\{ \begin{array}{ll}
0 & \text{if } 0 < \alpha < 1, \\
\alpha^{-(1/\alpha)} \cdot n^{-1-1/\alpha} \cdot E\xi & \text{if } 1 < \alpha < 2.
\end{array} \right. \quad (3.4.14)
\]

**Theorem 3.2** For \(0 < \alpha < 2, \ \alpha \neq 1\) and \(\tilde{A}_n\) given by formula (3.4.14)
\[
P\left(\frac{S_n}{(\alpha n)^{1/\alpha}} - \tilde{A}_n < x\right) \Rightarrow \hat{S}_\alpha(x). \quad (3.4.15)
\]

Here \(\hat{S}_\alpha\) is a distribution function which depends only on parameter \(\alpha\) and the function \(\hat{\alpha}_\alpha(s) = \int_{-\infty}^{+\infty} e^{-sx} d\hat{S}_\alpha(x), \ s \in R^+, \) is given by its logarithm
\[
\ln \hat{\alpha}_\alpha(s) = \left\{ \begin{array}{ll}
\int_{0-}^{+\infty} (e^{-sx} - 1)x^{-\alpha-1} dx & \text{if } 0 < \alpha < 1, \quad (3.4.13'')
\int_{0-}^{+\infty} (e^{-sx} - 1 + sx)x^{-\alpha-1} dx & \text{if } 1 < \alpha < 2. \quad (3.4.13''')
\end{array} \right.
\]
Let us make two remarks.

In the case $\alpha = 1$ it is possible to find asymptotic behavior of $A_n$ given by formula (3.4.11) as $n \to +\infty$. Namely,

$$A_n = \ln n + o(\ln n), \ n \to +\infty \ \text{(see, 8., 9, 215, [54])}. \tag{3.4.16}$$

However the last expression follows from the results of general theory too.

Next, with the help of integration by parts from either (3.4.13) or (3.4.13')-(3.4.13") we may come to following Canonical Representations of Right-side Stable Laws. For instance, from (3.4.13) we obtain

$$\ln \hat{\varphi}_\alpha(s) = \begin{cases} \frac{cs - \Gamma(1 - \alpha)s^\alpha}{s} & \text{if } \alpha \neq 1, \\ cs + s \log s & \text{if } \alpha = 1, \end{cases} \tag{3.4.17}$$

where $c = \begin{cases} -\int_0^\infty \frac{x}{1+x^2} dx^{-\alpha}, & 0 < \alpha < 1, \\ \int_0^\infty \frac{x^3}{1+x^2} dx^{-\alpha}, & 1 < \alpha < 2, \\ 1, & \alpha = 1, \end{cases}$

$\bullet$ $\ln \hat{\varphi}_\alpha(s) = \begin{cases} \frac{cs - \Gamma(1 - \alpha)s^\alpha}{s} & \text{if } \alpha \neq 1, \\ cs + s \log s & \text{if } \alpha = 1, \end{cases}$

$$\ln \hat{\varphi}_\alpha(s) = \begin{cases} \frac{cs - \Gamma(1 - \alpha)s^\alpha}{s} & \text{if } \alpha \neq 1, \\ cs + s \log s & \text{if } \alpha = 1, \end{cases} \tag{3.4.17}$$

$3.5$ Method of Laplace Transforms: Convergence to Stable Laws II

In this Section a technical proof to Theorems 3.1 and 3.2 is done based on different representations of the function

$$n \cdot (1 - \varphi_\xi(s/B_n)), \ n = 1, 2, \cdots, \ s \in R^+ \tag{3.5.1}$$

(see, (3.4.6) and (3.4.9)), which under the corresponding choice of a sequence $\{A_n\}$ leads to different forms of limit distribution function.

3.5.1 Representations of Function (3.5.1)

For $s \in R^+$ let us introduce a function

$$\delta_s(x) = \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right)\frac{1+x^2}{x^2}, \ x \in [0, +\infty). \tag{3.5.2}$$

For any fixed $s \in R^+$ the function $\delta_s(x)$ is continuous and bounded by $x$. Indeed,

$$\lim_{x \to +\infty} \delta_s(x) = -1, \ s \in R^+, \tag{3.5.3}$$

and by L’Hopital’s rule, $\lim_{x \to 0} \delta_s(x) = \lim_{x \to 0} (e^{-sx} - 1) + \lim_{x \to 0} \frac{e^{-sx} - 1 + sx}{x^2} = \frac{s^2}{2}, \ s \in R^+$.

Thus, $\delta_s(x)$ is continuous. That is why there is some $x_0 \in R^+$ such that, due to (3.5.3), $|\delta_s(x)| < M$ for $x \in (x_0, +\infty)$, where $M$ is some positive and finite number. The continuous in $[0, x_0]$ function $\delta_s(x)$ achieves its maximal and minimal value. All together implies the boundedness of $\delta_s(x)$. 

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Let us introduce also a function

$$F_\alpha(x) = \int_{0^-}^{x} \frac{u^{1-\alpha}}{1+u^2} du, \quad 0 < \alpha < 2, \quad x \in \mathbb{R}^+.$$  \hfill (3.5.4)

Note that

$$F_\alpha(+\infty) < +\infty. \hfill (3.5.5)$$

Indeed, $0 < \int_{x}^{\infty} \frac{u^{1-\alpha}}{1+u^2} du < \int_{x}^{\infty} \frac{du}{u^{2\alpha}} = \frac{1}{\alpha x^\alpha} \to 0$ as $x \to +\infty$.

Now, for $0 < \alpha < 2$, $s \in \mathbb{R}^+$ and $n = 1, 2, \cdots$ we have

$$n \cdot (\varphi_\xi\left(\frac{s}{B_n}\right) - 1) = n \cdot \int_{0^-}^{+\infty} (e^{-sx} - 1)dR_\alpha(xB_n) = \int_{0^-}^{+\infty} \delta_s(x)dF_{n,\alpha}(x) - s \cdot n \cdot \int_{0^-}^{+\infty} \frac{x}{1+x^2} dR_\alpha(xB_n), \hfill (3.5.6)$$

where

$$F_{n,\alpha}(x) = \int_{0^-}^{x} \frac{nu^2}{1+u^2} dR_\alpha(u \cdot B_n), \quad x \in \mathbb{R}^+, \quad n = 1, 2, \cdots. \hfill (3.5.7)$$

Note that for $n = 1, 2, \cdots$

$$F_{n,\alpha}(+\infty) < +\infty, \hfill (3.5.8)$$

which is proved similarly to (3.5.5) with the help of (3.4.1).

Finally, by the notation (3.4.11), we transform (3.5.6) into the first representation being suitable for all cases $0 < \alpha < 2$:

$$s \cdot A_n - n \cdot (1 - \varphi_\xi\left(\frac{s}{B_n}\right)) = \int_{0^-}^{+\infty} \delta_s(x)dF_{n,\alpha}(x). \hfill (3.5.9)$$

The case $0 < \alpha < 1$ For $s \in \mathbb{R}^+$ let us introduce a function

$$\tilde{\delta}_s(x) = (e^{-sx} - 1)\frac{1+x}{x}, \quad x \in [0, +\infty), \hfill (3.5.2')$$

which for any fixed $s \in \mathbb{R}^+$, obviously, is continuous and bounded by $x$. Indeed,

$$\lim_{x \to +\infty} \tilde{\delta}_s(x) = -1 \quad \text{and} \quad \lim_{x \to 0} \tilde{\delta}_s(x) = \lim_{x \to 0} (e^{-sx} - 1) + \lim_{x \to 0} \frac{e^{-sx} - 1}{x} = -s,$$

which implies the boundedness of $\tilde{\delta}_s(x)$.

Let us introduce also a function

$$\tilde{F}_\alpha(x) = \int_{0^-}^{x} \frac{u^{-\alpha}}{1+u} du, \quad 0 < \alpha < 1, \quad x \in \mathbb{R}^+. \hfill (3.5.4')$$

Note that

$$\tilde{F}_\alpha(+\infty) < +\infty. \hfill (3.5.5')$$

Indeed, $0 < \int_{x}^{\infty} \frac{u^{-\alpha}}{1+u} du < \int_{x}^{\infty} \frac{du}{u^{1+\alpha}} = \frac{1}{\alpha x^\alpha} \to 0$ as $x \to +\infty.$
Now, for $0 < \alpha < 2$, $s \in R^+$ and $n = 1, 2, \cdots$ we have

$$n \cdot \varphi_\xi \left( \frac{s}{B_n} \right) - 1 = n \cdot \int_{0-}^{+\infty} (e^{-sx} - 1) dR_\alpha(xB_n) = \int_{0-}^{+\infty} \delta_s(x)d\bar{F}_{n,\alpha}(x), \quad (3.5.6')$$

where

$$\bar{F}_{n,\alpha}(x) = \int_{0-}^{x} \frac{nu}{1+u} dR_\alpha(uB_n), \quad x \in R^+, \quad n = 1, 2, \cdots. \quad (3.5.7')$$

Note that

$$\bar{F}_{n,\alpha}(+\infty) < +\infty, \quad (3.5.8')$$

which is proved similarly to (3.5.5) with the help of (3.4.1).

The case $1 < \alpha < 2$ For $s \in R^+$ let us introduce a function

$$\delta_s(x) = (e^{-sx} - 1 + sx) \frac{1+x}{x^2}, \quad x \in [0, +\infty). \quad (3.5.2'')$$

For any fixed $s \in R^+$ the function $\delta_s(x)$ is continuous and bounded. Indeed,

$$\lim_{x \to +\infty} \delta_s(x) = s \quad \text{and} \quad \lim_{x \to 0} \delta_s(x) = \frac{s^2}{2}, \quad s \in R^+.$$

Let us introduce also a function

$$\bar{F}_\alpha(x) = \int_{0-}^{x} \frac{u^{1-\alpha}}{1+u} du, \quad 1 < \alpha < 2, \quad x \in R^+. \quad (3.5.4'')$$

Note that

$$\bar{F}_\alpha(+\infty) < +\infty. \quad (3.5.5'')$$

Indeed, for $x \in R^+$ taking into account that $1 < \alpha < 2$ we have

$$0 < \int_{x}^{+\infty} \frac{u^{1-\alpha}}{1+u} du < \int_{x}^{+\infty} \frac{du}{u^\alpha} = \frac{1}{(\alpha - 1)x^{\alpha-1}} \to 0 \quad \text{as} \quad x \to +\infty.$$

Now, for $1 < \alpha < 2$, $s \in R^+$ and $n = 1, 2, \cdots$ we have

$$n \cdot \varphi_\xi \left( \frac{s}{B_n} \right) - 1 = n \cdot \int_{0-}^{+\infty} (e^{-sx} - 1 + sx) dR_\alpha(xB_n) \frac{n\alpha}{B_n} E_\xi =$$

$$= n \cdot \int_{0-}^{+\infty} \delta_s(x)d\bar{F}_{n,\alpha}(x) - \frac{s}{\alpha^{1/\alpha}} n^{-1/(1/\alpha)} E_\xi, \quad (3.5.6'')$$

where

$$\bar{F}_{n,\alpha}(x) = \int_{0-}^{x} \frac{nu^2}{1+u} dR_\alpha(uB_n), \quad x \in R^+, \quad n = 1, 2, \cdots. \quad (3.5.7'')$$

In (3.5.6'') we used (3.4.2) and the following known fact: for a distribution function, which has a regularly varying tail with exponent $(-\alpha)$ and is concentrated on $[0, +\infty)$, all moments of orders $\nu \in (0, \alpha)$ are finite. Note that

$$\bar{F}_{n,\alpha}(+\infty) < +\infty, \quad (3.5.8'')$$

which is proved similarly to (3.5.5) with the help of (3.4.1).

From (3.5.6'') in case $1 < \alpha < 2$ we get the analog of (3.9):

$$sA_n - n \cdot \varphi_\xi \left( \frac{s}{B_n} \right) = \int_{0-}^{+\infty} \delta_s(x)d\bar{F}_{n,\alpha}(x). \quad (3.5.9'')$$

The preparatory work is over.
3.5.2 Technical Realization of The Method

Let us explain the idea of the proof of Theorems 3.1 and 3.2 from this point.

We constructed in all cases $0 < \alpha < 2$ and separately in cases $0 < \alpha < 1, 1 < \alpha < 2$ integral representations (3.5.9), (3.5.6') and (3.5.9'') for the expression under the limit at the left-hand-side of (3.4.9):

$$
\int_{0-}^{+\infty} \delta_s(x) dF_{n,\alpha}(x), \quad \int_{0-}^{+\infty} \bar{\delta}_s(x) d\bar{F}_{n,\alpha}(x), \quad \int_{0-}^{+\infty} \bar{\delta}_s(x) d\bar{E}_{n,\alpha}(x).
$$

Here the functions under integrals are continuous and bounded, the function under differentials are finite measures concentrated on $[0, +\infty)$ (see, (3.5.7)-(3.5.8), (3.5.7')-(3.5.8'), (3.5.7'')-(3.5.8'')). That is why

$$
\{(F_{n,\alpha}(x)/F_{n,\alpha}(+\infty)), (\bar{F}_{n,\alpha}(x)/\bar{F}_{n,\alpha}(+\infty)), (\bar{E}_{n,\alpha}(x)/\bar{E}_{n,\alpha}(+\infty))\}
$$

form sequences of distribution functions concentrated on $[0, +\infty)$. At the same time, in all cases $0 < \alpha < 2$ and separately in cases $0 < \alpha < 1, 1 < \alpha < 2$ finite measures $F_{\alpha}, \bar{F}_{\alpha}, \bar{E}_{\alpha}$ are introduced (see, (3.5.4)-(3.5.5), (3.5.4')-(3.5.5'), (3.5.4'')-(3.5.5'')). So, we get distribution functions $(F_{\alpha}(x)/F_{\alpha}(+\infty)), (\bar{F}_{\alpha}(x)/\bar{F}_{\alpha}(+\infty)), (\bar{E}_{\alpha}(x)/\bar{E}_{\alpha}(+\infty))$ concentrated on $[0, +\infty)$.

Now, everything is based on limit relations: for $x \in R^+$ and $x = +\infty$

$$
\lim_{n \to +\infty} F_{n,\alpha}(x) = F_{\alpha}(x), \quad 0 < \alpha < 2, \quad (3.5.10)
$$

$$
\lim_{n \to +\infty} \bar{F}_{n,\alpha}(x) = \bar{F}_{\alpha}(x), \quad 0 < \alpha < 1, \quad (3.5.10')
$$

$$
\lim_{n \to +\infty} \bar{E}_{n,\alpha}(x) = \bar{E}_{\alpha}(x), \quad 1 < \alpha < 2. \quad (3.5.10'')
$$

If these limit relations are proved, then, by the second Helly’s Theorem (see, for instance, [28]), for any fixed $s \in R^+$

$$
\lim_{n \to +\infty} \int_{0-}^{+\infty} \delta_s(x) d\left(\frac{F_{n,\alpha}(x)}{F_{n,\alpha}(+\infty)}\right) = \int_{0-}^{+\infty} \delta_s(x) d\left(\frac{F_{\alpha}(x)}{F_{\alpha}(+\infty)}\right),
$$

$$
\lim_{n \to +\infty} \int_{0-}^{+\infty} \bar{\delta}_s(x) d\left(\frac{\bar{F}_{n,\alpha}(x)}{\bar{F}_{n,\alpha}(+\infty)}\right) = \int_{0-}^{+\infty} \bar{\delta}_s(x) d\left(\frac{\bar{F}_{\alpha}(x)}{\bar{F}_{\alpha}(+\infty)}\right),
$$

$$
\lim_{n \to +\infty} \int_{0-}^{+\infty} \bar{\delta}_s(x) d\left(\frac{\bar{E}_{n,\alpha}(x)}{\bar{E}_{n,\alpha}(+\infty)}\right) = \int_{0-}^{+\infty} \bar{\delta}_s(x) d\left(\frac{\bar{E}_{\alpha}(x)}{\bar{E}_{\alpha}(+\infty)}\right),
$$

or

$$
\lim_{n \to +\infty} \int_{0-}^{+\infty} \delta_s(x) dF_{n,\alpha}(x) = \int_{0-}^{+\infty} \delta_s(x) dF_{\alpha}(x), \quad 0 < \alpha < 2, \quad (3.5.11)
$$

$$
\lim_{n \to +\infty} \int_{0-}^{+\infty} \bar{\delta}_s(x) d\bar{F}_{n,\alpha}(x) = \int_{0-}^{+\infty} \bar{\delta}_s(x) d\bar{F}_{\alpha}(x), \quad 0 < \alpha < 1, \quad (3.5.11')
$$
The integrals at the right-hand-sides of (3.5.11), (3.5.11′), (3.5.11″) are transformed exactly into the integrals at the right-hand-sides of (3.4.13), (3.4.13′), (3.4.13″).

In order to complete proofs of Theorems 3.1 and 3.2 it remains to establish limit relations (3.5.10), (3.5.10′), (3.5.10″). Since the proofs of these limit relations are realized by the same technical manner, let us establish, for instance, only (3.5.10′).

Using the equality $d(\frac{x}{1+x}) = \frac{dx}{(1+x)^2}$, where $d$ denotes the sign of differential, and integration by parts, from (3.5.7′) for technical manner, let us establish, for instance, only (3.5.10′).

By (3.4.1) and (3.4.2), we conclude: for $x \in R^+$ and $n = 1, 2, \ldots$ we obtain

$$F_{n,\alpha}(x) = n \cdot \int_{0-}^{x} \frac{u}{1+u} dR_{\alpha}(u \cdot B_n) = n \cdot \frac{u}{1+u} (1 - R_{\alpha}(uB_n)) \bigg|_{0-}^{x} + n \cdot \int_{0-}^{x} (1 - R_{\alpha}(uB_n)) \frac{du}{(1+u)^2}.$$

By (3.4.1) and (3.4.2), we conclude: for $x \in R^+$

$$\lim_{n \to +\infty} F_{n,\alpha}(x) = -\frac{1}{\alpha} \frac{x^{1-\alpha}}{1+x} + \lim_{n \to +\infty} \left( n \cdot \int_{0-}^{x} (1 - R_{\alpha}(uB_n)) \frac{du}{(1+u)^2} \right) =$$

$$= -\frac{1}{\alpha} \frac{x^{1-\alpha}}{1+x} + \int_{0-}^{x} \lim_{n \to +\infty} (n \cdot (1 - R_{\alpha}(uB_n))) \frac{du}{(1+u)^2} = -\frac{x^{1-\alpha}}{\alpha(1+x)} + \int_{0-}^{x} \frac{1}{\alpha(1+u)^2} du =$$

$$= -\frac{x^{1-\alpha}}{\alpha(1+x)} + \frac{1}{\alpha} \int_{0-}^{x} u^{-\alpha} d(\frac{u}{1+u}). \quad (3.5.12)$$

The legitimacy of limit transition under the sign of integral in (3.5.12) shall be substantiated later.

Making integration by parts in the second term at the right-hand-side of (3.5.12) for $x \in R^+$ we come to the following limit equality

$$\lim_{n \to +\infty} F_{n,\alpha}(x) = \int_{0-}^{x} \frac{u^{-\alpha}}{1+u} du, \quad (3.5.13)$$

which, due to (3.5.4′), proves (3.5.10′). Note that (3.5.13) holds for $x = +\infty$ too.

Finally, let us substantiate the equality

$$\lim_{n \to +\infty} \left( n \cdot \int_{0-}^{x} (1 - R_{\alpha}(uB_n)) \frac{du}{(1+u)^2} \right) = \int_{0-}^{x} \lim_{n \to +\infty} (n \cdot (1 - R_{\alpha}(uB_n))) \frac{du}{(1+u)^2} \quad (3.5.14)$$

for any $x \in (0, +\infty]$, which establishes the legitimacy of (3.5.12).

Let $x \in (0, +\infty]$ be fixed. Since $0 < \alpha < 1$, therefore we may choose $\gamma \in (1, 1/\alpha)$. Let an integer $n_0$ be chosen from the condition $n_0 > (1/x)^\gamma$. Then, for all $n = n_0, n_0 + 1, \ldots$ we have $(1/n^\gamma) < x$. Let us write down

$$n \int_{0-}^{x} (1 - R_{\alpha}(uB_n)) \frac{du}{(1+u)^2} = n \cdot \left( \int_{0-}^{1/n^\gamma} + \int_{1/n^\gamma}^{x} \right) (1 - R_{\alpha}(uB_n)) \frac{du}{(1+u)^2} \quad (3.5.15)$$

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for \( n = n_0, n_0+1, \cdots \) and estimate the asymptotic behavior of each integral at the right-hand-side of (3.5.15) separately. For \( n = n_0, n_0 + 1, \cdots \)

\[
0 < n \cdot \int_{0^-}^{(1/n^\gamma)} (1 - R_\alpha(uB_n)) \frac{du}{(1 + u)^2} < n \cdot \int_{0^-}^{(1/n^\gamma)} \frac{du}{(1 + u)^2} = \frac{1}{\alpha} \cdot \int_{0^-}^{(1/n^\gamma)} \frac{du}{(1 + u)^2} = \frac{1}{\alpha} = \frac{1}{\alpha} = \frac{1}{\alpha} \cdot n^1 = n \cdot (1 - \frac{n^-}{1+n^\gamma}) = n^{1-\gamma} \to 0 \text{ as } n \to +\infty
\]

(3.5.16)

because \( \gamma > 1 \). Given \( \varepsilon \in (0, 1) \), due to (3.4.1) and (3.4.2), there is an integer \( n_1 > n_0 \) such that for all \( u \in (1/n^\gamma, +\infty) \) the inequalities hold

\[
\frac{1 - \varepsilon}{\alpha u^\alpha} < n \cdot (1 - R_\alpha(uB_n)) < \frac{1 + \varepsilon}{\alpha u^\alpha}, \; n = n_1, n_1 + 1, \cdots
\]

(3.5.17)

because \((1/\alpha) - \gamma > 0\) and for all \( u \in (1/n^\gamma, +\infty) \) and \( n = n_1, n_1 + 1, \cdots \)

\[
uB_n > \frac{1}{n^\gamma} B_n = \frac{(\alpha\gamma)^{1/\alpha}}{n^\gamma} = \alpha^{1/\alpha} \cdot n^{(1/\alpha) - \gamma} \to +\infty \text{ as } n \to +\infty.
\]

Let us estimate the second integral at the right-hand-side in (3.5.15) with the help of (3.5.17). For \( n = n_1, n_1 + 1, \cdots \) we have

\[
(1 - \varepsilon) \cdot \frac{1}{\alpha} \cdot \int_{0^-}^{(1/n^\gamma)} \frac{u^{-\alpha}du}{(1 + u)^2} < n \cdot \int_{0^-}^{(1/n^\gamma)} \frac{du}{(1 + u)^2} < \frac{1 + \varepsilon}{\alpha} \cdot \int_{0^-}^{(1/n^\gamma)} \frac{u^{-\alpha}du}{(1 + u)^2}.
\]

(3.5.18)

Next, there is an integer \( n_2 > n_1 \) such that for \( n = n_2, n_2 + 1, \cdots \)

\[
0 < \int_{0^-}^{(1/n^\gamma)} \frac{u^{-\alpha}du}{(1 + u)^2} < \varepsilon
\]

(3.5.19)

and, due to (3.5.16), simultaneously the first integral at the right-hand-side in (3.5.15) is less than \( \varepsilon \). Taking this and (3.5.18)-(3.5.19) into account we obtain

\[
\frac{1 - \varepsilon}{\alpha} \cdot \int_{0^-}^{(1/n^\gamma)} \frac{u^{-\alpha}du}{(1 + u)^2} - \frac{1 - \varepsilon}{\alpha} < n \cdot \int_{0^-}^{(1/n^\gamma)} \frac{du}{(1 + u)^2} < \frac{1 + \varepsilon}{\alpha} \cdot \int_{0^-}^{(1/n^\gamma)} \frac{u^{-\alpha}du}{(1 + u)^2} + \varepsilon
\]

(3.5.20)

for \( n = n_2, n_2 + 1, \cdots \). Now, tending \( \varepsilon \downarrow 0 \) in (3.5.20) we obtain for any \( x \in (0, +\infty] \)

\[
\lim_{n \to +\infty} (n \cdot \int_{0^-}^{x} \frac{1 - R_\alpha(uB_n)}{(1 + u)^2} \frac{du}{(1 + u)^2}) = \frac{1}{\alpha} \cdot \int_{0^-}^{x} \frac{u^{-\alpha}du}{(1 + u)^2}.
\]

But, as we see in (3.5.12), \( \frac{1}{\alpha} \cdot \int_{0^-}^{x} \frac{u^{-\alpha}du}{(1 + u)^2} = \int_{0^-}^{x} \lim_{n \to +\infty} (n \cdot (1 - R_\alpha(uB_n))) \frac{du}{(1 + u)^2}.
\]

The equality (3.5.14) is proved.

Thus, we completed the proofs of Theorems 3.1 and 3.2.

We suggest the reader to prove limit relations (3.5.10) and (3.5.10') on one's own by using similar arguments and computations as presented above.
3.6 Statistical Approach Related to Stable Approximation

In this Section we revise our opinion on empirical frequency distributions in fractals of growing biomolecular network being the basement of the events occurrence number’s frequency distribution construction in whole growing network using stable approximation. In this situation, the necessary information on frequency distribution in order to apply stable approximation, as we have already seen, essentially changed. It implies the possibility of general statistical method application instead of searching different concrete new empirical frequency distributions. That is why in frame of fractals we discuss various statistical problems arising in different situations related with empirical frequency distributions.

Namely, the frequency distribution of events’ occurrence number in fractals may be:

- completely unknown;
- the type depending on parameters is given but parameters either partly or completely are unknown;
- the frequency distribution is completely known.

Until now we deal with the last situation and will continue to deal with the subsequent Sections. Below we discuss first two situations.

3.6.1 Empirical Distribution Function

Estimating the frequency distribution in whole growing biomolecular network, first of all, let us talk about events occurrence number’s frequency distribution \( \{p_n\} \) on fractals in the following case. Namely, according to some reasons, the approximation of \( \{p_n\} \) by suggested in literature empirical frequency distributions, or, for instance, by distributions generated by Stable Laws (see, Chapter 2) is not appropriate in frame of fractals of growing biomolecular network.

Thus, in this case we assume that \( \{p_n\} \) is unknown, only it satisfies all known statistical facts described in Section 1.1. Let \( \xi \) be a random number of events occurrence number in separately taken fractal. Denote by \( F(x) = P(\xi < x) \) a distribution function which corresponds to distribution \( \{p_n\} \). Naturally, \( F \) is unknown.

The Theory of Statistics suggests the following well-known procedure of the construction of estimation for unknown \( F \). The construction is based on observations

\[
X = (x_1, x_2, \cdots, x_n)
\]  

(3.6.1)

taken from general sample on random variable \( \xi \). Here \( x_i, i = 1, 2, \cdots, n \) is the i-th realization of random variable \( \xi \) with unknown distribution function \( F \). Forming order statistics \( x_{(1)} \leq \cdots \leq x_{(n)} \)
\( x_{(2)} \leq \cdots \leq x_{(n)} \) from (3.6.1) and variational series \((x_{(1)}, x_{(2)}, \cdots, x_{(n)})\) we introduce distribution function

\[
F_n(x) = \begin{cases} 
0 & \text{if } x \leq x_{(1)}, \\
\frac{K-1}{n} & \text{if } x_{(K-1)} < x \leq x_{(K)}, \ K = 2, 3, \cdots, n, \\
1 & \text{if } x > x_{(n)}. 
\end{cases}
\]

\(F_n(x)\) is so-called empirical distribution function, which estimates the unknown \(F(x)\).

Many theoretical results which prove that \(F_n\) is "good" approximation for \(F\) are obtained in the Theory of Statistics. For instance, in case when \(F\) is continuous, the following fact is proved (Glivenko’s Theorem): for any \(\varepsilon \in \mathbb{R}^+\)

\[
\lim_{n \to +\infty} P(\sup_{-\infty \leq x \leq +\infty} |F_n(x) - F(x)| \geq \varepsilon) = 0. \tag{3.6.2}
\]

There is an improvement of this fact which gives the rate of convergence \(F_n\) to \(F\) (Komolgorov - Smirnov’s Theorem), etc. Anyway, we may take as distribution function of events occurrence number in fractal the distribution function \(F_n\).

At once, following non-statistical (probabilistic) problems arise because \(F_n\) has to satisfy known statistical facts described in Section 1.1.

1. Show that \(1 - F_n(x)\) varies regularly at infinity with the same exponent as \(1 - F(x)\) has.
2. If \(1 - F\) exhibits constant slowly varying component, then show that \(1 - F_n\) exhibits the same constant slowly varying component.
3. If \(F\) is downward/upward convex, then \(F_n\) possesses the same property, etc.

Another type of non-statistical problems is concerned with parameters.

Let \(F\) depend on parameters \(a_1, a_2, \cdots, a_K\): \(F(x) = F(x; a_1, a_2, \cdots, a_K)\).

As a rule, the following properties for empirical frequency distributions which are transformed to their distribution functions have to be fulfilled.

\(F\) is monotone by each parameter and continuous by all parameters, etc.

If some property of such a "smoothness" type takes place for \(F\), then it requires to prove that \(F_n\) possess the same property.

Similar problems may be settled down for empirical moment \(\hat{m}_\nu = \frac{1}{n} \sum_{K=1}^{n} x_K^\nu\) of any positive order \(\nu\) being the estimate for \(E\xi^\nu\).

In case when additional information on class \(F\) of distribution functions, which includes unknown \(F\), is given, then improvements for rate of convergence in (3.6.2) is of interest.
3.6.2 Statistical Problems

Now, let us forget about empirical distribution function and empirical moments \( \hat{m}_\nu \).

Remind that our purpose consists in approximation of distribution function \( F \) corresponding to distribution \( \{p_n\} \) in whole growing biomolecular network of the type being described in 3.1.1 of Section 3.1. In such a situation for any \( F \) satisfying general statistical facts results of Theory of Sums of Independent, Identically Distributed Random Variables are of interest. Limit Theorems of this Theory lead to Stable Laws, which allow even in case of completely unknown \( \{p_n\} \), i.e. \( F \), give stable approximation for distribution function \( F \). In this way several statistical problems have to be solved.

Here is the list of corresponding statistical problems.

(a) Since \( \{p_n\} \) has to vary regularly at infinity, therefore the representation holds

\[
p_n = n^{-\rho} \cdot L(n), \quad 1 < \rho < +\infty, \quad n = 1, 2, \ldots
\]

(3.6.3)

(the case \( \rho = 1 \) is of special interest and requires separate consideration).

From the point of view of stable approximation essential nonintersected cases of the exponent \( (-\rho) \) of regular variation of \( \{p_n\} \), which lead to different Stable Laws in Limit Theorems, are

\[
1 < \rho < 2, \quad 2 < \rho < 3, \quad \rho = 3 \quad \text{and} \quad 3 < \rho < +\infty.
\]

(3.6.4)

In case \( 3 < \rho < +\infty \) first two moments of \( \{p_n\} \) are finite; in case \( \rho = 3 \) the first moment is finite but the finiteness of the second moment depends on asymptotic behavior of \( \{L(n)\} \) as \( n \to +\infty \); in case \( 2 < \rho < 3 \) the first moment is finite but the second one is infinite; in case \( \rho = 2 \) the finiteness of the first moment depends on asymptotic behavior of \( \{L(n)\} \) as \( n \to +\infty \); in case \( 1 < \rho < 2 \) the first moment is infinite.

A verification of hypothesis on essential cases (3.6.4) is needed in order to determine the exponent \( \alpha = \rho - 1 \) of corresponding Limit Stable Law in Scheme (3.1.2)-(3.1.3).

(b) After solving above formulated problem on hypothesis testing the point estimation of parameter \( \rho \) by different statistical methods has to be done (minimal square, maximal likelihood, moment’s method, etc.) based on observations (3.6.1) on random variable \( \xi \).

(c) Next, we need to clear up the asymptotical behavior of a slowly varying sequence \( \{L(n)\} \) in (3.6.3) as \( n \to +\infty \). As a first step we may check out statistically the hypothesis on belonging of \( \{L(n)\} \) to several typical growths to infinity for \( \{L(n)\} \) of types, for instance \( \{\ln n\} \), \( \{(\ln n)^\alpha\} \) with either \( \alpha \in \mathbb{R}^+ \) or \( \alpha \in (-\infty, 0) \). Here we have to take into attention opinions of experts in bioinformatics for the concrete growing network on type of \( \{L(n)\} \).

This is the most complicate and ”foggy” part of statistical programm.
(d) The situation is more clear if we assume that \( \{p_n\} \) exhibits constant slowly varying component, i.e. the limit exists

\[
L := \lim_{n \to +\infty} L(n) \in R^+.
\]  

(3.6.5)

Remind that well-known empirical frequency distribution being regularly varying at infinity exhibit constant slowly varying component (see, Section 1.7). In this case only the statistical problem on point estimation of unknown parameter \( L \) in (3.6.5) arises. More general, we solve point estimation problem for vector \((\rho, L)\) at once, where \( \rho \) is the exponent and \( L \) is the constant slowly varying component of unknown \( \{p_n\} \) of type (3.6.3).

The problem becomes more specific if solving the problem we use \( F_n(x) \) as an estimate for \( F(x) \), where \( F \) corresponds to \( \{p_n\} \).

Note that in conditions (3.6.5) in essential cases \( \rho = 2 \) and \( \rho = 3 \) in (3.6.4) the first moment is infinite and the first moment is finite, the second infinite, respectively.

(e) Let us assume that the hypothesis on, for instance, \( 1 < \rho \leq 3 \) and \( 3 < \rho < +\infty \) gives positive result for case \( 3 < \rho < +\infty \). Then, it is not necessary to make point estimation on \( \rho \) because it is obvious that the mean value and the variance of \( \xi \) are finite. At once we have to estimate statistically \( M_\xi \) and \( D_\xi \) because we are in situation when \( F \) belongs to the domain of normal attraction of Standard Normal Law.

Speaking simpler the Central Limit Theorem for sums of independent, identically distributed random variables takes place and we get normal approximation for \( F \). We need only in \( E\xi \) and \( D\xi \) which can be, for instance, estimated with the help of empirical moments (see, 3.6.1).

(f) Let the hypothesis, for instance \( 1 < \rho \leq 2, \ 2 < \rho < 3, \ 3 \leq \rho < +\infty \) gives positive result for case \( 2 < \rho < 3 \). Now, the mean value is finite but the variance of \( \xi \) is infinite. We have to estimate the mean value of \( \xi \) statistically and, in addition to this, solve the equation (3.2.15) in order to establish a sequence \( \{B_n\} \) of norming constants in Scheme (3.1.2)-(3.1.3).

(g) Let the hypothesis, for instance, \( 1 < \rho < 2, \ 2 \leq \rho < +\infty \) give positive result for case \( 1 < \rho < 2 \). Then the mean value of \( \xi \) is infinite and we deal only with equation (3.2.15).

If \( \{p_n\} \) exhibits constant slowly varying component, then in situations (f) and (g) we are in frame of Limit Theorems 1 and 2 formulated in 3.3.1 of Section 3.3.

Let us make one remark. If the type of \( \{p_n\} \) is known, but, parameters of \( \{p_n\} \) are unknown, then all statistical problems being formulated in (a)-(g) stay unchanged in this new situation, but become more simple.
3.7 Stable Approximation: The Power Laws

In this Section we improve the stable approximation’s application procedure to empirical frequency distributions defined in fractals of growing biomolecular network which leads to determination of the frequency distribution in whole systems. Here the case is considered when the frequency distribution in fractals is completely known. This case is discussed on example of Power Law which is interesting from the methodological point of view because has very simple form. There is only one problem: How to apply results of General Theory to empirical frequency distributions?

3.7.1 Normal Approximation

Let the random variable $\xi$ has distribution $\{p_n\}$, where

$$p_n = c(\rho) \cdot n^{-\rho}, \quad 1 < \rho < +\infty, \quad n = 1, 2, \cdots,$$  \hspace{1cm} (3.7.1)

and

$$c(\rho) = \left( \sum_{n \geq 1} n^{-\rho} \right)^{-1}.$$ \hspace{1cm} (3.7.2)

Simultaneously, we consider the random variable $\hat{\xi}$ with density $f_\rho$ of the form

$$f_\rho(x) = (\rho - 1) \cdot x^{-\rho}, \quad 1 < \rho < +\infty, \quad 1 \leq x < +\infty,$$ \hspace{1cm} (3.7.3)

which is a continuous analog of distribution $\{p_n\}$.

Let us denote by $R_\rho(x)$ and $\hat{R}_\rho(x)$ distribution functions of $\{p_n\}$ and $f_\rho(x)$, respectively:

$$R_\rho(x) = c(\rho) \cdot \sum_{n < x} n^{-\rho}, \quad \hat{R}_\rho(x) = \int_{1}^{x} f_\rho(u) du = 1 - \frac{1}{x^{\rho - 1}}.$$  

We are in situation when the form of distribution and all parameters (in this case there is only one parameter $\rho$) are known.

The normal approximation here is possible, in particular, in case $3 < \rho < +\infty$, when first two moments of random variables $\xi$ and $\hat{\xi}$ are finite. Let us evaluate these moments.

$$E \xi = \sum_{n \geq 1} n \cdot p_n = c(\rho) \cdot \sum_{n \geq 1} n^{-\rho + 1} = \frac{c(\rho)}{c(\rho - 1)},$$  \hspace{1cm} (3.7.4)

$$E \xi^2 = \sum_{n \geq 1} n^2 \cdot p_n = c(\rho) \sum_{n \geq 1} n^{-\rho + 2} = \frac{c(\rho)}{c(\rho - 2)}.$$  

and, as a consequence of this

$$D \xi = E \xi^2 - (E \xi)^2 = \frac{c(\rho)}{c(\rho - 2)} - \left( \frac{c(\rho)}{c(\rho - 1)} \right)^2 > 0.$$ \hspace{1cm} (3.7.5)

Here (3.7.1)-(3.7.2) were used.
Similarly,

\[ E\hat{\xi} = \int_{1}^{+\infty} x \cdot f_\rho(x) dx = (\rho - 1) \cdot \int_{1}^{+\infty} x^{-\rho+1} dx = \frac{\rho - 1}{\rho - 2} \int_{1}^{+\infty} f_{\rho-1}(x) dx = \frac{\rho - 1}{\rho - 2}, \quad (3.7.6) \]

\[ E\hat{\xi}^2 = \int_{1}^{+\infty} x^2 f_\rho(x) dx = (\rho - 1) \cdot \int_{1}^{+\infty} x^{-\rho+2} dx = \frac{\rho - 1}{\rho - 3} \int_{1}^{+\infty} f_{\rho-2}(x) dx = \frac{\rho - 1}{\rho - 3}, \]

and, as a result of this,

\[ D\hat{\xi} = E\hat{\xi}^2 - (E\hat{\xi})^2 = \frac{\rho - 1}{\rho - 3} - \left(\frac{\rho - 1}{\rho - 2}\right)^2 = \frac{\rho - 1}{(\rho - 2)^2(\rho - 3)} > 0. \quad (3.7.7) \]

Here (3.7.3) was used.

We have several representations of \( c(x) \) in form of improper integrals (see, Sections 1.7 and 1.8), which are a result of following relationship \( c(\rho) = (\zeta(\rho))^{-1} \), where \( \zeta(\rho) \) is Weierstrass’ Zeta Function.

According to Central Limit Theorem for independent, identically distributed random variables \( \{\xi_n\} \): \( \lim_{n \to +\infty} P\left(\frac{S_n - nE\xi_n}{\sqrt{nD\xi}} < x\right) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \) uniformly on \( x \in R^1 \), where \( S_n = \xi_1 + \xi_2 + \cdots + \xi_n, n = 1, 2, \cdots \) (see, for instance, [28], [29], [54], [56]).

If \( \{\xi_n\} \) have the same distribution function as \( \xi \) or \( \hat{\xi} \), then we get a normal approximation for distribution function \( F \) of events’ occurrence number in whole growing network with the corresponding distribution function \( R_\rho \) or \( \hat{R}_\rho \) in fractals of frequency distribution \( \{p_n\} \) or of its continuous analog \( f_\rho(x) \).

The approximation for large \( n \) is presented, for instance, in case of \( \{p_n\} \) in the form

\[ F(x) \approx \Phi\left(\frac{x - n \cdot E\xi}{\sqrt{n \cdot D\xi}} \right) < x, \quad x \in R^+. \quad (3.7.8) \]

This is just the situation when distribution functions \( R_\rho \) and \( \hat{R}_\rho \) belong to the domain of normal attraction of standard Normal Law \( \phi \).

The normal approximation holds also in case \( \rho = 3 \). Indeed, let us denote \( \mu_2(x) = \sum_{n<x} n^2 \cdot p_n \), \( \bar{\mu}_2(x) = \int_{1}^{x} u^2 \cdot f_\rho(u) du = \ln x \) (\( \bar{\mu}_2(x) \) varies slowly at infinity). Let us show that \( \mu_2(x) \) varies slowly at infinity: \( \mu_2(x) = c(\rho) \cdot \sum_{n<x} \frac{1}{n} \approx c(\rho) \cdot \ln x, \quad x \to +\infty \), which proves the statement.

It means that in case \( \rho = 3 \) distribution functions \( R_\rho \) and \( \hat{R}_\rho \) belong to the domain of non-normal attraction of \( \Phi \). This case, as well as the case \( \rho = 2 \), is not so interesting because estimating practically the value of parameters one never can be sure that this value being obtained as an integer is true taking into account estimation’s errors.

### 3.7.2 Stable Approximation

Now, let us consider the cases when \( R_\rho \) and \( \hat{R}_\rho \) belong to the domain of normal attraction of standard Stable Law with exponent \( \alpha = \rho - 1 \in (0, 2) \). Naturally, we deal with Right-side Stable Laws.
Let us start from the case $2 < \rho < 3$. The tail

$$1 - R_{\rho}(x) = \sum_{n \geq x} p_n = c(\rho) \cdot \sum_{n \geq x} n^{-\rho} \approx c(\rho) \cdot \int_{x}^{+\infty} \frac{du}{u^\rho} = \frac{c(\rho)}{\rho-1} \cdot \frac{1}{x^{\rho-1}}, \ x \to +\infty, \quad (3.7.9)$$

varies regularly at infinity with exponent $(-\rho+1)$ and exhibits constant slowly varying component $c(\rho)/\rho-1$.

Let $\{\xi_n\}$ be a sequence of independent, identically distributed random variables with distribution function 

$(3.7.9)$

and $(3.4.2)$, $(3.4.14)-(3.4.15)$ were used and

$$\lim_{n \to +\infty} P(S_n - nE\xi_1 < x) = S_{\rho-1}(x), \quad (3.7.10)$$

where $(3.7.9)$ was used. Moreover,

$$E\eta_n = \left(\frac{\rho-1}{c(\rho)}\right)^{1/(\rho-1)} \cdot E\xi_1, \ n = 1, 2, \ldots \quad (3.7.11)$$

By Theorem 3.2., uniformly on $x \in R^1$,

$$\lim_{n \to +\infty} P(\frac{T_n - nE\eta_1}{((\rho-1)n)^{1/(\rho-1)}} < x) =$$

$$= \lim_{n \to +\infty} P(\frac{S_n - nE\xi_1}{c(\rho) \cdot n^{1/(\rho-1)}} < x) = \lim_{n \to +\infty} P(\frac{c(\rho-1)S_n - n \cdot c(\rho)}{c(\rho-1)(c(\rho)n)^{1/(\rho-1)}} < x) = S_{\rho-1}(x),$$

where $(3.7.4)$ and $(3.7.11)$, $(3.4.2)$, $(3.4.14)-(3.4.15)$ were used and

$$T_n = \eta_1 + \eta_2 + \cdots + \eta_n, \ n = 1, 2, \ldots$$

Thus, the following approximation for distribution function $F$, when $n$ is large,

$$F(x) \approx S_{\rho-1}\left(\frac{c(\rho-1)x - n \cdot c(\rho)}{c(\rho-1)(c(\rho)n)^{1/(\rho-1)}}\right), \ x \in R^+, \quad (3.7.12)$$

is obtained. Here we are in situation when $R_{\rho}$ belongs to the domain of normal attraction of the standard Right-side Stable Law $S_{\rho-1}$.

In case $\rho = 2$ we apply Theorem 3.1. Similar arguments as above, with the help of $(3.7.9)$-$(3.7.11)$ and $(3.4.12)$, $(3.4.16)$ imply that: uniformly on $x \in R^1$

$$\lim_{n \to +\infty} P(\frac{T_n - n \ln n}{n} < x) = \lim_{n \to +\infty} P(\frac{S_n}{c(2)n} - \ln n < x) = S_1(x).$$

Before the approximation in this case shall be written down let us evaluate $c(\rho)$. Due to $(1.8.4)-(1.8.5)$, we have

$$\frac{1}{c(\rho)} = \int_{0}^{+\infty} \frac{\mu^{-1} e^{-t}}{e^{-t}-1} dt, \ 1 < \rho < +\infty.$$
Thus we obtain stable approximation for distribution function $F$, when $n$ is large, in the form

$$F(x) \approx \hat{S}_1\left(\frac{x}{c(2)n} + \ln n\right) = \hat{S}_1\left((\frac{\pi^2}{6} - 1) \cdot \frac{x}{n} + \ln n\right), \quad x \in R^+.$$  

(3.7.13)

Here we are in situation when $R_2$ belongs to the domain of normal attraction of the standard Right-side Stable Law $\hat{S}_1$.

Let us consider the case $1 < \rho < 2$. By Theorem 3.2, uniformly on $x \in R^+$

$$\lim_{n \to +\infty} P\left(\frac{T_n}{((\rho - 1)n)^{1/(\rho - 1)}} < x\right) = \lim_{n \to +\infty} P\left(\frac{S_n}{(c(\rho) \cdot n)^{1/(\rho - 1)}} < x\right) = S_{\rho - 1}(x).$$

This is the most interesting case when $S_{\rho - 1}(x)$ is concentrated on $[0, +\infty)$. Here (3.7.10), (3.4.2), (3.4.14)-(3.4.15) were used.

Thus, the following approximation for distribution function $F$, when $n$ is large, is obtained

$$F(x) \approx S_{\rho - 1}\left(\frac{x}{(c(\rho) \cdot n)^{1/(\rho - 1)}}\right), \quad x \in R^+.$$  

(3.7.14)

Here we are in situation when $R_\rho$ belongs to the domain of normal attraction of the standard Right-side Stable Law $S_{\rho - 1}$.

Finally, we conclude. The complete description on example of Power Law how to apply stable approximation in order to estimate a frequency distribution of events’ occurrence number in growing biomolecular network is done.

3.8 Stable Approximation: The Pareto Distributions

In this Section the situation when frequency distribution in fractals of growing biomolecular network belongs to known families of empirical frequency distribution is considered. Now we deal with Pareto Distributions assuming that their parameters are known.

We automatically realize the way of obtaining stable approximation for distribution function $F$ of events’ occurrence number in whole growing biomolecular network.

We are familiar with how to get stable approximation. But we don’t know how large has to be the size of a network, i.e. how many fractals it has to have in order to get "good" approximation. Thus, the necessity of estimation of rates of convergence in applying Limit Theorems on Stable Laws arises. This problem is deeply investigated in Probability Theory. We’ll give only brief information on this topic.
3.8.1 Evaluating The Characteristics

Remind that the Pareto Distribution \( \{p_n\} \) is defined in Section 1.2:

\[
p_n = c(\rho, b) \cdot (n + b)^{-\rho}, \quad n = 1, 2, \ldots, \tag{3.8.1}
\]

\[
c(\rho, b) = \left( \sum_{n \geq 1} (n + b)^{-\rho} \right)^{-1} \tag{3.8.2}
\]

under constraints on parameters \(-1 < b < +\infty, \ 1 < \rho < +\infty\).

The continuous analog of \( \{p_n\} \) - density \( f_{\rho,b}(x) \) takes the form

\[
f_{\rho,b}(x) = (\rho - 1)(1 + b)^{\rho-1} \cdot (x + b)^{-\rho}, \quad x \in [1, +\infty). \tag{3.8.3}
\]

Denote by

\[
R_{\rho,b}(x) = c(\rho, b) \sum_{n < x} (n + b)^{-\rho}, \quad x \in R^1, \tag{3.8.4}
\]

for \( x \in [1, +\infty) \)

\[
\hat{R}_{\rho,b}(x) = \int_1^x f_{\rho,b}(u) du = (\rho - 1)(1 + b)^{\rho-1} \cdot \int_{1+b}^{x+b} v^{-\rho} dv = 1 - \frac{(1 + b)^{\rho-1}}{(x + b)^{\rho-1}}, \quad x \to +\infty, \tag{3.8.5}
\]

and \( \hat{R}_{\rho,b}(x) = 0 \) for \( x < 1 \), distribution functions of \( \{p_n\} \), \( f_{\rho,b}(x) \), respectively.

Here the formulas (3.8.1)-(3.8.3) were used.

The initial information is done. Let us turn to characteristics’ evaluation.

The functions \( 1 - R_{\rho,b}(x) \) and \( 1 - \hat{R}_{\rho,b}(x) \), \( x \in R^+ \) vary regularly at infinity with the same exponent \((-\rho + 1)\). Moreover, they have constant slowly varying components. Namely,

\[
1 - R_{\rho,b}(x) \approx c(\rho, b) \int_x^{+\infty} \frac{du}{(u + b)^{\rho}} = c(\rho, b) \frac{1}{\rho - 1} \frac{1}{(x + b)^{\rho-1}} \approx \frac{c(\rho, b)}{\rho - 1} x^{-\rho + b}, \quad x \to +\infty, \tag{3.8.6}
\]

and

\[
1 - \hat{R}_{\rho,b}(x) = (\rho - 1)(1 + b)^{\rho-1} \int_x^{+\infty} \frac{du}{(u + b)^{\rho}} \approx (1 + b)^{\rho-1} \cdot x^{-\rho + 1}, \quad x \to +\infty. \tag{3.8.7}
\]

Here (3.8.4)-(3.8.5) were used.

Let us evaluate the moments. If

\[
2 < \rho < +\infty, \tag{3.8.8}
\]

then, due to (3.8.1)-(3.8.2), for \( \{p_n\} \) we have

\[
a := \sum_{n \geq 1} n \cdot p_n = c(\rho, b) \cdot \sum_{n \geq 1} \frac{n}{(n + b)^{\rho}} = c(\rho, b) \cdot \left( \frac{1}{c(\rho - 1, b)} - \frac{b}{c(\rho, b)} \right) = \frac{c(\rho, b)}{c(\rho - 1, b)} \tag{3.8.9}
\]

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If

\[ 3 < \rho + \infty, \]

then

\[
\sum_{n \geq 1} n^2 \cdot p_n = c(\rho, b) \sum_{n \geq 1} \frac{(n + b)^2 - 2 \cdot b \cdot (n + b) + b^2}{(n + b)\rho} = \\
= c(\rho, b) \left( \frac{1}{c(\rho - 2, b)} - \frac{2b}{c(\rho - 1, b)} + \frac{b^2}{c(\rho, b)} \right) = \frac{c(\rho, b)}{c(\rho - 2, b)} - 2b \cdot \frac{c(\rho, b)}{c(\rho - 1, b)} + b^2.
\]

As a result of this and of (3.8.9), we conclude that

\[
\sigma^2 := \sum_{n \geq 1} n^2 \cdot p_n - a^2 = \frac{c(\rho, b)}{c(\rho - 2, b)} - \left( \frac{c(\rho, b)}{c(\rho - 1, b)} \right)^2.
\]

Next, if (3.8.8) holds, then, due to (3.8.3), for \( f_{\rho,b} \) we have

\[
a := \int_1^{+\infty} x \cdot f_{\rho,b}(x)dx = (\rho - 1)(1 + b)\rho - 1 \cdot \int_0^{+\infty} \frac{x dx}{(x + b)\rho} = \\
= (\rho - 1)(1 + b)\rho - 1 \cdot \left( \int_1^{+\infty} \frac{dx}{(x + b)\rho - 1} - b \cdot \int_1^{+\infty} \frac{dx}{(x + b)\rho} \right) = (1 + b) \frac{\rho - 1}{\rho - 2} - b = \\
= \frac{\rho - 1 + b}{\rho - 2}.
\]

If (3.8.10) holds, then

\[
\int_1^{+\infty} x^2 \cdot f_{\rho,b}(x)dx = (\rho - 1)(1 + b)\rho - 1 \cdot \int_1^{+\infty} \frac{(x + b)^2 - 2b(x + b) + b^2}{(x + b)\rho} dx = \\
= (1 + b)^2 \cdot \frac{\rho - 1}{\rho - 3} - 2(1 + b) \frac{\rho - 1}{\rho - 2} + b^2.
\]

As a result of this and of (3.8.12), we conclude that

\[
\sigma^2 := \int_1^{+\infty} x^2 \cdot f_{\rho,b}(x)dx - a^2 = (1 + b)^2 \cdot \frac{\rho - 1}{\rho - 3} - (1 + b)^2 \cdot \left( \frac{\rho - 1}{\rho - 2} \right)^2 = \\
= (1 + b)^2 \cdot \frac{\rho - 1}{(\rho - 2)^2 \cdot (\rho - 3)}.
\]

Easily seen that (3.8.9) and (3.8.12), (3.8.11) and (3.8.13) for \( b = 0 \) are reduced to (3.7.4) and (3.7.6), (3.7.5) and (3.7.7) respectively.

Finally, let us remind that we have several representations of \( c(\rho, b) \) in form of improper integrals (see, Sections 1.7 and 1.8), which are a result of following relationships

\[
c(\rho, b) = (\zeta(\rho, b) - (1/b))^{-1} \text{ for } b \in \mathbb{R}^+ \text{ and } c(\rho, b) = (\zeta(\rho, 1 - b))^{-1} \text{ for } -1 < b < 0, \text{ where } \zeta(\rho, b) \text{ is Riemann’s Zeta Function.}
\]

### 3.8.2 Applying The Stable Approximation

Since the procedure of stable approximation’s application is described in detail in Section 3.7, therefore below we’ll only formulate the results for Pareto Distributions.
Let $F$ be a distribution function of the number of events’ occurrence in whole growing biomolecular network and the corresponding distribution function in fractals of this network is $R_{\rho,b}(x) = P(\xi < x)$.

If (3.8.10) takes place, then the first two moments of random variable $\xi$ are finite, i.e. $a < +\infty, \sigma^2 < +\infty$. The Central Limit Theorem takes place and, as a result of this, the following approximation for $F$, when $n$ is large, holds (see, (3.7.8) and (3.8.9), (3.8.11))

$$F(x) \approx \Phi\left(\frac{x - n \cdot \left(\frac{c(\rho,b)}{c(\rho-1,b)} - b\right)}{\sqrt{n \cdot \left(\frac{c(\rho,b)}{c(\rho-2,b)} - \left(\frac{c(\rho,b)}{c(\rho-1,b)}\right)^2\right)}}\right). \quad (3.8.14)$$

In this situation the distribution function $R_{\rho,b}$ belongs to the domain of normal attraction of standard Normal Law $\Phi$.

We illustrated how to use normal approximation, let us discuss when.

The answer gives results on rate of convergence in Central Limit Theorem. Such results have more simple formulations and less restrictions when the distribution of $\{\xi_n\}$ in Central Limit Theorem doesn’t belong to lattice distributions.

**Definition 20.** We say that random variable $\xi$ has a lattice distribution if it is restricted to values of the form $b, b \pm h, b \pm 2h, \cdots$.

Empirical frequency distributions, in particular, Pareto Distributions are lattice distributions.

In order to avoid this difficulty we may consider the continuous analog of Pareto Distributions, i.e. the density $f_{\rho,b}(x)$, and formulate the analog of approximation (3.8.14) in this case. Then, $a$ and $\sigma^2$ taken from (3.8.9) and (3.8.11) and serving as centring and norming constants being multiplied by $n$ in (3.8.14), are replaced by $a$ and $\sigma^2$ taken from (3.8.12) and (3.8.13). After that we’ll found for approximation in this last case the required number $n$, starting from which the approximation is ”good”. We may declare that the same number is ”suitable” for the case of approximation (3.8.14) because $a$ and $\sigma^2$ in (3.8.14) are discretizations of $a$ and $\sigma^2$ of the last case. Thus, we may deal with results on rates of convergence assuming that $\{\xi_n\}$’s distribution function is not generated by lattice distribution.

Let us assume that $E\xi_n = 0, n = 1, 2, \cdots$, which is not true in case of empirical frequency distribution’s continuous analog, but we’ll consider instead of $\{\xi_n\}$ a sequence $\{\xi_n - E\xi_n\}$.

The following result was discovered by A. Berry and G. Esseen independently (for information, see, [23], [55], [56]).

**The Berry-Esseen’s Inequality**

Let $\{\xi_n\}$ be a sequence of independent, identically distributed random variables with

$$E\xi_1 = 0, \; \hat{\rho} := E|\xi_1|^3 < +\infty. \quad (3.8.15)$$

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Then, denoting by $\sigma^2 = E\xi_1^2$, for all $x$ and $n$ the inequality holds
\[
|P\left(\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{\sigma \sqrt{n}} < x \right) - \Phi(x)| < \frac{33}{4} \frac{\hat{\rho}}{\sigma^3 \sqrt{n}}.
\] (3.8.16)

Here the upper bound is independent on individual distribution function of $\{\xi_n\}$.

Thus, in case $4 < \rho < +\infty$ for Pareto Distributions this statement allows to find out required number $n$ in (3.8.14).

There is a huge number of publications devoted to the constant’s improvement in (3.8.16), to reduction of order of finite absolute moment in (3.8.15) until any order $2 + \varepsilon$, $\varepsilon > 0$, etc.

The normal approximation holds also in case $\rho = 3$.

Indeed, let us denote $\mu_2(x) = \sum_{n<x} n^2 \cdot p_n$, $\hat{\mu}_2(x) = \int_1^x u^2 \cdot f_{\rho,b}(u)du$. Then
\[
\mu_2(x) = c(\rho, b) \cdot \sum_{n<x} \frac{n^2}{(n+b)^3} = c(\rho, b) \sum_{n<x} \frac{(n + b)^2 - 2b(n + b) + b^2}{(n + b)^3} \approx \\
\approx c(\rho, b) \cdot \sum_{n<x} \frac{1}{n + b} \approx c(\rho, b) \ln x, \ x \rightarrow +\infty,
\] (3.8.17)

and
\[
\hat{\mu}_2(x) = (\rho - 1)(1 + b)\rho^{-1} \cdot \int_1^x \frac{1}{u + b} \left(1 - \frac{2b}{u + b} + \frac{b^2}{(u + b)^2}\right)du \approx \\
\approx (\rho - 1)(1 + b)\rho^{-1} \cdot \ln x, \ x \rightarrow +\infty.
\]

Thus, $\mu_2(x)$ and $\hat{\mu}_2(x)$ vary slowly at infinity, which means that $R_{\rho,b}$ and $\hat{R}_{\rho,b}$ belong to the domain of non-normal attraction of $\phi$.

In case $2 < \rho < 3$, similarly to Power Law, uniformly on $x \in R^1$
\[
\lim_{n \rightarrow +\infty} P\left(\frac{S_n - n \cdot E\xi_1}{c(\rho, b) \cdot n^{1/(\rho - 1)}} < x \right) = \\
= \lim_{n \rightarrow +\infty} P\left(\frac{c(\rho - 1, b)S_n - n \cdot (c(\rho, b) - b \cdot c(\rho - 1, b))}{c(\rho - 1, b) \cdot (c(\rho, b) \cdot n)^{1/(\rho - 1)}} < x \right) = S_{\rho - 1}(x),
\]
where applying Theorem 3.2 the formulas (3.8.9) and (3.8.17) were used.

Thus, the following approximation for distribution function $F$, when $n$ is large and $2 < \rho < 3$,
\[
F(x) \approx S_{\rho - 1}\left(\frac{c(\rho - 1, b) \cdot x - n \cdot (c(\rho, b) - b \cdot c(\rho - 1, b))}{c(\rho - 1, b) \cdot (c(\rho, b) \cdot n)^{1/(\rho - 1)}}\right), \ x \in R^+,
\] (3.8.18)
is obtained. Here we are in situation when $R_{\rho,b}$ belongs to the domain of normal attraction of the standard Right-side Stable Law $S_{\rho - 1}$.

In case $\rho = 2$, by Theorem 3.2, similarly to Power Law, we obtain the following approximation for distribution function $F$, when $n$ is large:
\[
F(x) \approx \hat{S}_1\left(\frac{x}{c(2,b)n} + \ln n\right), \ x \in R^+.
\] (3.8.19)
Results on rates of convergence for these cases also exist in corresponding literature. The remained case \(1 < \rho < 2\) leads to the approximation (3.7.13), where \(c(\rho)\) simply is replaced by \(c(\rho, b)\).

### 3.9 Waring Distributions: The Moments of Integer Order

Waring Distributions are widely used as frequency distributions of various events’ occurrence number in many large-scale biomolecular systems. For Waring Distributions conditions on moments’ convergence/divergence easily follows from their property on regular variation with constant slowly varying component.

In this section under the conditions on moments convergence a method on integer order moments’ evaluation for Waring Distributions is suggested.

#### 3.9.1 On Moments’ Existence

We say that the random variable \(\xi\) of events exhibits the Waring Distribution \(\{p_n(p, q)\}\) if

\[
p_0(p, q) = P(\xi = 0) = \left(1 + \sum_{n \geq 1} \prod_{K=1}^{n} \frac{p + K - 1}{q + K}\right)^{-1},
\]

\[
p_n(p, q) = P(\xi = n) = p_0 \prod_{K=1}^{n} \frac{p + K - 1}{q + K}, \quad n = 1, 2, \ldots,
\]

under the constraints

\[
0 < p < q < +\infty.
\]

In section 1.2 it was shown that

\[
p_0 = (1 - \frac{p}{q}) = p_0(p, q).
\]

Let us denote

\[
\rho = q - p + 1.
\]

We already know (see, sections 1.3 and 1.7) that the Waring Distribution \(\{p_n(p, q)\}\) given by formulas (3.9.1)-(3.9.3) varies regularly at infinity with exponent \((-\rho)\), where number \(\rho\) is given by (3.9.4) and exhibits constant slowly varying component

\[
L = \lim_{n \to +\infty} L(n) = (\rho - 1) \frac{\Gamma(q)}{\Gamma(p)}.
\]

Here \(L(n) = n^\rho \cdot p_n(p, q)\) varies slowly and \(\Gamma(\cdot)\) is Euler’s Gamma Function.
Let us denote \( q_n = p_n + p_{n+1} + \cdots, \ n = 1, 2, \cdots \). Then
\[
q_n \approx \frac{1}{n^{\rho-1}} \frac{\Gamma(q)}{\Gamma(p)}, \ n \to +\infty,
\]
(3.9.6)
because from (3.9.5) we have \( p_n \approx \frac{e^{-1}}{n^\rho} \frac{\Gamma(q)}{\Gamma(p)}, \ n \to +\infty. \)

Now, for any \( \nu \in R^+ \) let us denote by \( m_{\nu}(p, q) = \sum_{n \geq 1} n^{\nu} \cdot p_n(p, q) \) the moment of order \( \nu \) for Waring Distribution \( \{p_n(p, q)\} \). Then, from (3.9.6) it follows that
\[
m_{\nu}(p, q) \begin{cases} < +\infty & \text{if } 0 < \nu < \rho - 1, \\ = +\infty & \text{if } \rho - 1 \leq \nu < +\infty. \end{cases}
\]
(3.9.7)

At once, due to (3.9.7), a problem of moments’ explicit evaluation arises, when the condition \( 0 < \nu < \rho - 1 \) of moments’ convergence holds.

In this Section a manner is suggested which allows to evaluate \( m_{\nu}(p, q) \) for
\[
\nu = 1, 2, \cdots, \lfloor \rho - 1 \rfloor \ \text{if } \rho \text{ is not an integer},
\]
(3.9.8)
and for
\[
\nu = 1, 2, \cdots, \rho - 1 \ \text{if } \rho \text{ is an integer},
\]
(3.9.9)
so, (3.9.8) includes (3.9.9). Here \( \lfloor x \rfloor \) denotes an integer part of positive number \( x \).

### 3.9.2 Mean Value and Variance

Let us denote by \( \sigma^2 = D\xi = m_2(p, q) - (m_1(p, q))^2 \) (if \( m_1(p, q) \) is finite) the variance of Waring Distribution \( \{p_n(p, q)\} \).

**Theorem 3.3**

(a) Let the condition hold
\[
0 < p < q - 1 < +\infty \ (\text{or } 0 < p < +\infty \text{ and } 2 < \rho < +\infty).
\]
(3.9.10)

Then
\[
0 < m_1(p, q) = \frac{p}{q - p - 1} = \frac{p}{\rho - 2} < +\infty.
\]
(3.9.11)

(b) Let the condition hold
\[
0 < p < q - 2 < +\infty \ (\text{or } 0 < p < +\infty \text{ and } 3 < \rho < +\infty).
\]
(3.9.12)

Then
\[
\sigma^2(p, q) = \frac{(\rho - 1) \cdot p \cdot (p + \rho - 2)}{(\rho - 3) \cdot (\rho - 2)^2}.
\]
(3.9.13)
In order to prove Theorem 3.3, a manner is suggested which is based on two simple ideas. The first one is easy to explain on example of mean value’s evaluation.

Let us assume that the condition (3.9.10) holds and with the help of formulas (3.9.1) and (3.9.3) present the probability \( p_{n+1}(p, q - 1) \), \( n = 0, 1, 2 \cdots \), in the following form

\[
p_{n+1}(p, q - 1) = (1 - \frac{p}{q - 1})^\frac{1}{q} (p + n) \cdot \prod_{K=1}^{n} \frac{p + K - 1}{q + K}.
\]  

(3.9.14)

In order to get the equality (3.9.14), from the product at the right-hand-side of (3.9.1) the last multiplier of numerator and the first one of denominator are extracted.

Due to (3.9.1), (3.9.3) and (3.9.14), for \( n = 0, 1, 2 \cdots \) we obtain the following equality

\[
(1 - \frac{p}{q - 1})p_{n+1}(p, q - 1) = (1 - \frac{p}{q - 1})^\frac{1}{q} (p + n) \cdot p_n(p, q).
\]  

(3.9.15)

Taking into account formula (3.9.3) and norming conditions

\[
\sum_{n \geq 0} p_n(p, q - 1) = \sum_{n \geq 0} p_n(p, q) = 1,
\]

which for Waring Distributions \( \{p_n(p, q - 1)\} \) and \( \{p_n(p, q)\} \) are fulfilled under the constraints (3.9.10), we come to the equality

\[
(1 - \frac{p}{q - 1})\frac{p}{q - 1} = (1 - \frac{p}{q - 1})^\frac{1}{q} (p + n) \cdot p_n(p, q)
\]

by summarizing both sides of (3.9.15) over \( n = 1, 2, \cdots \).

Thus, we obtain \( \frac{p}{q - 1} = (1 - \frac{p}{q - 1})m_1(p, q) \), which implies (3.9.11).

By the same way, just a little longer, the variance of Waring Distributions may be evaluated. Indeed, let us assume that the condition (3.9.12) holds. Multiplying both sides of equality (3.9.15) by \( n \) and summarizing over \( n = 1, 2, \cdots \) we come to the following equality

\[
(1 - \frac{p}{q - 1})\sum_{n \geq 1} n \cdot p_{n+1}(p, q - 1) = (1 - \frac{p}{q - 1})^\frac{1}{q} m_1(p, q) + (1 - \frac{p}{q - 1})^\frac{1}{q} m_2(p, q).
\]  

(3.9.16)

The evaluation of the left-hand-side in (3.9.16) already shall include an element of the second simple idea (namely, number \( n \) is presented in the form \( (n + 1) - 1 \)). By (3.9.3) and (3.9.11), we proceed

\[
\sum_{n \geq 1} n \cdot p_{n+1}(p, q - 1) = \sum_{n \geq 2} np_n(p, q - 1) - \sum_{n \geq 2} p_n(p, q - 1) = m_1(p, q - 1) - \sum_{n \geq 1} p_n(p, q - 1) = m_1(p, q - 1) - (1 - p_0(p, q - 1)) = \]

\[
= \frac{p}{q - p - 2} - \frac{p}{q - 1} = \frac{p(p + 1)}{(q - p - 2)(q - 1)}.
\]

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That is why we have

$$\left(1 - \frac{p}{q}\right) \sum_{n \geq 1} n \cdot p_{n+1}(p, q - 1) = \frac{q - p}{q - p - 2} \cdot \frac{p + 1}{q}. \quad (3.9.17)$$

Formulas (3.9.11), (3.9.16) and (3.9.17) imply that the following equality holds

$$m_2(p, q) = \frac{p}{q - p - 1} (\frac{(q - p)(p + 1)}{q - p - 2} - p) = m_1(p, q) \cdot \frac{q + p}{q - p - 2}. \quad (3.9.18)$$

It is of interest to mention that under the condition (3.9.12) if \(q - p \downarrow 2\) (which is equivalent to \(\rho \downarrow 2\)), then \(m_1(p, q)\) stays finite (it converges to \(m_1(p, p + 2)\)) but, due to (3.9.18), \(m_2(p, q) \uparrow +\infty\).

From (3.9.11) and (3.9.18) the expression for the variance of Waring Distribution is derived

$$m_2(p, q) - (m_1(p, q))^2 = m_1(p, q) \left(\frac{q + p}{q - p - 2} - m_1(p, q)\right) =$$

$$= m_1(p, q) \frac{(q - p)(q - 1)}{(q - p - 2)(q - p - 1)} = \frac{p(q - p)(q - 1)}{(q - p - 2)(q - p - 1)^2},$$

which implies (3.9.13). Theorem 3.3 is proved.

### 3.9.3 Moments of Integer Orders

Let us present a manner of moments’ evaluation, elements of which already were illustrated on examples of mean value and variance. Let us assume that

$$0 < p < q - s < +\infty \text{ (or } 0 < p < +\infty \text{ and } s + 1 < \rho < +\infty), \quad (3.9.19)$$

where \(s \geq 3\) is a finite integer.

Multiplying both sides of (3.9.15) by \(n^{s-1}\) and summarizing over \(n = 1, 2, \cdots\) we come to the following equality

$$\left(1 - \frac{p}{q + 1}\right) \cdot \frac{1}{q} m_s(p, q) = \left(1 - \frac{p}{q}\right) \sum_{n \geq 1} n^{s-1} \cdot p_{n+1}(p, q - 1) - \left(1 - \frac{p}{q - 1}\right) \cdot \frac{p}{q} \cdot m_{s-1}(p, q). \quad (3.9.20)$$

Thus, due to (3.9.20), in order to get recurrent relation for moments of Waring Distributions \(\{p_n(p, q)\}\) and \(\{p_n(p, q - 1)\}\), the first term at the right-hand-side of (3.9.20) has to be transformed. Here we use the second idea being presented before. We write \(n^{s-1}\) in the form \(n^{s-1} = ((n + 1) - 1)^{s-1} = \sum_{r=0}^{s-1} \binom{s-1}{r}(-1)^{s-r-1}(n + 1)^r\), where

$$\binom{s-1}{r} = \frac{(s-1)!}{r!(s-1-r)!}, \quad r = 0, 1, \cdots, s-1,$$
are so-called binomial coefficients. Then
\[
(1 - \frac{p}{q}) \sum_{n \geq 1} n^{s-1} \cdot p_{n+1}(p, q - 1) = (1 - \frac{p}{q}) \cdot \sum_{n \geq 2} \left( \sum_{r=0}^{s-1} \left( \frac{s-1}{r} \right) (-1)^{s-r-1} \cdot n^r \right) \cdot p_n(p, q - 1) =
\]
\[
= (1 - \frac{p}{q}) \cdot \sum_{r=0}^{s-1} \left( \frac{s-1}{r} \right) (-1)^{s-r-1} \cdot (m_r(p, q - 1) - p_n(p, q - 1)) =
\]
\[
(1 - \frac{p}{q}) \cdot \sum_{r=0}^{s-1} \left( \frac{s-1}{r} \right) (-1)^{s-r-1} \cdot m_r(p, q - 1),
\]
(3.9.21)

because \(\sum_{r=0}^{s-1} \left( \frac{s-1}{r} \right) (-1)^{s-r-1} = (1 - 1)^{s-1} = 0\). In (3.9.21) we have to define \(m_0(p, q - 1)\). Namely, by (3.9.3), we obtain \(m_0(p, q - 1) = \sum_{n \geq 1} p_n(p, q - 1) = 1 - p_0(p, q - 1) = \frac{q}{q-1}\).

That is why (3.9.21) may be rewritten in the form
\[
(1 - \frac{p}{q}) \sum_{n \geq 1} n^{s-1} \cdot p_{n+1}(p, q - 1) = (-1)^{s-1}(1 - \frac{p}{q}) \cdot \frac{p}{q - 1} + (1 - \frac{p}{q}) m_{s-1}(p, q - 1) +
\]
\[
+ (1 - \frac{p}{q}) \cdot \sum_{r=0}^{s-2} \left( \frac{s-1}{r} \right) (-1)^{s-r-1} \cdot m_r(p, q - 1).
\]

Substituting the last equality into (3.9.20) we come to the following recurrent equation

for moments
\[
(1 - \frac{p}{q}) \frac{1}{q} m_s(p, q) = (-1)^{s-1}(1 - \frac{p}{q}) \cdot \frac{p}{q - 1} + ((1 - \frac{p}{q}) m_{s-1}(p, q - 1) - (1 - \frac{p}{q}) \frac{p}{q - 1} m_{s-1}(p, q)) +
\]
\[
+ (1 - \frac{p}{q}) \cdot \sum_{r=1}^{s-2} \left( \frac{s-1}{r} \right) (-1)^{s-r-1} \cdot m_r(p, q - 1).
\]
(3.9.22)

This recurrent equation already has been obtained for \(s = 2\) in evaluation of the variance. It allows to evaluate step by step all finite moments of integer order of Waring Distributions starting from the value \(s = 3\). Then, as initial conditions for this equation the conditions (3.9.11) and (3.9.18) have to be taken.

The manner presented above may also be used for the truncated moments evaluation of integer order. It is of interest when the usual moments of the same order are infinite. This is just the object of our attention in the next Section.

### 3.10 Waring Distributions: The Truncated Moments

Let \(R_{p,q}(x)\) be a distribution function which corresponds to Waring Distribution \(\{p_n(q, p)\}\) defined by (3.9.1)-(3.9.3). For a given \(x \in \mathbb{R}^+\) let us denote

\[
\mu_1(x) := \int_0^x u dR_{p,q}(x) = \sum_{n=1}^{x} n \cdot p_n(p, q),
\]
(3.10.1)
and
\[
\mu_2(x) := \int_{0-}^{x+} u^2 dR_{p,q}(x) = \sum_{n=1}^{[x]} u^2 \cdot p_n(p,q).
\tag{3.10.2}
\]

If \( \rho = 2 \) (or \( q = p + 1 \)), then the first moment of distribution \( \{p_n(p,p+1)\} \) is infinite and the asymptotic behavior of the truncated moment of order 1 given by (3.10.1) as \( x \to +\infty \) is of interest. If \( \rho = 3 \) (or \( q = p + 2 \)), then the second moment of distribution \( \{p_n(p,p+2)\} \) is infinite and the asymptotic behavior of the truncated moment of order 2 given by (3.10.2) is of interest.

In this Section by using the manner being suggested in Section 3.9 we evaluate introduced truncated moments and investigate their asymptotic behavior.

### 3.10.1 Evaluation of \( \mu_1(x) \), \( x \in R^+ \)

From (3.9.1) and (3.9.3) we conclude for case \( \rho = 2 \) (or \( q = p + 1 \))
\[
\begin{cases}
p_0(p,p+1) = \frac{1}{p+1}, \\
p_n(p,p+1) = \frac{p}{(p+n)(p+n+1)}, \quad n = 1, 2, \ldots
\end{cases}
\tag{3.10.3}
\]

Below \( s \geq 3 \) denotes a finite integer. We need to evaluate the sum \( \sum_{n=1}^{s} n \cdot p_n(p,p+1) \), which may be done with the help of formula (3.10.3). Indeed,
\[
\sum_{n=1}^{s} n \cdot p_n(p,p+1) = p \cdot \sum_{n=1}^{s} \frac{p+n}{(p+n)(p+n+1)} - p^2 \cdot \sum_{n=1}^{s} \frac{1}{(p+n)(p+n+1)} = \\
= p \cdot \sum_{n=1}^{s} \frac{1}{p+n+1} - p \cdot \sum_{n=1}^{s} p_n(p,p+1).
\]

Next,
\[
\sum_{n=1}^{s} p_n(p,p+1) = \sum_{n=1}^{s} \frac{p}{(p+n)(p+n+1)} = p \cdot \sum_{n=1}^{s} \left( \frac{1}{p+n} - \frac{1}{p+n+1} \right) = \\
= \frac{p}{p+1} - \frac{p}{p+s+1}.
\]

The obtained equalities imply
\[
\mu_1(s) = \sum_{n=1}^{s} n \cdot p_n(p,p+1) = p \cdot \left( \sum_{n=1}^{s} \frac{1}{p+n+1} - \frac{p}{p+1} + \frac{p}{s+p+1} \right) = \\
= p \cdot \left( \sum_{n=1}^{s+1} \frac{1}{p+n} - 1 + \frac{p}{s+p+1} \right),
\tag{3.10.4}
\]

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It is clear that the main term of asymptotics of $\mu_1(s)$ as $s \to +\infty$ in representation (3.10.4) is included into expression

$$p \cdot \sum_{n=1}^{s+1} \frac{1}{p + n}. \quad (3.10.5)$$

Using the well-known equivalency $\sum_{n=1}^{s} \frac{1}{n} \approx \ln s, \ s \to +\infty$, we find out two-side estimations of (3.10.5). Namely,

$$\left( p \right) + s + 2 \sum_{n=1}^{s+1} \frac{1}{n + n} < \sum_{n=1}^{s+1} \frac{1}{p + n} < \sum_{n=1}^{s+1} \frac{1}{p + n} = \sum_{n=1}^{s+1} \frac{1}{n} - \sum_{n=1}^{s} \frac{1}{n},$$

or for any given $\varepsilon \in (0, 1)$ there is an integer $s_0 > 1$ such that for $s = s_0, s_0 + 1, \cdots$

$$(1 - \varepsilon) \ln([p] + s + 2) \leq \sum_{n=1}^{s+1} \frac{1}{p + n} < (1 + \varepsilon) \ln([p] + s + 1).$$

Since $\lim_{s \to +\infty} \frac{\ln([p] + s + 1)}{\ln s} = \lim_{s \to +\infty} \frac{\ln([p] + s + 2)}{\ln s} = 1$, therefore there is an integer $s_1 > s_0$ such that $(1 - \varepsilon)^2 \ln s < \sum_{n=1}^{s+1} \frac{1}{p + n} < (1 + \varepsilon)^2 \ln s$ for $s = s_1, s_1 + 1, \cdots$ and $(1 - \varepsilon)^2 \leq \lim_{s \to +\infty} \left( (\sum_{n=1}^{s+1} \frac{1}{p + n}) / \ln s \right) \leq \lim_{s \to +\infty} \left( (\sum_{n=1}^{s+1} \frac{1}{p + n}) / \ln s \right) < (1 + \varepsilon)^2$.

Tending $\varepsilon \downarrow 0$ we prove the following equivalency

$$\mu_1(s) \approx p \cdot \ln s, \ s \to +\infty. \quad (3.10.6)$$

In case when $p$ is an integer more terms of asymptotics of $\mu_1(s)$ as $s \to +\infty$ is possible to derive. Indeed, let us assume that:

$$p \text{ is an integer}. \quad (3.10.7)$$

Then (3.10.4) may be rewritten as follows

$$\mu_1(s) = p \cdot \sum_{n=1}^{s+p+1} \frac{1}{n} - p \cdot A_1(p) + \frac{p}{s + p + 1}. \quad (3.10.8)$$

where

$$A_1(p) = 1 + \sum_{n=1}^{p} \frac{1}{n}. \quad (3.10.9)$$

Due to 0.131, p.2, [20], for any positive integer $s$

$$\sum_{n=1}^{s} \frac{1}{n} = \gamma + \ln s + \frac{1}{2s} + O\left( \frac{1}{s^2} \right), \quad (3.10.10)$$
where $\gamma$ is the famous Euler’s constant and $O(x)$ means that we have an expression for which there is an upper bound of the form $x \cdot \text{const.}

Let us make preliminary estimations

$$
\ln(s + p + 1) = \ln s + \ln(1 + \frac{p + 1}{s}), \quad \ln(s + p + 1) - \ln s \approx -\frac{p + 1}{s}, \quad s \to +\infty,
$$

$$
\frac{1}{s + p + 1} - \ln s \approx \frac{1}{s}, \quad s \to +\infty.
$$

Basing on these equivalencies, from (3.10.8) and (3.10.10) we obtain

$$
\mu_1(s) - p \cdot \ln s \approx p \cdot (\gamma - A_1(p)), \quad s \to +\infty,
$$

$$
\mu_1(s) - p \cdot \ln s - p \cdot (\gamma - A_1(p)) \approx \frac{p}{2s}, \quad s \to +\infty.
$$

Combining the obtained results we may formulate the following

**Theorem 3.4**

Let $q - p = 1$ (or $\rho = 2$). Then:

1) $\mu_1(s) = p \cdot \sum_{n=1}^{s+p+1} \frac{1}{n} - p \cdot A_1(p) + \frac{p}{s+p+1}, \quad s = 1, 2, \cdots$, where $A_1(p)$ is given by (3.10.9),

2) $\mu_1(s) \approx p \cdot \ln s, \quad s \to +\infty$;

3) If condition (3.10.7) holds, then

$$
\mu_1(s) = p \cdot \left(\ln s + (\gamma - A_1(p)) + \frac{1}{2s}\right) + O\left(\frac{1}{s^2}\right), \quad s \to +\infty,
$$

where $\gamma$ is the Euler’s constant.

### 3.10.2 Evaluation of $\mu_2(x), \ x \in R^+$

From (3.9.1) and (3.9.3) we conclude

$$
\begin{cases}
p_0(p, p + 2) = \frac{2}{p+2}, \\
p_n(p, p + 2) = \frac{2p(p+1)}{(p+n)(p+n+1)(p+n+2)}, \quad n = 1, 2, \cdots
\end{cases}
$$

(3.10.11)

For $n = 0, 1, 2, \cdots$ the equality (3.9.15) is transformed into the following one

$$
2(p + 1)p_{n+1}(p, p + 1) = (p + n)p_n(p, p + 2),
$$

(3.10.12)

which may also be verified directly with the help of (3.10.3) and (3.10.11).

With the help of (3.10.12) we may write down the following equalities

$$
\sum_{n=1}^{s} n^2 \cdot p_n(p, p + 2) = 2(p + 1)\left(\sum_{n=1}^{s+1} n \cdot p_n(p, p + 1) - \sum_{n=1}^{s+1} p_n(p, p + 1)\right) -
$$

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\[-p \cdot \sum_{n=1}^{s} n \cdot p_n(p, p + 2)\]

(at the right-hand-side under the signs of sums in brackets the first terms coincide), and

\[
\sum_{n=1}^{s} n \cdot p_n(p, p + 2) = 2(p + 1) \left( \sum_{n=1}^{s+1} p_n(p, p + 1) - \frac{p}{(p + 1)(p + 2)} \right) - p \cdot \sum_{n=1}^{s} p_n(p, p + 2)
\]

(at the right-hand-side under the sign of a sum in brackets the first term, due to (3.10.3), equals to \(\frac{p}{(p+1)(p+2)}\)). The obtained equalities imply

\[
\sum_{n=1}^{s} n^2 \cdot p_n(p, p + 2) = p^2 \cdot \sum_{n=1}^{s} p_n(p, p + 2) + 2(p + 1) \sum_{n=1}^{s} n \cdot p_n(p, p + 1) - 2(p + 1)^2 \cdot \sum_{n=1}^{s+1} p_n(p, p + 1) + \frac{2p^2}{p + 2},
\]

\[\text{(3.10.13)}\]

We have

\[
\sum_{n=1}^{s} p_n(p, p + 2) = 2p(p + 1) \sum_{n=1}^{s} \frac{1}{(p + n)(p + n + 1)(p + n + 2)} =
\]

\[
= 2p(p + 1) \sum_{n=1}^{s} \left( \frac{1}{p + n} - \frac{1}{p + n + 1} \right) \frac{1}{p + n + 2} =
\]

\[
= 2p(p + 1) \left( \frac{1}{2p + 1} + \frac{1}{2p + 2} - \frac{1}{2p + s + 1} - \frac{1}{2p + s + 2} \right) =
\]

\[
= p(p + 1) \left( \frac{1}{p + 1} - \frac{1}{p + 2} - \frac{1}{p + s + 1} + \frac{1}{p + s + 2} \right) =
\]

\[
= \frac{p}{p + 2} - \frac{p(p + 1)}{(s + p + 1)(s + p + 2)},
\]

\[\text{(3.10.14)}\]

where (3.10.10) was used.

The sums \(\sum_{n=1}^{s} n \cdot p_n(p, p + 1)\) and \(\sum_{n=1}^{s} p_n(p, p + 1), s = 1, 2, \cdots\), were evaluated in 3.10.1 of this Section. According to expressions for these sums, where \(s\) has to be replaced by \(s + 1\), and to (3.10.14), we evaluate \(\mu_2(s)\) given by formula (3.10.13).

\[
\mu_2(s) = 2p \cdot (p + 1) \left( \sum_{n=1}^{s+2} \frac{1}{p + n} - \frac{3p + 4}{2(p + 1)} + \frac{2p + 1}{s + p + 2} \right) - \frac{p^2}{2(s + p + 1)(s + p + 2)}.\]

\[\text{(3.10.15)}\]

Similarly to case \(\mu_1(s)\), here also the main term of asymptotics of \(\mu_2(s)\) as \(s \to +\infty\) in representation (3.10.15) is included into expression \(2p \cdot (p + 1) \cdot \sum_{n=1}^{s+2} \frac{1}{p + n}\).
Similarly to $\mu_1(s)$, we come to equivalency (see, (3.10.6))

$$\mu_2(s) \approx 2p(p + 1) \cdot \ln s, \ s \to +\infty. \quad (3.10.16)$$

If condition (3.10.7) holds, then more terms of asymptotics of $\mu_2(s)$ as $s \to +\infty$ is possible to derive. Indeed, let us rewrite (3.10.15) in the form

$$\mu_2(s) = 2p(p + 1) \cdot \sum_{n=1}^{s+p+2} \frac{1}{n} - 2p(p + 1) \cdot A_2(p) + \frac{1}{s} \cdot 2p(p + 1)(2p + 1) + O\left(\frac{1}{s^2}\right), \ s \to +\infty, \quad (3.10.17)$$

where

$$A_2(p) = \frac{3p + 4}{2(p + 1)} + \sum_{n=1}^{p} \frac{1}{n}. \quad (3.10.18)$$

Now, by (3.10.10), similarly to $\mu_1(s)$, from (3.10.17) we obtain following equivalencies

$$\mu_2(s) - 2p(p + 1) \ln s \approx 2p(p + 1)(\gamma - A_2(p)), \ s \to +\infty,$$

$$\mu_2(s) - 2p(p + 1) \ln s - 2p(p + 1)(\gamma - A_2(p)) \approx 2p(p + 1)(2p - 1), \ s \to +\infty.$$

Combining the obtained results we may formulate the following

**Theorem 3.5**

Let $q - p = 2$ (or $\rho = 3$). Then:

1) $\mu_2(s) = 2p(p + 1)\left(\sum_{n=1}^{s+p+2} \frac{1}{n} - A_2(p) + \frac{2p+1}{s+p+2} - \frac{p^2}{2(s+p+1)(s+p+2)}\right)$, where $A_2(p)$ is given by (3.10.18);

2) $\mu_2(s) \approx 2p(p + 1) \ln s, \ s \to +\infty$;

3) If condition (3.10.17) holds, then

$$\mu_2(s) = 2p(p + 1)(\ln s + (\gamma - A_2(p)) + \frac{2p - 1}{2s}) + O\left(\frac{1}{s^2}\right), \ s \to +\infty.$$

### 3.11 Stable Approximation: The Waring Distributions

In this Section the situation when frequency distribution in fractals of growing biomolecular network belongs to known families of empirical frequency distributions is considered. Now we deal with the family of Waring Distributions assuming that its parameters are known. We automatically realize the way of obtaining stable approximation for distribution function $F$ of events’ occurrence number in whole growing biomolecular network.
3.11.1 Applying The Normal Approximation

Let us formulate the results for distribution function $F$ in case of Waring Distributions taken as empirical frequency distribution in fractals.

Let $3 < \rho < +\infty$. Then the first moment $a$ (mean value) and variance $\sigma^2$ are finite, the Central Limit Theorem takes place and $F(x) \approx \Phi\left(\frac{x - na}{\sigma\sqrt{n}}\right)$ for large $n$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$, $a$ and $\sigma$ are given by Theorem 3.3.

Thus, $R$ belongs to the domain of normal attraction of $\Phi$, where $R_{p,q}$ is a distribution function of Waring Distribution $\{p_n(p,q)\}$.

Let $\rho = 3$. In this case for Waring Distribution $\{p_n(p,q)\}$ we have $q = p + 2$ and the truncated moment of order 2, due to Theorem 3.5, exhibits the following asymptotic behavior

$$\mu_2(s) \approx 2p(p+1) \ln s, \ s \to +\infty. \quad (3.11.1)$$

Asymptotic relation (3.11.1) says that $\mu_2(s)$ varies slowly at infinity. At the same time, the distribution function $R_{p,p+2}$ of Waring Distribution $\{p_n(p,p+2)\}$ is not concentrated at one point. So, the conditions of following statement formulated below in our particular case (see, Section 3.3) are fulfilled.

$$R_{p,p+2} \text{ being not concentrated at one point belongs to the domain of attraction of } \Phi \text{ iff } \mu_2(x) \text{ varies slowly at infinity.}$$

Thus, $R_{p,p+2}$ belongs to the domain of non-normal attraction of $\phi$.

But until now in case $\rho = 3$, where $(-\rho)$ is the exponent of regular variation of Power Laws, Parero, Waring Distributions, we didn’t give the form of approximation for $F$. The reason is that in all these situations the mentioned form is similar because $\mu_2(x) \approx c(\rho) \ln x, \ x \to +\infty$ for Power Laws, $\mu_2(x) \approx c(\rho, b) \ln x, \ x \to +\infty$ for Pareto Distributions and $\mu_2(x)$ for Waring Distribution $\{p_n(p,p+2)\}$ is given by (3.11.1).

Thus, in all situations we have the same form

$$\mu_2(x) \approx \text{const} \cdot \ln x, \ x \to +\infty. \quad (3.11.2)$$

According to Solution to Problem 3 in Section 3.2, in case $\rho = 3$ in the frame of Scheme (3.1.2)-(3.1.3) the centring constants $A_n = n \cdot E\xi_1, n = 1, 2, \cdots$, and $\lim_{n \to +\infty} \frac{n \cdot \mu_2(B_n)}{B_n^2} = 1$.

It means that in our situations we must find out $\{B_n\}$ from limit relation (see, (3.11.2))

$$\lim_{n \to +\infty} \frac{n \cdot \ln B_n}{B_n^2} = \text{const.} \quad (3.11.3)$$

It is natural to search $\{B_n\}$ under the condition: $\{B_n\}$ is strictly increasing for large $n$. Let $\varphi(t)$ be a linear continuous analog of $\{B_n\}$. Then $\varphi(t)$ is strictly increasing too, and $\varphi(t) \to +\infty$ as $t \to +\infty$. Then, there exists an inverse function, say $\varphi^{-1}(t)$, for $\varphi(t)$ which is strictly increasing, and $\lim_{t \to +\infty} \varphi^{-1}(t) = +\infty$. In equation (see, (3.11.3)) $\frac{n \cdot \ln B_n}{B_n^2} \approx \text{const.}$ we replace
\{B_n\} by its linear continuous analog \(\frac{t}{\ln t} \ln (\varphi(t)) \approx \text{const}\). This last equation implies \(\frac{\varphi^{-1}(t) \cdot \ln t}{t^2} \approx \text{const}\). Having \(\varphi^{-1}(t) \approx \text{const} \cdot \frac{\ln t}{t^2}\), we may easily obtain its inverse function, even graphically.

Easily seen that
\[
\frac{d}{dt}[\varphi^{-1}(t)] = \frac{2t}{\ln t} \cdot \left(2 - \frac{1}{\ln t}\right) > 0 \quad \text{for } t \geq e,
\]
which means that \(\varphi^{-1}(t)\) strictly increases for \(t \in [e, +\infty)\).

Thus, for large \(n\) in case \(\rho = 3\) we have
\[
F(x) \approx \Phi\left(\frac{x - na}{B_n}\right), \quad x \in R^+,
\]
where \(a\) is a mean value of considering distribution.

### 3.11.2 Applying The Stable Approximation

Let \(2 < \rho < 3\). Then, the mean value \(a\) is finite, the second moment is infinite. Remind that
\[
1 - R_{p,q}(x) \approx \frac{1}{x^\rho} \cdot \frac{\Gamma(q)}{\Gamma(p)}, \quad x \to +\infty,
\]
where \(\rho = q - p + 1\) (see, (3.9.4) and (3.9.6)).

In this case instead of sequence \(\{\xi_n\}\) of independent random variables with the same distribution function \(R_{p,q}\) we consider a sequence \(\{\eta_n\} = \left\{\frac{\Gamma(p)}{\Gamma(q)}^{1/(\rho - 1)} \cdot \xi_n\right\}\). For \(\{\eta_n\}\) the conditions of Theorem 3.2 are fulfilled and with the help of Theorem 3.3 we obtain the following approximation, when \(n\) is large,
\[
F(x) \approx S_{\rho - 1}\left(\frac{x - \frac{np}{q - p - 1}}{(\rho - 1)n \cdot \frac{\Gamma(q)}{\Gamma(p)}^{1/(\rho - 1)}}\right),
\]
(3.11.4)

In this situation \(R_{p,q}\) belongs to the domain of normal attraction of standard Right-side Stable Law \(S_{\rho - 1}\) given by (3.4.13") in terms of logarithm of its Laplace-Stieltjes Transform.

In cases \(\rho = 2\) and \(1 < \rho < 2\), by using similar arguments, we come to corresponding approximations for \(F(x)\), when \(n\) is large, in terms of standard Right-side Stable Laws \(S_1\) and \(S_{\rho - 1}\) given by (3.4.13) and (3.4.13") respectively.

In the first case the centring constants are equal to
\[
(n \cdot \ln n) \cdot \frac{\Gamma(q)}{\Gamma(p)} = (n \cdot \ln n) \cdot \frac{\Gamma(p + 1)}{\Gamma(p)} = p \cdot n \ln n,
\]
because in this case \(q = p + 1\) and \(\Gamma(p + 1) = p \cdot \Gamma(p)\).

In the second case, the centring constants are put equal to zero.

The norming constants in both cases are equal to denominator of argument of Stable Law \(S_{\rho - 1}\) in (3.11.4). In particular, for \(1 < \rho < 2\) the \(F\)'s approximation, when \(n\) is large, takes the form
\[
F(x) \approx S_{\rho - 1}\left(\frac{x}{((\rho - 1)n \cdot \Gamma(q))^{1/(\rho - 1)}}\right), \quad x \in R^+.
\]
Chapter 4

Biomolecular Birth-Death Models

4.1 Standard Stochastic Birth-Death Process

4.1.1 The Process

Let us describe the standard stochastic birth-death process, say

\[ \{\xi(t) : t \geq 0\}, \tag{4.1.1} \]

where \( t \) denotes the time. The transition probabilities \( P_{ij}(t) := P(\xi(t+s) = j/\xi(s) = i) = P_{ij}(s,t) \) of the process (4.1.1) for any numbers \( s \in R^+, t \in R^+ \) and for any integers \( i = 0, 1, 2, \cdots, j = 0, 1, 2, \cdots \) do not depend on \( s \). Here \( P(A/B) \) denotes the conditional probability of event \( A \) under the condition of occurrence of event \( B \).

Moreover, as \( t \downarrow 0 \)

\[ P_{ij}(t) = o(t) \quad \text{for} \quad 1 < |i - j| < +\infty, \quad P_{i+1}(t) = \lambda_i \cdot t + o(t), \quad P_{i+1}(t) = \mu_{i+1} \cdot t + o(t). \]

The assumptions imply \( P_{ii}(t) = 1 - (\lambda_i + \mu_i)t + o(t) \) as \( t \downarrow 0 \).

Due to the assumptions, the state probabilities, say \( p_K(t) := P(\xi(t) = K) \) with \( K = 0, 1, 2, \cdots \) at moment \( t \in [0, +\infty) \), satisfy differential equations

\[
\begin{aligned}
\frac{dp_0(t)}{dt} &= -\lambda_0 p_0(t) + \mu_1 p_1(t), \\
\frac{dp_n(t)}{dt} &= - (\lambda_n + \mu_n) p_n(t) + \lambda_{n-1} p_{n-1}(t) + \mu_{n+1} p_{n+1}(t), \quad n = 1, 2, \cdots,
\end{aligned}
\tag{4.1.2}
\]

with arbitrary initial conditions \( p_i(0) \geq 0, \quad i = 0, 1, 2, \cdots, \quad \sum_{i \geq 0} p_i(0) = 1. \)

Without loss of generality we assume \( p_0(0) = 1, \quad p_K(0) = 0, \quad K = 1, 2, \cdots. \)

It seems impossible to express the solution of the infinite system of differential equations (4.1.2) in finite radicals. There are various particular cases of intensities \( \lambda_i, \mu_{i+1}, i = 0, 1, 2, \cdots \), which exhibit solution to system. For instance, \( \lambda_i = \lambda, \mu_{i+1} = (i+1)\mu, i = 0, 1, 2, \cdots \). In this case the solution is presented in form of weights of Bessel functions of imaginary argument (see, [57]).

The standard procedure for obtaining (4.1.2) is to form equilibrium (or balance) equations for state probabilities at \( t + \Delta t \) and \( t \) and letting \( \Delta t \downarrow 0. \)
It is possible to investigate the asymptotic behavior of system (4.1.2) as \( t \to +\infty \) by using different methods. It was done directly in [57].

It is important to notice that (4.1.1) is a Markovian Process with continuous time and countable number of states which lends the tools of Markovian Processes for its study.

It is well-known that for the existence of a stationary solution of the system of differential equations (4.1.2) the convergence of series

\[
T := \sum_{m\geq 1} \prod_{i=1}^{m} \frac{\lambda_{i-1}}{\mu_{i}} < +\infty
\]

is necessary and sufficient (see, for instance, [57]). Always, the limits exist

\[
p_{i} = \lim_{t \to +\infty} p_{i}(t), \quad i = 0, 1, 2, \ldots,
\]

where

\[
p_{i} \begin{cases} > 0 & \text{with } \sum_{i \geq 0} p_{i} = 1 \quad \text{if } T < +\infty, \\ = 0 & \text{with } \sum_{i \geq 0} p_{i} = 0 \quad \text{if } T = +\infty. \end{cases}
\]

Moreover, if \( T < +\infty \), then

\[
\begin{cases}
    p_{0} = (1 + \sum_{m\geq 1} \prod_{i=1}^{m} \frac{\lambda_{i-1}}{\mu_{i}})^{-1}, \\
p_{K} = p_{0} \prod_{i=1}^{K} \frac{\lambda_{i-1}}{\mu_{i}}, \quad K = 1, 2, \ldots.
\end{cases}
\]

### 4.1.2 Stationary Distributions

We may suggest a direct manner for substantiation of formulas (4.1.4)-(4.1.6) which is a mixture of the usual approach and one simple idea. Let us denote by \( \pi \) the random duration of time-interval, which begins from the crossing moment from state 0 to state 1 and finishes at the first crossing moment from state 1 to state 0 after that.

Sometimes, in general stochastic processes (we conserve notation (4.1.1) for these processes) depending on continuous time it is possible to figure out a random moment, say \( \tau_{1} > 0 \), such that the processes \( \{\xi(t) : t \geq 0\} \) and \( \{\xi(t + \tau_{1}) : t \geq 0\} \) have the same finite-dimensional distribution functions. Roughly speaking, the second stochastic process is an explicit "probabilistic copy", or duplicate on the first one. In such a situation, obviously, for the process there is even a sequence of increasing (with probability one) moments \( 0 < \tau_{1} < \tau_{2} < \cdots < \tau_{n} < \cdots \), where \( \eta_{n} := \tau_{n} - \tau_{n-1}, \quad n = 1, 2, \ldots, \quad \eta_{0} = 0 \), are independent, identically distributed, positive with probability one random variables, such that for any \( m = 1, 2, \cdots \) the stochastic process \( \{\xi(t + \tau_{m}) : t \geq 0\} \) has the same finite-dimensional distribution functions as the initial one.

The sequence \( \{\tau_{n}\} \) forms a so-called renewal process.

In case when the number of states of initial stochastic process is no more than countable (numerable) for corresponding state probabilities \( p_{n}(t), \quad n = 0, 1, 2, \cdots, \quad t \in \mathbb{R}^{+} \), the following statement is well-known (see, XI, 8, [23]):

\[
The \limits (4.1.4) \text{ exist and for } n = 0, 1, 2, \cdots$

\[
p_{n} \begin{cases} > 0 & \text{if } E\tau_{1} < +\infty, \\
= 0 & \text{if } E\tau_{1} = +\infty. \end{cases}
\]
Thus, \( p_n > 0 \) for all \( n = 0, 1, 2, \cdots \) iff \( E\tau_1 < +\infty \).

Next, let us make the ”enlargement” of the states of general stochastic process (4.1.1) by conserving state 0 and combining all other states into one ”enlarged” state, i.e. states 1, 2, \cdots in one ”large” state, say \( E \). The obtained new enlarged process, say

\[
\{ \eta(t) : t \geq 0 \}
\]

(4.1.8)

with two states 0 and \( E \) exhibits the same as before imbedded renewal process \( \{\tau_n\} \). For the new process (4.1.8) initially at moment 0 we are in state 0. Let us additionally assume that the sojourn time in state 0, say \( \delta_1 \), has exponential distribution function

\[
1 - \exp(-\lambda_0 \cdot t), \quad t \in R^+, \quad \lambda \in R^+,
\]

(4.1.9)

and doesn’t depend on the sojourn time in state \( E \).

At the random moment \( \delta_1 \) we cross to the state \( E \) for the first time and stay in state \( E \) random time \( \pi_1 > 0 \). Denote the distribution function of the sojourn time \( \pi_1 \) (in \( E \)) by \( F \).

The successively chosen sojourn times \( \delta_1, \delta_2, \cdots \) and \( \pi_1, \pi_2, \cdots \) in states 0 and \( E \), respectively, alternate each other and form independent sequences \( \{\varepsilon_n\} \) and \( \{\pi_n\} \) of independent, identically distributed random variables with distribution functions (4.1.9) and \( F \), respectively.

Here the distribution function \( F \) is unknown.

Under above introduced assumptions the process (4.1.8) form a so-called two-stage renewal process. It is easy to see that in this case \( \eta_n = \varepsilon_n + \pi_n \) for any integer \( n = 1, 2, \cdots \), and \( p_0 \), i.e. the limit as \( t \to +\infty \) of probability at stage 0, for the initial general stochastic process and the constructed particular one under the above introduced assumptions is the same.

The distribution function \( F \) of random variable \( \pi \) (and \( \{\pi_n\} \)) is unknown.

Its mean value \( E\pi = \int_0^{+\infty} x dF(x) = \int_0^{+\infty} (1 - F(x)) dx \) only, due to XI, 8, [23], is important, in particular, for the probability \( p_0 \) of considered two-stage renewal process

\[
p_0 = \frac{E\delta_1}{E\delta_1 + E\pi_1} = \frac{1/\lambda_0}{(1/\lambda_0) + E\pi},
\]

where, due to (4.1.9), we use the equality \( E\delta_1 = \int_0^{+\infty} \exp(-\lambda_0 x) dx = \frac{1}{\lambda_0} \).

Thus, under our assumptions for the general stochastic process (4.17) holds and

\[
p_0 = \frac{1/\lambda_0}{(1/\lambda_0) + E\pi}. \quad (4.1.10)
\]

Applying this result to the standard birth-death process we conclude that the condition \( E\pi < +\infty \) is necessary and sufficient for the existence of stationary solution of the system of differential equations (4.1.2).

Now, let us assume that \( E\pi < +\infty \) and evaluate \( p_n, \ n = 0, 1, 2, \cdots \) and \( E\pi \) from the system of differential equations (4.1.2). First of all, let us show that the limits exist

\[
\lim_{t \to +\infty} \frac{dp_n(t)}{dt} = 0, \quad n = 0, 1, 2, \cdots. \quad (4.1.11)
\]
Since we already proved that the limits (4.1.4) exist, therefore from (4.1.2) we conclude: the limits exist
\[\varepsilon_n := \lim_{t \to +\infty} \frac{dp_n(t)}{dt}, \, n = 0, 1, 2, \ldots\]
So, we have to prove that \(\varepsilon_n = 0\) for all \(n = 0, 1, 2, \ldots\). Let us assume the opposite, i.e. there is an index \(n_0\) such that \(|\varepsilon_{n_0}| \in R^+\), because, due to (4.1.2), the equality \(|\varepsilon_{n_0}| = +\infty\) can not be true. The inclusion \(|\varepsilon_{n_0}| \in R^+\) means that for large \(t\) the probability \(p_{n_0}(t)\) is linear with non-zero coefficient. Then, necessarily it becomes either more than one or less than 0. This contradiction proves the statement.

Next, tending \(t \to +\infty\) in system (4.1.2) we come to the system of equations
\[
\begin{align*}
-\lambda_0 p_0 + \mu_1 p_1 &= 0, \\
-(\lambda_n + \mu_n)p_n + \lambda_{n-1}p_{n-1} + \mu_{n+1}p_{n+1} &= 0.
\end{align*}
\]
(4.1.12)

Introducing new variables \(z_n = -\lambda_n p_n + \mu_{n+1} p_{n+1}, \, n = 0, 1, 2, \ldots\), from (4.1.12) we get a following system \(z_0 = 0, \, z_n - z_{n-1} = 0, \, n = 1, 2, \ldots\), which implies \(z_0 = z_1 = \cdots = z_n = \cdots = 0\). Thus,
\[
p_n = p_{n-1} \cdot \frac{\lambda_{n-1}}{\mu_n} = \cdots = p_0 \cdot \prod_{K=1}^{n} \frac{\lambda_{K-1}}{\mu_K}, \, n = 1, 2, \ldots
\]
and using the equality \(\sum p_n = 1\), we come to (4.1.6).

Finally, from (4.1.6) and (4.1.10) we obtain \(1 + \lambda_0 \cdot E\pi = 1 + \sum_{m \geq 1} \prod_{i=1}^{m} \frac{\lambda_{i-1}}{\mu_i}\), or (see, (4.1.3))
\[
\lambda_0 E\pi = T.
\]
(4.1.13)

### 4.1.3 Interpretation

In Queueing Models described by the standard birth-death process the random variable \(\xi(t), \, t \in R^+\), presents the number of customers being in the system at moment \(t\). Then, the random variable \(\pi\) is a time-interval, which begins from the moment of arrival of a customer into the empty system and finishes at the first after that moment when the system becomes empty. In Queueing Theory the random variable \(\pi\) is called a busy period.

In Populational Genetic Theory the random variable \(\pi\) may be interpreted as a life-time (life-duration) of whole population, functioning in accordance with standard birth-death process, where, under the condition of Population size equal to \(n\), the coefficients \(\lambda_n\) and \(\mu_n\) denote the instantaneous probabilities of birth and death of one representative of the Population. For this important characteristic, due to (4.1.14) and (4.1.3) the following statement may be formulated.

**Theorem 4.1** Put \(\prod_{i=2}^{1} = 1\). For the mean value of the population life-time the following formula holds
\[
E\pi = \begin{cases}
\frac{1}{\mu_1} \sum_{m \geq 1} \prod_{i=2}^{m} \frac{\lambda_{i-1}}{\mu_i} & \text{if } T < +\infty, \\
+\infty & \text{if } T = +\infty,
\end{cases}
\]
(4.1.14)

where \(T\) is given by formula (4.1.3).
4.2 Birth-Death Process with Coefficients of Moderate Growth

The standard stochastic birth-death process with various forms of intensities (coefficients) is an excellent source for obtaining natural skewed distributions which in turn are important in modelling different phenomena in many large-scale biomolecular systems. For instance, the number of transcripts in eukaryotic cells, the number of protein domain occurrences in protein coding DNA sequences for different species, etc. Such skewed distributions are of interest in other fields. They are modelling the number of the words in the text, the number of citations of an author, etc. In all cases mentioned above the coefficients of the respective birth-death process have moderate growth by $n$.

Of the numerous publications on such distributions we wish to point out the pioneering works of J. Yule [58] and H. Simon [19] and some more recent ones W. Granzel and A. Shubert [59], S. Bornholdt and H. Ebel [60], V. Oluie-Vicovic [61], V. Kuznetsov [4], J. Astola and E. Danielian [16], [62], [63]. The most general assumptions on moderate growth of coefficients are considered in [63].

In this Section we are going to present general assumptions on moderate growth of coefficients of standard birth-death process, simplify the necessary and sufficient conditions on steady-state existence, and classify the forms of corresponding stationary distributions.

Finally, we have to notice that for biomolecular systems’ needs there is no other models rather than birth-death models which can explain the mechanism of these biomolecular systems’ functioning.

4.2.1 The Assumptions

Let us give a characterization of moderate growth for, for instance, a sequence $\{\mu_n\}$ of coefficients.

1. $\{\mu_n\}$ is increasing, $\lim_{n \to +\infty} \mu_n = +\infty$, $\lim_{n \to +\infty} \frac{\mu_{n+1}}{\mu_n} = 1$.

   Here only the second and third conditions (the limits) say that $\{\mu_n\}$ exhibits a moderate growth. The first one simplifies future consideration and is quite natural.

Since we talk about moderate growth of both sequences $\{\mu_n\}$ and $\{\lambda_n\}$, therefore the next assumption implies together with 1. a moderate growth for sequence $\{\lambda_n\}$ of coefficients.

2. The limit exists $\lim_{n \to +\infty} \frac{\lambda_n}{\mu_n} = \Theta \in R^+$.

   From assumptions 1. and 2. it follows that

   $\lim_{n \to +\infty} \lambda_n = \Theta \cdot \lim_{n \to +\infty} \mu_n = +\infty$, $\lim_{n \to +\infty} \frac{\lambda_{n+1}}{\lambda_n} = \lim_{n \to +\infty} \frac{\mu_{n+1}}{\mu_n} \cdot \lim_{n \to +\infty} \frac{\mu_n}{\mu_{n+1}} = 1$. 

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So, the *moderate growth* for \( \{\lambda_n\} \) is guarantied.

Let us notice also that

\[
\lim_{n \to +\infty} \frac{\lambda_{n+m}}{\mu_n} = \lim_{n \to +\infty} \frac{\lambda_n}{\mu_n} \cdot \prod_{K=n}^{n+m-1} \frac{\lambda_{K+1}}{\lambda_K} = \Theta \quad \text{for all } m = 1, 2, \cdots. \tag{4.2.1}
\]

Let us consider a sequence of *non-negative* numbers \( \{\gamma_n\} \) depending on some apriori given and fixed integer \( n_0 \geq 0 \) defined by equality

\[
\frac{1}{\gamma_n} = |1 - \frac{1}{\Theta} \lambda_{n+n_0}|, \quad n = 1, 2, \cdots. \tag{4.2.2}
\]

By (4.2.1), \( \lim_{n \to +\infty} \gamma_n = +\infty \) (\( \lim_{n \to +\infty} \frac{1}{\gamma_n} = 0 \)), where we put \( \frac{1}{+\infty} = 0 \), or \( +\infty = \frac{1}{0} \).

The next assumption sounds as follows.

3. **Starting from some integer** \( l \geq 1 \)

\[
1 - \frac{1}{\Theta} \frac{\lambda_{n+n_0}}{\mu_n} = \frac{\alpha}{\gamma_n}, \quad n = l, l+1, \cdots, \tag{4.2.3}
\]

where for all \( n = l, l+1, \cdots \) \( \alpha \) is constant: \( +1, -1 \) or \( 0 \).

In particular, if \( \alpha = 0 \) it means that

\[
\frac{1}{\Theta} \lambda_{n+n_0} = \mu_n, \quad n = l, l+1, \cdots. \tag{4.2.4}
\]

In other cases, the assumption 3. means that the sequence \( \left\{ 1 - \frac{1}{\Theta} \frac{\lambda_{n+n_0}}{\mu_n} \right\} \) starting from some index \( l \) *converses* the sign: \( \alpha = \text{sign} \left( 1 - \frac{1}{\Theta} \frac{\lambda_{n+n_0}}{\mu_n} \right), \quad n = l, l+1, \cdots. \)

We’ll see that the case \( \Theta = 1 \) in assumption 2. is the *most interesting* for biomolecular applications. For this case additional information on \( \{\gamma_n\} \)’s asymptotic behavior as \( n \to +\infty \) in terms of coefficients \( \{\lambda_n\} \) and \( \{\mu_n\} \) is needed.

Let us introduce the following *specific* assumption

4. **If** \( \Theta = 1 \) **then** the limit exists

\[
0 < \beta := \lim_{n \to +\infty} \frac{\gamma_n}{(\mu_n)^{n_0+1}} < +\infty. \tag{4.2.5}
\]

Denote \( \nu_n = (\mu_n)^{n_0+1}, \quad n = 1, 2, \cdots. \)

Later we’ll see that before considered in literature birth-death models, for which the stationary distributions form well-known frequency distributions of large-scale biomolecular systems, satisfy assumptions 1.-4.

The following particular cases play an *exceptional* role in future.

In assumption 1. \( \Theta = 1 \). In assumption 2. \( n_0 = 0 \).
Then $\frac{1}{\gamma_n} = |1 - \frac{\lambda_n}{\mu_n}|$, $n = 1, 2, \cdots$, and the assumption 3. sounds as follows: either

$$\lambda_n = \mu_n, \quad n = 1, 2, \cdots, \quad (4.2.4')$$

or (4.2.3) becomes an identity, including situation (4.2.4') for $\alpha = 0$, when $\gamma_n = +\infty$.

In assumption 4. $\mu_n = \nu_n$, $n = 1, 2, \cdots$, and (4.2.5) now means that

$$\lim_{n \to +\infty} |\mu_n - \lambda_n| = \frac{1}{\beta} \in \mathbb{R}^+. \quad (4.2.5')$$

We’ll call the birth-death model with such assumptions a Special Model.

### 4.2.2 Simplification of a Steady-state Condition

Under the assumptions 1.-4. it is possible to simplify the steady-state condition (4.1.3) for the stationary distributions’ existence.

In the steady-state condition (4.1.3) write $T = T(\Theta)$ and in the case $\Theta = 1$, where also assumptions 3.-4. hold, write $T = T(1, \alpha)$ with $\alpha = -1, 0, 1$. Remind that $\alpha = 0$ corresponds to (4.2.4).

Our goal is to present, for the case of moderate growth of coefficients $\{\lambda_n\}$ and $\{\mu_n\}$ satisfying the above assumptions, a simple criterion for the existence of the steady-state distributions in terms of $\alpha, \Theta$ and convergence/divergence of the series

$$I = \sum_{n \geq 1} \frac{1}{\nu_n}. \quad (4.2.6)$$

Let us formulate the main result.

**Theorem 4.2** 1. Let assumptions 1.-2. hold. Then

$$T = T(\Theta) \begin{cases} < +\infty & \text{if } 0 < \Theta < 1, \\ = +\infty & \text{if } 1 < \Theta < +\infty. \end{cases} \quad (4.2.7)$$

2. Let assumptions 1.-3. hold and $\Theta = 1$. Then

$$T(1, 0) < +\infty \quad \text{iff} \quad I < +\infty. \quad (4.2.8)$$

3. Let assumptions 1.-4. hold (naturally, $\Theta = 1$). Then:
   a) $T(1, 1) < +\infty$; \quad (4.2.9)
   b) $T(1, -1) < +\infty \quad \text{iff} \quad I < +\infty. \quad (4.2.10)$
Due to Theorem 4.2, quite a complete simplification of the condition (4.1.3) is presented in our case.

In the next Section the proof and supplement to Theorem 4.2 shall be done.

Now, let us derive stationary distributions with Moderate Growth. (We guess that assumptions of Moderate Growth of coefficients leads to stationary distributions with Moderate Growth. Let us find out the steady-state (stationary) distributions, under some simplifying conditions. First we assume that assumption 3. holds from \( n = 1 \) and denote \( b = \frac{1}{\beta}, \ d = \lambda_0 \cdot \lambda_1 \cdots \lambda_{n_0} \).

Next, let us introduce the sequences \( \{ \varepsilon_n \} \) and \( \{ \delta_n \} \), where
\[
\varepsilon_n = \mu_n \mu_{n+1} \cdots \mu_{n+n_0}, \quad \delta_n = b \cdot \gamma_n, \quad n = 1, 2, \ldots. \tag{4.2.11}
\]

Due to assumption 1.,
\[
\{ \varepsilon_n \} \text{ is increasing, } \lim_{n \to +\infty} \varepsilon_n = +\infty, \quad \lim_{n \to +\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n} = 1, \tag{4.2.12}
\]

For the Special Model we have
\[
d = \lambda_0, \quad \varepsilon_n = \mu_n, \quad \delta_n = b \cdot |1 - \frac{\lambda_n}{\mu_n}|^{-1} (= b \cdot \gamma_n), \quad n = 1, 2, \ldots. \tag{4.2.11'}
\]

For simplicity we assume that \( \{ \gamma_n \} \) is an increasing sequence.

If \( \alpha = 1 \), then, due to assumption 3., \( \gamma_n > 0 \) and so \( \delta_n > b \). Now
\[
\{ \delta_n \} \text{ is increasing, } \lim_{n \to +\infty} \delta_n = +\infty, \quad \lim_{n \to +\infty} \frac{\delta_{n+1}}{\delta_n} = 1, \tag{4.2.13}
\]
and, by definition of \( b \),
\[
\lim_{n \to +\infty} \frac{\delta_n}{\varepsilon_n} = 1. \tag{4.2.14}
\]

Thus,
\[
\sum_{n=1}^{n_0} \prod_{m=1}^{n} \frac{\lambda_{m-1}}{\mu_m} = (\sum_{n=1}^{n_0} + \sum_{n=n_0+1}^{n} \prod_{m=1}^{n} \frac{\lambda_{m-1}}{\mu_m}) = \]
\[
= \sum_{n=1}^{n_0} \prod_{m=1}^{n} \frac{\lambda_{m-1}}{\mu_m} + d \cdot \sum_{n=n_0+1}^{n_0} \frac{1}{\mu_n \mu_{n+1} \cdots \mu_{n+n_0}} \cdot \prod_{m=n_0+1}^{n_0} \frac{\lambda_n}{\mu_m-n_0} = \]
\[
= \sum_{n=1}^{n_0} a_n + d \cdot \sum_{s=1}^{n_0} \varepsilon_s \cdot \prod_{m=1}^{s-1} (1 - \frac{\alpha}{\gamma_n}) = \sum_{n=1}^{n_0} a_n + d \cdot \sum_{n=1}^{n_0} \Theta_n \cdot \prod_{m=1}^{n-1} (1 - \frac{\alpha \cdot b}{\delta_m}),
\]
where \( \sum_{n=1}^{0} = 0, \prod_{n=1}^{0} = 1, \quad a_n = \prod_{m=1}^{n_0} \frac{\lambda_{m-1}}{\mu_m}, \quad n = 1, 2, \ldots, n_0. \)

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Therefore, by (4.1.6), for $\alpha = 0, -1, +1$ and $n_0 \geq 0$ for steady-state distribution $\{p_n(\alpha)\}$ we obtain

$$
\begin{cases}
p_0(\alpha) = \left(\sum_{n=1}^{n_0} a_n + d \cdot \sum_{n\geq1} \frac{\Theta^n}{\varepsilon_n} \prod_{m=1}^{n-1} (1 - \frac{\alpha \cdot b}{\delta_m})\right)^{-1}, \\
p_K(\alpha) = a_K \cdot p_0(\alpha), \quad K = 1, 2, \cdots, n_0, \\
p_{n+n_0}(\alpha) = d \cdot \frac{\Theta^n}{\varepsilon_n} \prod_{m=1}^{n-1} (1 - \frac{\alpha \cdot b}{\delta_m}), \quad n = 1, 2, \cdots.
\end{cases}
$$

(4.2.15)

By Theorem 4.2, the ranges of parameters are:

$$
\begin{align*}
\alpha &= 1 \quad 0 < \Theta \leq 1, \quad 0 < b < 1; \\
\alpha &= 0 \text{ or } 1 \quad \text{either } 0 < \Theta < 1, \quad 0 < b < +\infty, \quad \text{or if also} \\
\sum_{n\geq1} \frac{1}{\varepsilon_n} < +\infty,
\end{align*}
$$

(4.2.16)

then $\Theta = 1, \quad 0 < b < +\infty$.

In all cases $a_K, \quad K = 1, 2, \cdots, n_0$ and $d$ are arbitrary positive numbers. We call this class of stationary distributions the class of distributions with Moderate Growth.

Let us return to the Special Model. Since $n_0 = 0$, so the terms $a_n$ at the right-hand-side of (4.2.15) must be omitted. We may rewrite (4.2.5') in the form

$$
\lim_{n \to +\infty} |\mu_n - \lambda_n| = b
$$

(4.2.5'')

and (4.2.11') as $d = \lambda_0, \quad \mu_n = \varepsilon_n, \quad n = 1, 2, \cdots$, and

either $\lambda_n = \varepsilon_n \cdot (1 - \frac{b}{\delta_n})$ with $0 \leq b < 1$,  
(4.2.17)

or $\lambda_n = \varepsilon_n \cdot (1 + \frac{b}{\delta_n})$ with $0 \leq b < +\infty$,  
(4.2.18)

where the conditions (4.2.12)-(4.1.14) have to be fulfilled.

Thus, now we are familiar with two forms of coefficients (4.2.17) and (4.2.18) where $\varepsilon_n$ is replaced by $\mu_n$, which lead to the Special Model.

### 4.3 Theorem 4.2: Proof and Supplement

In the present Section the proof and supplement to Theorem 4.3 shall be done.
4.3.1 The Remarks

The proof of Theorem 4.2 is based on the following remarks.

Remark 3.1 Write for series in condition (4.1.3) \( T(\Theta) = \sum_{n \geq 1} \prod_{m=1}^{n} \frac{\lambda_{m-1}}{\mu_{m}} = \sum_{n \geq 1} t_{n} \), where 
\[
t_{n} = \prod_{m=1}^{n} \frac{\lambda_{m-1}}{\mu_{m}}, \quad n = 1, 2, \ldots, \]
and 
\[
r_{n} = \frac{1}{\nu_{n}} \prod_{m=1}^{n} \frac{\lambda_{m+n}}{\mu_{m}}, \quad n = 1, 2, \ldots.\]
Then \( R(\Theta) \leq +\infty \) is equivalent to \( T(\Theta) < +\infty \). This equivalency is expressed as
\[
R(\Theta) \sim T(\Theta). \tag{4.3.1}
\]

This can be seen as follows. The parameter \( n_{0} \) is a non-negative integer and since \( \lim_{n \to +\infty} \frac{\mu_{n+1}}{\mu_{n}} = 1 \), we have for \( n \) large enough \( \frac{1}{2} < \frac{\mu_{n} - \mu_{n-1}}{\mu_{n} - \mu_{n-1}} \frac{\mu_{n-1}}{\mu_{n}} < 2 \). Consider the general term of \( T(\Theta) \)
\[
t_{n} = \frac{\lambda_{0}\lambda_{1} \cdots \lambda_{n-1}}{\mu_{1}\mu_{2} \cdots \mu_{n}} = \frac{\lambda_{0}\lambda_{1} \cdots \lambda_{n}}{\mu_{n-n_{0}} \mu_{n-n_{0}-1} \cdots \mu_{n}} = \frac{\lambda_{0}\lambda_{1} \cdots \lambda_{n}}{\mu_{n-n_{0}} \mu_{n-n_{0}-1} \cdots \mu_{n}} \prod_{m=1}^{n-n_{0}-1} \frac{\lambda_{m+n_{0}}}{\mu_{m}}.
\]
Thus, we have the inequalities
\[
\frac{\lambda_{0}\lambda_{1} \cdots \lambda_{n}}{2 \cdot (\mu_{n-n_{0}-1})^{n_{0}+1}} \prod_{m=1}^{n-n_{0}-1} \frac{\lambda_{m+n_{0}}}{\mu_{m}} = \frac{1}{2} (\lambda_{0}\lambda_{1} \cdots \lambda_{n}) \cdot r_{n-n_{0}} < t_{n} < 2 \cdot (\lambda_{0}\lambda_{1} \cdots \lambda_{n}) \cdot r_{n-n_{0}}-1
\]
implies that (4.3.1) holds.

Since leaving out a finite number of terms or multiplication of all terms by a constant have no effect on convergence we also have \( T(\Theta) \sim R(\Theta) = \sum_{n \geq K} \frac{1}{\nu_{n}} \prod_{m=K}^{n} \frac{\lambda_{m+n_{0}}}{\mu_{m}} \) for any \( K \).

Remark 3.2 Let assumptions 1.-3. hold. Now we write
\[
T(1) = T(1, \alpha), \quad R_{K}(1) = R_{K}(1, \alpha), \quad K = 1, 2, \ldots
\]
and denote for \( K = 1, 2, \ldots \) and either \( \alpha = 1 \) or \( \alpha = -1 \)
\[
F_{K}(1, \alpha) = \sum_{n \geq K} \frac{1}{\nu_{n}} \exp(-\alpha \sum_{m=K}^{n} \frac{1}{\gamma_{m}}).
\]
Then \( T(1, \alpha) \sim F_{K}(1, \alpha) \) for any \( K = 1, 2, \ldots \).

Indeed, for \( K \) large enough, using standard approximations, we get
\[
R_{K}(1, \alpha) = \sum_{n \geq K} \frac{1}{\nu_{n}} \prod_{m=K}^{n} (1 - \frac{\alpha}{\gamma_{m}}) = \sum_{n \geq K} \frac{1}{\nu_{n}} \exp\left( \sum_{m=K}^{n} \ln(1 - \frac{\alpha}{\gamma_{m}}) \right) \sim \sum_{n \geq K} \frac{1}{\nu_{n}} \exp(-\alpha \cdot \sum_{m=K}^{n} \frac{1}{\gamma_{m}}) = F_{K}(1, \alpha)
\]
and apply Remark 3.1.
Remark 3.3 For any positive sequence, say $\{\frac{1}{\nu_n}\}$, and $c \in \mathbb{R}^+$
\[
\varphi(K) := \sum_{n \geq K} \frac{1}{\nu_n} \exp(-c \cdot \sum_{m=K}^{n} \frac{1}{\nu_m}) < +\infty.
\]

From the graph, it is obvious that the sum is bounded by $\int_0^\infty \exp(-cx)dx = \frac{1}{c}$.

![Figure 8.](image)

### 4.3.2 Proof of Theorem 4.2

1. If $\Theta \neq 1$, then (4.2.7) follows immediately by the ratio test.

2. By assumption 3., $R_K(1,0) = \sum_{n \geq K} \frac{1}{\nu_n} \sim I$, where $I$ is given by (4.2.6), and, using Remark 3.1, we get (4.2.8).

3. By assumption 4., from some index $K_0$, say on, $\frac{1}{2\beta \nu_n} < \frac{1}{\gamma_n} < \frac{3}{2\beta \nu_n}$, where (4.2.5) is used.

Writing $c^\pm = \frac{\alpha}{2}(1 + \frac{1}{2} \text{ sign } \alpha)$ for $\alpha = \pm 1$ we have
\[
\sum_{n \geq K_0} \frac{1}{\nu_n} \exp(-c^+ \sum_{m=K_0}^{n} \frac{1}{\nu_m}) < F_{K_0}(1, \alpha) < \sum_{n \geq K_0} \frac{1}{\nu_n} \exp(-c^- \sum_{m=K_0}^{n} \frac{1}{\nu_m}). \tag{4.3.2}
\]

If $\alpha = 1$, then (4.2.9) follows at once from (4.3.2).

Let then $\alpha = -1$. If $I = +\infty$, then, by (4.3.2), $\sum_{n \geq K_0} \frac{1}{\nu_n} \exp((\frac{1}{2\beta} \sum_{m=K_0}^{n} \frac{1}{\nu_m})) < F_{K_0}(1,-1)$ implying (4.2.10) for $I = +\infty$.

If $I < +\infty$, then using (4.3.2) we get $F_{K_0}(1,-1) < \sum_{n \geq K_0} \frac{1}{\nu_n} \exp((\frac{3}{2\beta} \sum_{m=K_0}^{n} \frac{1}{\nu_m})) < +\infty$ implying (4.2.10) for $I < +\infty$. **Theorem 4.2** is proved.
4.3.3 Supplement to Theorem 4.2

Let us notice that the limit cases $\Theta = 0$ and $\Theta = +\infty$ may be included in (4.2.7). Let us consider the limit cases for $\beta$ defined in (4.2.5).

The following Supplement to Theorem 4.2 may be formulated

1. Let $\beta = 0$. Then: a) (4.2.8) holds; b) $T(1, -1) = +\infty$ if $I = +\infty$.

2. Let $\beta = +\infty$ and $J := \sum_{n \geq 1} \frac{1}{\gamma_n}$. Then:
   a) (4.2.9) holds; b) $I < +\infty$ implies $T(1, 1) < +\infty$;
   c) $I = +\infty$, $J < +\infty$ imply $T(1, 1) = +\infty$.

Proof. 1. Let $\beta = 0$. For any $A \in R^+$ we have $\gamma_n^{-1} > A \cdot \nu_n^{-1}$ starting from some index $K_0$. So, by Remarks 3.2 - 3.3, $T(1, 1) \sim F_{K_0}(1, 1) < A^{-1} \sum_{n \geq K_0} \frac{1}{\gamma_n} \cdot \exp(- \sum_{m=K_0}^n \frac{1}{\gamma_m}) < +\infty$ and so (4.2.9) holds.

If $I = +\infty$, then, by Remark 3.2

$$T(1, -1) \sim \sum_{n \geq 1} \frac{1}{\nu_n} \exp(\sum_{m=1}^n \frac{1}{\gamma_m}) > I = +\infty.$$  \hspace{1cm} (4.3.3)

2. Let $\beta = +\infty$. For any $A \in R^+$ we have $\gamma_n^{-1} < A \cdot \nu_n^{-1}$ starting from some index $K_0$.

Thus, $I < +\infty$ implies $J < +\infty$. By Remark 3.2, if $I < +\infty$, then we have

$$T(1, -1) \sim \sum_{n \geq 1} \frac{1}{\nu_n} \exp(\sum_{m=1}^n \frac{1}{\gamma_m}) < I \cdot \exp(J) < +\infty.$$  

Let then $I = +\infty$. Then, by Remark 3.2, also (4.3.3) takes place and thus (4.2.10) holds in this case. Consider then $T(1, +1)$. If $I < +\infty$, then, by Remark 3.2,

$$T(1, 1) \sim \sum_{n \geq 1} \frac{1}{\nu_n} \exp(- \sum_{m=1}^n \frac{1}{\gamma_m}) < I < +\infty.$$  

If then $I = +\infty$ and $J < +\infty$, we have, by Remark 3.2,

$$T(1, 1) \sim \sum_{n \geq 1} \frac{1}{\nu_n} \exp(+ \sum_{m=1}^n \frac{1}{\gamma_m}) > I \cdot \exp(+J) = +\infty.$$  

The Supplement doesn’t give a complete picture of situation as becomes clear through the following examples satisfying assumptions 1.-3. with $n_0 = 0$ but $\beta$ is not finite and nonzero.

Example 1. Put $\mu_n = n$, $n = 1, 2, \cdots$, with two variations:

a) $\lambda_n = n - (\ln n)^{-1}$; \hspace{1cm} and \hspace{1cm} b) $\lambda_n = n - 2 \cdot (\ln n)^{-1}$.

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Thus, $\alpha = 1$ and $\beta = +\infty$. Let us show that in the case a) we have $T(1, 1) = +\infty$ and in the case b) we have $T(1, 1) < +\infty$.

Indeed, consider the case a). Since $\gamma_n = n \cdot \ln n, n = 1, 2, \cdots$, we have, by Remark 3.2,

$$T(1, 1) \sim \sum_{n \geq 1} \frac{1}{n} \cdot \exp\left(-\sum_{m=1}^{n} \frac{1}{m \cdot \ln m}\right) \sim \sum_{n \geq 1} \frac{1}{n} \exp\left(-\ln \ln n\right) = \sum_{n \geq 1} \frac{1}{n \ln n} = +\infty.$$ 

Consider then the case b). Since $\gamma_n = \frac{1}{2} n \cdot \ln n, n = 1, 2, \cdots$, as above, we get

$$T(1, 1) \sim \sum_{n \geq 1} \frac{1}{n(n \ln n)^2} < +\infty.$$ 

Example 2. Put $\mu_n = n(\ln n)^2, n = 1, 2, \cdots$, again with two variations:

a) $\lambda_n = n \cdot (\ln n)^2 + \ln n$; and b) $\lambda_n = n(\ln n)^2 + \frac{1}{2} \ln n$.

Now, $\alpha = -1$ and $\beta = 0$. In case a) we have

$$T(1, -1) \sim \sum_{n \geq 1} \frac{1}{n(\ln n)^2} \cdot \exp\left(-\sum_{m=1}^{n} \frac{1}{m \cdot \ln m}\right) \sim \sum_{n \geq 1} \frac{1}{n \ln n} = +\infty,$$

and in case b) we have, since $\gamma_n = 2n \ln n, n = 1, 2, \cdots$, $T(1, -1) \sim \sum_{n \geq 1} \frac{1}{n(\ln n)^{3/2}} < +\infty$.

Thus, in cases $\beta = +\infty$, $J = +\infty$ and $\beta = 0$ and $I < +\infty$, $J < +\infty$ to conclude that $T(1, 1) < +\infty$ and $T(1, -1) < +\infty$, respectively, additional information is needed.

### 4.4 Hypergeometric Distributions

In this Section as an example of construction of empirical frequency distributions, by using standard birth-death process with coefficients of moderate growth, a new four-parametric family of Hypergeometric Distributions is introduced. This family is included into the Scheme being considered in Sections 4.2 - 4.3, but the picture given by results of Sections 4.2 - 4.3 in this particular case is not complete. That is why we have a reason to study the introduced family separately. Moreover, it is possible to investigate the properties of the family of Hypergeometric Distributions more deeply than it allows a general case.

#### 4.4.1 Definition and Existence

Consider a population of $n$ elements where $n_1$ are red and $n_2 = n - n_1$ are black. A group of $r$ elements is chosen. Denote by $\hat{p}_K, K = 0, 1, \cdots, n_1$, the probability that the group contains exactly $K$ red elements. Then, for $K = 0, 1, \cdots, n_1$ we have $\hat{p}_K = \binom{n_1}{K} \binom{n-n_1-r}{n-r-K} \binom{n}{r}$. A sequence $\{\hat{p}_n\}$ form a Hypergeometric Distribution well-known in Probability Theory (see, II.6, p.41-42, [27]).
Let us consider a generalization of this distribution. Namely, let $p_1, p_2, q, \Theta$ be finite positive numbers which have to serve as parameters of below introduced family of distributions $\{\hat{p}_n\}$, i.e.

$$\hat{p}_n = \hat{p}_n(p_1, p_2, q, \Theta), \quad n = 1, 2, \ldots.$$ \hfill (4.4.1)

Now, let us write down a four-parametric family $\{\hat{p}_n(p_1, p_2, q, \Theta)\}$

$$\hat{p}_0 = (1 + \sum_{n \geq 1} \Theta^n \prod_{K=0}^{n-1} \frac{(p_1 + K)(p_2 + K)}{(1 + K)(q + K)})^{-1},$$ \hfill (4.4.2)

$$\hat{p}_n = \hat{p}_0 \cdot \Theta^n \prod_{K=0}^{n-1} \frac{(p_1 + K)(p_2 + K)}{(1 + K)(q + K)}, \quad n = 1, 2, \ldots,$$ \hfill (4.4.3)

which we call a family of Hypergeometric Distributions.

The first problem arising here sounds as follows.

Find conditions on $p_1, p_2, q, \Theta$ implying that $\{\hat{p}_n\}$ is a proper distribution, which in other words means that $\sum_{n \geq 0} \hat{p}_n(p_1, p_2, q, \Theta) = 1$.

In our case norming condition, according to (4.4.2)-(4.4.3), takes the form

$$1 + \sum_{n \geq 1} \Theta^n \prod_{K=0}^{n-1} \frac{(p_1 + K)(p_2 + K)}{(1 + K)(q + K)} < +\infty.$$ \hfill (4.4.4)

Due to 9.100, p.1039, [20], the expression (4.4.4) presents a series expansion for hypergeometric function $F(p_1, p_2, q, \Theta)$, which is one of special functions of Mathematical Analysis. It explains the name taken for introduced family. We'll see also that Generating Function of distribution $\{\hat{p}_n(p_1, p_2, q, \Theta)\}$ is expressed in terms of hypergeometric function.

Now, due to 9.102, p.1040, [20], a Hypergeometric Distribution $\{\hat{p}_n(p_1, p_2, q, \Theta)\}$ is proper

\begin{align*}
\text{either} & \quad 0 \leq p_1 + p_2 - q < 1, \quad 0 < \Theta < 1, \quad (4.4.5) \\
\text{or} & \quad -\infty < p_1 + p_2 - q < 0, \quad 0 < \Theta < +\infty. \quad (4.4.6)
\end{align*}

In remained cases, namely,

\begin{align*}
\text{either} & \quad 1 \leq p_1 + p_2 - q < +\infty \quad \text{or} \quad 0 \leq p_1 + p_2 - q < 1 \quad \text{and} \quad 1 \leq \Theta < +\infty
\end{align*}

the distribution $\{\hat{p}_n(p_1, p_2, q, \Theta)\}$ is improper. Moreover, $\hat{p}_n(p_1, p_2, q, \Theta) = 0$ for all $n = 0, 1, 2 \ldots$.

Thus, complete solution to problem is done.

The solution allows following reformulation. Due to (4.4.4)-(4.4.6),

The inequality
\[ T(p_1, p_2, q, \Theta) := \sum_{n \geq 1} \Theta^n \cdot \prod_{K=0}^{n-1} \frac{(p_1 + K)(p_2 + K)}{(1 + K)(q + K)} < +\infty \] (4.4.7)

holds iff (4.4.5) or (4.4.6) hold.

Comparing the forms (4.1.3) and (4.4.7) we come to the conviction that the family of Hypergeometric Distributions may be obtained as stationary distributions of standard birth-death process by choosing corresponding form of coefficients.

### 4.4.2 Connection with Birth-Death Models

In standard birth-death model given by system (4.1.2) for \( n = 0, 1, 2 \cdots \) let us put

\[ \lambda_n = (p_1 + n)(p_2 + n)\Theta, \mu_{n+1} = (1 + n)(q + n). \] (4.4.8)

Stationary distribution (4.1.6) in this case transforms into (4.4.2)-(4.4.3), and steady-state condition (4.1.3) transforms into (4.4.7), or into (4.4.7)’s simplification (4.4.5) and (4.4.6). Thus, the four-parametric family of Hypergeometric Distributions is generated by standard birth-death process with coefficients of the form (4.4.8).

In case \( p_2 = 1, p_1 = p \) (or \( p_1 = 1, p_2 = p \)), so, \( \hat{p}_n(p, 1, q, \Theta) = \hat{p}_n(1, p, q, \Theta), \ n = 0, 1, 2 \cdots \) we obtain a following family of Komolgorov-Waring Distributions (see, (1.2.13)-(1.2.14)) introduced by V.Kuznetsov [4]:

\[ \hat{\theta}_0 = (1 + \sum_{n \geq 1} \Theta^n \cdot \prod_{K=0}^{n-1} \frac{p + K}{q + K})^{-1}, \hat{\theta}_K = \theta_0 \cdot \Theta \cdot \prod_{m=0}^{K-1} \frac{p + m}{q + m}, K = 1, 2, \cdots. \]

The conditions of a steady-state existence (4.4.5)-(4.4.6) transforms into following ones:

- either \( 0 < \Theta < 1, \ 0 < p < +\infty, \ 1 < q < +\infty, \)
- or \( \Theta = 1, \ 0 < p < q - 1 < +\infty. \)

So, the separate proof on equivalency of these conditions to (4.1.3) for above mentioned case given in [4] is not needed, because these conditions are nothing else but series expansion’s convergence conditions for hypergeometric function \( F(p_1, p_2, q, \Theta). \)

Here case \( \Theta = 1 \) presents a famous family of Waring Distributions.

The conditions on moderate growth of coefficients of standard birth-death process are introduced in Section 4.2 (see, conditions 1. and 2. in 4.2.1):

1. \( \{\mu_n\} \) is increasing, \( \lim_{n \to +\infty} \mu_n = +\infty, \lim_{n \to +\infty} \frac{\mu_{n+1}}{\mu_n} = 1; \) 2. \( \lim_{n \to +\infty} \frac{\lambda_n}{\mu_n} := \Theta \in R^+. \)

From (4.4.8) easily seen that conditions 1.-2. in our case are fulfilled. Moreover, the limit value in condition 2. coincides with parameter \( \Theta \) of Hypergeometric Distribution \( \{\hat{p}_n(p_1, p_2, q, \Theta)\}. \)
The assumptions 3. and 4. in Section 4.2 are technical. Anyway let us verify them in our case. Denote \( \frac{1}{\gamma_n} = |1 - (p_1 + n)(p_2 + n)| \), for \( n = 1, 2, \cdots \) which is an analog of (4.2.3) for Hypergeometric Distribution \( \{\hat{p}_n(p_1, p_2, q, \Theta)\} \) when \( n_0 = 0 \) in (4.2.3). The sequence \( \{\gamma_n\} \) here, as we see, doesn’t depend on \( \Theta \). Since

\[
1 - \frac{(p_1 + n)(p_2 + n)}{n \cdot (q + n - 1)} = \frac{q - p_1 - p_2 - 1}{q + n - 1} - \frac{p_1 \cdot p_2}{n(q + n - 1)} =
\]

\[
\begin{cases}
-\frac{1}{\gamma_n} \text{ for all } n = 2, 3, \cdots \text{ in case (4.4.5) and in subcase of (4.4.6), namely, } 0 < q - p_1 - p_2 \leq 1; \\
\frac{1}{\gamma_n} \text{ starting from some index in subcase of (4.4.6), namely, } 1 < q - p_1 - p_2 < +\infty,
\end{cases}
\]

therefore the assumption 3. is fulfilled with either \( \alpha = -1 \) or \( \alpha = 1 \) (so, the value \( \alpha = 0 \) doesn’t arise). Easily seen that \( \lim_{n \to +\infty} \frac{(\gamma_n)^{1/2}}{\gamma_n} = |q - p_1 - p_2 - 1| \in [0, +\infty) \).

It means that the assumption 4. in case \( |q - p_1 - p_2 - 1| \in R^+ \) doesn’t hold. Moreover, in this case \( \lim_{n \to +\infty} \frac{\gamma_n}{\gamma_n} = 0 \) and we are in conditions of supplement to the model, described in Section 4.3. But the supplement doesn’t make clear situation with steady-state condition in some cases.

Thus, we have a reason to study the family of Hypergeometric Distributions separately.

### 4.4.3 The Generating Function

Consider a Generating Function of Hypergeometric Distribution \( \{\hat{p}_n(p_1, p_2, q, \Theta)\} \)

\[
\hat{G}(x; p_1, p_2, q, \Theta) = \sum_{n \geq 0} x^n \cdot \hat{p}_n(p_1, p_2, q, \Theta), \quad 0 \leq x \leq 1. \tag{4.4.9}
\]

By 9.111, p.1040, [20], for \( 0 < p_2 < q < +\infty \) we have

\[
F(p_1, p_2; q, z) = \frac{1}{B(p_2, q - p_2)} \int_0^1 t^{p_2-1}(1-t)^{q-p_2-1}(1-tz)^{-p_2}dt, \tag{4.4.10}
\]

where \( B(x, y) \) is the well-known Beta Function

\[
B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{4.4.11}
\]

By symmetry of parameters \( p_1 \) and \( p_2 \) in series expansion (4.4.4) for \( F(p_1, p_2; q, z) \), where \( \Theta \) is replaced by \( z \), we may change the places of \( p_1 \) and \( p_2 \) in (4.4.10), so for \( 0 < p_1 < q < +\infty \) we have

\[
F(p_1, p_2; q, z) = \frac{1}{B(p_1, q - p_1)} \int_0^1 t^{p_1-1}(1-t)^{q-p_1-1}(1-tz)^{-p_1}dt. \tag{4.4.10'}
\]

Taking into account the series expansion (4.4.4) of \( F(p_1, p_2; q, \Theta) \), the forms (4.4.8) and (4.4.9) of Generating Function, and integral representations (4.4.10), (4.4.10'), we
obtain

\[
\hat{G}(x; p_1, p_2, q, \Theta) = (F(p_1, p_2, q\Theta x)/F(p_1, p_2, q, \Theta)) = \int_0^1 t^{p_2-1}(1 - t)^{q-p_2-1}(1 - t\Theta x)^{-p_1}dt \\
= \int_0^1 t^{p_2-1}(1 - t)^{q-p_2-1}(1 - t\Theta)^{-p_1}dt \quad \text{(if } 0 < p_2 < q < +\infty),
\]

\[
= \int_0^1 t^{p_1-1}(1 - t)^{q-p_1-1}(1 - t\Theta x)^{-p_2}dt \quad \text{(if } 0 < p_1 < q < +\infty).
\]

The constraints \(0 < p_2 < q < +\infty\) and \(0 < p_1 < q < +\infty\), under which formula (4.4.12) is derived, in case (4.4.6) both take place.

In frame of (4.4.5) formula (4.4.12) works not always. Indeed, for \(p_1 = p_2 = 0, 5; q = 0, 3; \Theta \in (0, 1)\) (4.4.5) takes place but \(p_1 > q\) and \(p_2 > q\).

### 4.5 Empirical Facts for Hypergeometric Distributions

In order to suggest Hypergeometric Distributions as new empirical frequency distributions, it is necessary to verify the fulfilment of known empirical (statistical) facts for them. The facts are presented and discussed in Sections 1.1-1.4 on examples of well-known frequency distributions. We’ll conserve only the part of four-parametric family of Hypergeometric Distributions which satisfies all empirical facts. This is a description of the present Section’s contents.

#### 4.5.1 Regular Variation

Let us find conditions when Hypergeometric Distribution \(\{\hat{p}_n(p_1, p_2, q, \Theta)\}\) varies regularly at infinity, i.e. is presented in the form (see, [21]) \(\hat{p}_n(p_1, p_2, q, \Theta) = n^{-\rho}L(n), \ n = 1, 2, \ldots\), where \(\rho \in R^+\) and for \(s = 2, 3, \ldots\) the limit exists \(\lim_{n \to +\infty}(L(sn)/L(n)) = 1\).

Let us apply to (4.4.3) the following formula (see, 8.322, p.936, [20])

\[
\prod_{k=1}^n \frac{K}{x+K} \approx \frac{\Gamma(x+1)}{n^x}, \ n \to +\infty, \ x \in R^+.
\]

Thus,

\[
\hat{p}_n = \hat{p}_0 \cdot \Theta^n \cdot \frac{p_1 p_2}{q} \prod_{k=1}^{n-1} \left( \frac{K}{1+K} \cdot \frac{p_1 + K}{K} \cdot \frac{p_2 + K}{K} \cdot \frac{K}{q+K} \right) \approx
\]

\[
\approx \hat{p}_0 \cdot \Theta^n \cdot \frac{p_1 p_2}{n \cdot q} \cdot n^{p_1+p_2-q} \frac{\Gamma(q+1)}{\Gamma(p_2+1)\Gamma(p_2+1)} = \hat{p}_0 \cdot \Theta^n \cdot n^{p_1+p_2-q-1} \frac{\Gamma(q)}{\Gamma(p_1)\Gamma(p_2)}, \ n \to +\infty,
\]

where the formula \(x \cdot \Gamma(x) = \Gamma(x+1)\) was used. So,

\[
\hat{p}_n \approx \hat{p}_0 \cdot \Theta^n \cdot \frac{\Gamma(q)}{\Gamma(p_1)\Gamma(p_2)} \cdot \frac{1}{n^\rho}, \ n \to +\infty, \quad (4.5.1)
\]
where
\[ \rho = q + 1 - p_1 - p_2. \]  
(4.5.2)

According to constraints (4.4.5) and (4.4.6) on parameters of Hypergeometric Distributions, \( \rho \in R^+ \).

The equivalency (4.5.1) says that:

\( \text{Distribution } \{ \hat{p}_n(p_1, p_2, q, \Theta) \} \text{ varies regularly at infinity iff } \Theta = 1. \)

If \( \Theta = 1 \), then the exponent \( (-\rho) \) of regular variation (see, (4.5.2)) satisfies condition
\[ 1 < \rho < +\infty \quad (\text{then}, \ q - p_1 - p_2 > 0), \] 
(4.5.3)
which is a consequence of (4.4.6) and (4.5.2). Thus, from the four-parametric family of Hypergeometric Distributions a three-parametric subfamily of regularly varying distributions is extracted: \( \{ P_n(p_1, p_2, q) \} = \{ \hat{p}_n(p_1, p_2, q, 1) \} \), where
\[ P_0(p_1, p_2, q) = (1 + \sum_{n \geq 1} \prod_{K=0}^{n-1} \frac{(p_1 + K)(p_2 + K)}{(1 + K)(q + K)})^{-1}, \] 
(4.5.4)
\[ P_n(p_1, p_2, q) = P_0(p_1, p_2, q) \cdot \prod_{K=0}^{n-1} \frac{(p_1 + K)(p_2 + K)}{(1 + K)(q + K)}, \quad n = 1, 2, \cdots, \] 
(4.5.5)
under constraints
\[ 0 < p_1 < +\infty, \quad 0 < p_2 < +\infty, \quad p_1 + p_2 < q < +\infty. \] 
(4.5.6)

In case \( p_2 = 1 \) it reduces to two-parametric family of Waring Distributions.

### 4.5.2 Probability \( P_0(p_1, p_2, q) \)

Let us find more suitable expression than (4.5.4) for probability \( P_0(p_1, p_2, q) \).

By series expansion for \( F(p_1, p_2; q, 1) \), form (4.5.4) of \( P_0(p_1, p_2, q) \), integral representation (4.4.10), and expression (4.4.11) for Beta Function, we conclude:
\[ \frac{1}{P_0(p_1, p_2, q)} = F(p_1, p_2; q, 1) = \frac{1}{B(p_2, q - p_2)} \int_0^1 t^{p_2-1}(1 - t)^{q-p_1-p_2-1} dt = \]
\[ = (B(p_2, q - p_1 - p_2)/B(p_2, q - p_2)). \] 
(4.5.7)

With the help of well-known formula
\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \] 
(4.5.8)
(4.5.7) transforms into a following symmetric by parameters \( p_1 \) and \( p_2 \) form

\[
(P_0(p_1, p_2, q))^{-1} = \frac{\Gamma(q - p_1 - p_2)\Gamma(q)}{\Gamma(q - p_1)\Gamma(q - p_2)}.
\tag{4.5.9}
\]

Consider particular cases of parameters.

(a) For family of Waring Distributions

\[
P_0 = (1 + \sum_{n \geq 1} \prod_{K=0}^{n-1} \frac{p + K}{q + K})^{-1}, \quad P_n = P_0 \cdot \prod_{K=0}^{n-1} \frac{p + K}{q + K}, \quad n = 1, 2, \ldots,
\]

with constraints \( 0 < p < q - 1 < +\infty \) putting in (4.5.9) \( p_1 = p, p_2 = 1 \), (or \( p_1 = 1, p_2 = p \)) we come to known formula (see, [6]): \( P_0 = 1 - \frac{p}{q - 1} \).

(b) Generalize (a) assuming that: \( p_1 \) is an integer. By (4.5.1),

\[
\frac{\Gamma(q - p_1)}{\Gamma(q)\Gamma(q - p_2)} = \frac{1}{(q - 1)(q - 2) \cdots (q - p_1)},
\]

\[
\frac{\Gamma(q - p_2)}{\Gamma(q - p_1 - p_2)} = (q - p_2 - 1)(q - p_2 - 2) \cdots (q - p_1 - p_2),
\]

and, as a result of this, because of \( q > p_1 + p_2 \), from (4.5.9) we obtain

\[
P_0(p_1, p_2, q) = \prod_{K=1}^{p_2} \frac{q - p_2 - K}{q - K} = \prod_{K=1}^{p_1} \left(1 - \frac{p_2}{q - K}\right).
\tag{4.5.10}
\]

By symmetry, under the assumption: \( p_2 \) is an integer we conclude

\[
P_0(p_1, p_2, q) = \prod_{K=1}^{p_1} \left(1 - \frac{p_1}{q - K}\right).
\tag{4.5.10'}
\]

(c) Assume that: \( q \) and \( p_1 + p_2 \) are integers. Assumption doesn’t imply that \( p_1 \) or \( p_2 \) are integers. For instance, \( q = 3, p_1 = p_2 = (1/2) \). We have

\[
P_0(p_1, p_2, q) = \frac{\Gamma(q - p_1)\Gamma(q - p_2)}{(q - p_1 - p_2)!!(q - 1)!!}.
\tag{4.5.11}
\]

In particular, if \( p_1 = s_1 - \frac{1}{2}, p_2 = s_2 - \frac{1}{2} \), where \( s_1 \) and \( s_2 \) are integers, then, due to 8.339, p.938, [20], \( \Gamma(n + \frac{1}{2}) = 2^{-n} \cdot \sqrt{\pi} \cdot (2n - 1)!! \), \( n = 1, 2, \ldots \), and from (4.5.11) we conclude

\[
P_0(s_1 - \frac{1}{2}, s_2 - \frac{1}{2}, q) = \pi \cdot \frac{(2(q - s_1) - 1)!!(2(q - s_2) - 1)!!}{(q - s_1 - s_2)!!(q - 1)!}.
\]

Finally, present an infinite-product representation for \( P_0(p_1, p_2, q) \) based on formula (see, 8.325, p.936, [20])

\[
\frac{\Gamma(x)\Gamma(y)}{\Gamma(x + z)\Gamma(y - z)} = \prod_{K \geq 0} \left(1 + \frac{z}{x + K}\right)\left(1 - \frac{z}{y + K}\right).
\tag{4.5.12}
\]
Putting in (4.5.12) \( x = q - p_1, \ y = q - p_2, \ z = p_1 \) we transform (4.5.9) into the following equality (the second one is written by symmetry)

\[
P_0(p_1, p_2, q) = \prod_{K \geq 0} \left(1 + \frac{p_1}{q - p_1 + K} \right) \left(1 - \frac{p_1}{q - p_2 + K} \right)
\]

\[
= \prod_{K \geq 0} \left(1 + \frac{p_2}{q - p_2 + K} \right) \left(1 - \frac{p_2}{q - p_1 + K} \right).
\]

Now, from (4.5.1) with \( \Theta = 1 \) we conclude that \( \{P_n(p_1, p_2, q)\} \) exhibits a constant slowly varying component

\[
L = \lim_{n \to +\infty} L(n) = \lim_{n \to +\infty} n^\rho P_n(p_1, p_2, q) = P_0(p_1, p_2, q) \cdot \frac{\Gamma(q)}{\Gamma(p_1) \Gamma(p_2)} =
\]

\[
= \frac{\Gamma(q - p_1)}{\Gamma(p_1)} \cdot \frac{\Gamma(q - p_2)}{\Gamma(p_2)} \cdot \frac{1}{\Gamma(q - p_1 - p_2)},
\]

where (4.5.9) was used.

If \( p_1 \) is an integer, then putting \( p_2 = p \) and \( p_1 = m \) we obtain

\[
L = \left(\frac{\rho - 1}{(m - 1)!}\right) \cdot \frac{\Gamma(q - m)}{\Gamma(p)}.
\]

which for \( m = 1 \) coincides with the constant slowly varying component of Waring Distribution (see, (1.7.13), where \( q \) is replaced by \( q - 1 \)).

### 4.5.3 The Log-Log Plot of Hypergeometric Distribution

Below we deal with \( \ln P_n \) versus \( \ln n \). Due to (4.5.4)-(4.5.6), we write down

\[
x_n := \frac{\ln P_n}{\ln n} = \frac{1}{\ln n} \left( c_1 + \sum_{K=1}^{n-1} \ln \left( \frac{(p_1 + K)(p_2 + K)}{(1 + K)(q + K)} \right) \right), \ n = 2, 3, \ldots,
\]

where \( c_1 = \ln \left( \frac{p_1 p_2}{q} \cdot P_0 \right) \). Continue calculations

\[
(ln n) x_n = c_1 + \sum_{K=1}^{n-1} (f_{p_1}(K) + f_{p_2}(K) - f_1(K) - f_q(K)), \ n = 2, 3, \ldots,
\]

where for \( \varepsilon \in R^+ \) and \( x \in [1, +\infty) \) we denote \( f_\varepsilon(x) = \ln(1 + \frac{x}{\varepsilon}) \). The function \( f_\varepsilon(x) \) is bounded and increasing. By [24], p.284, \( \sum_{K=1}^{n} f_\varepsilon(K) = \int_1^n f_\varepsilon(x)dx + a(\varepsilon) + \alpha_n(\varepsilon) \) for \( n = 1, 2, \ldots, \) where \( a(\varepsilon) \) is constant and \( \lim_{n \to +\infty} \alpha_n(\varepsilon) = 0 \). Substituting the last equality into (4.5.15) we obtain

\[
(ln n) x_n = c_2(1 + o(1)) + \int_1^{n-1} (f_{p_1}(x) + f_{p_2}(x) - f_1(x) - f_q(x))dx, \ n \to +\infty,
\]

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where \( c_2 = c_1 + a(p_1) + a(p_2) - a(1) - a(q) \). Further, for \( n = 1, 2, \ldots \) and \( \varepsilon \in R^+ \)

\[
\int_{0^-}^{n} f_{\varepsilon}(x)dx = n \ln(\varepsilon + n) - n \ln n - \int_{0^-}^{n} \frac{xdx}{x + \varepsilon} + \int_{0^-}^{n} dx = (n + \varepsilon) \ln(n + \varepsilon) - n \ln n - \varepsilon \ln \varepsilon + n.
\]

Substituting the last equality into (4.5.16) after not complicate calculations we get an equality

\[
\ln \frac{P_n}{\ln n} = p_1 + p_2 - q - 1 + \frac{c_3}{\ln n} (1 + o(1)), \quad n \to +\infty,
\]

(4.5.17)

where \( c_3 \) is some real constant. Due to (4.5.17), the deviations of \( \ln P_n \) versus \( \ln n \) from the straight line \( y = p_1 - p_2 - q - 1 = \text{const} \) are small, at least, for large values of \( n \) and show upward/downward convexity.

### 4.5.4 Convexity and Unimodality

Form the ratio

\[
\frac{P_{n+1}(p_1, p_2, q)}{P_n(p_1, p_2, q)} = \frac{(p_1 + n)(p_2 + n)}{(1 + n)(q + n)}, \quad n = 0, 1, 2, \ldots.
\]

(4.5.18)

If \( \min(p_1, p_2) < 1 \), for instance, \( p_1 < 1 \), then \( \frac{p_1 + n}{1 + n} \) is less than one and increases as \( n \) increases. Since \( p_2 < q \), therefore \( \frac{p_2 + n}{q + n} \) is less than one and increases as \( n \) increases. It means that their product \( \frac{(p_1 + n)(p_2 + n)}{(1 + n)(q + n)} \) is less than one and increases as \( n \) increases. Thus, if one of parameters \( p_1, p_2 \) is less than one, then, due to (4.5.18), \( \{P_n(p_1, p_2, q)\} \) decreases and is downward convex.

Automatically, it implies the unimodality of sequence \( \{P_n(p_1, p_2, q)\} \).

The remained case, when simultaneously \( p_1 > 1, p_2 > 2 \) is more complicate.

Transforming the right-hand-side of (4.5.18) we come to the equality

\[
\frac{P_{n+1}(p_1, p_2, q)}{P_n(p_1, p_2, q)} = \frac{(p_1 - 1)(p_2 + n)}{(1 + n)(q + n)} + \frac{p_2 + n}{q + n} = \frac{(p_1 - 1)(p_2 - 1)}{(1 + n)(q + n)} + \frac{p_1 - 1}{q + n} + \frac{p_2 - q}{q + n} + 1 = 1 - \frac{\rho}{q + n} + \frac{(p_1 - 1)(p_2 - 1)}{(1 + n)(q + n)},
\]

(4.5.19)

where \( \rho \) is given by (4.5.2).

If \( (p_1 - 1)(p_2 - 1) \leq \rho \), then, due to (4.5.19), we are in previous situation.

Let \( (p_1 - 1)(p_2 - 1) > \rho \). For \( n \) large enough it is clear that the sequence \( \{P_n(p_1, p_2, q)\} \) increases and is downward convex. Moreover, it is true starting from index \( n_0 = [x] \), where \( x \) is a solution to equation (it follows from (4.5.19))

\[
\frac{(p_1 - 1)(p_2 - 1)}{(1 + x)(q + x)} = \frac{\rho}{q + x}, \quad \text{or} \quad x = \frac{(p_1 - 1)(p_2 - 1)}{\rho} - 1.
\]

The empirical facts for distributions \( \{P_n(p_1, p_2, q)\} \) given by (4.5.4)-(4.5.6) are fulfilled. We call the extracted family a family of Regular Hypergeometric Distributions.
4.6 Generating Function and Moments of Hypergeometric Distributions

In this Section the form of Generating Function of Hypergeometric Distributions derived in Section 4.4 is simplified for Regular Hypergeometric Distributions. It allows to evaluate factorial moments and, in particular, a mean value and a variance of these distributions.

Finally, the most important results of Sections 4.4.-4.5 and of this Section are combined in one statement, which gives a complete description of a structure of family of Regular Hypergeometric Distributions.

4.6.1 The Generating Function

Denote

\[ G(x; p_1, p_2, q) = \sum_{n \geq 0} x^n P_n(p_1, p_2, q) = \hat{G}(x; p_1, p_2, q, 1), \]  

(4.6.1)

where \( \hat{G} \) is defined by (4.4.9) and has a form of improper integrals’ ratio (4.4.12).

From (4.4.12) without any restrictions (because we are in frame of constraints (4.4.6) for our case, and for (4.4.6) the formula (4.4.12) is true) we come to equalities

\[ G(x; p_1, p_2, q) = \int_0^1 t^{p_2-1}(1 - t)^{q-p_2-1}(1 - tx)^{-p_2} dt = \frac{\Gamma(q - p_1)\Gamma(q - p_2)}{\Gamma(q - p_1 - p_2)\Gamma(q)} F(p_1, p_2; q, x) = P_0 \cdot F, \]  

(4.6.2)

where \( \Gamma(x) \) and \( B(x, y) \) are Gamma and Beta Functions, respectively.

Obtaining the last equality in (4.6.2) we used (4.5.7)-(4.5.9).

Basing on Generating Functions of Regular Hypergeometric Distributions it is possible to evaluate factorial moments of distribution \( \{P_n(p_1, p_2, q)\} \). Let the assumption hold

\[ K + 1 < \rho < +\infty, \]  

(4.6.3)

where \( K \geq 1 \) is an integer. Since \( (-\rho) \) is the exponent of regular variation of Regular Hypergeometric Distribution \( \{P_n(p_1, p_2, q)\} \), then, according to Theory of Regularly Varying Functions, the moments of integer orders

\[ m_i = m_i(p_1, p_2, q) = \sum_{n \geq 1} n^i P_n(p_1, p_2, q), \quad i = 1, 2, \ldots, K, \]  

(4.6.4)

are finite. It implies the finiteness of factorial moments

\[ M_i = \sum_{j \geq i} n(n - 1) \cdots (n - j + 1) P_n(p_1, p_2, q), \quad i = 1, 2, \ldots, K, \]  

(4.6.5)
which may be evaluated by using following well-known formula

\[ M_i = \frac{d^i}{dx^i} G(x; p_1, p_2, q) / x = 1, \quad i = 1, 2, \ldots, K. \]  

(4.6.6)

The goal of this Section consists in evaluation of factorial moments and, in particular, a mean value \( m_1 = M_1 \), and a variance \( \sigma^2(p_1, p_2, q) \) of distribution \( \{P_n(p_1, p_2, q)\} \). Due to (4.6.4)-(4.6.5), we have

\[
M_2 = \sum_{n \geq 2} n(n - 1)P_n = \sum_{n \geq 2} n^2 \cdot P_n - \sum_{n \geq 2} nP_n = \sum_{n \geq 1} n^2 P_n - \sum_{n \geq 1} nP_n = m_2 - m_1,
\]
or

\[
m_2 = M_2 + m_1. \tag{4.6.7}
\]

Then, due to (4.6.7),

\[
\sigma^2(p_1, p_2, q) = m_2 - (m_1)^2 = M_2 + m_1(1 - m_1). \tag{4.6.8}
\]

### 4.6.2 Factorial Moments

Due to (4.6.2) and (4.6.3)-(4.6.6) there are two ways. We may differentiate \( i \) times, \( i = 1, 2, \ldots, K \), by \( x \) either the expression (4.4.4) with \( \Theta = x \), or integral representation (4.6.2) for \( G(x; p_1, p_2, q) \) and put after that \( x = 1 \).

The first way may be based on following formula for hypergeometric function (see, (9.2.3), 9.2, p.241, [64]): for \( m = 1, 2, \ldots \)

\[
\frac{d^m}{dx^m} F(p_1, p_2; q, x) = (\prod_{i=0}^{m-1} \frac{(p_1 + i)(p_2 + i)}{q + i}) F(p_1 + m, p_2 + m; q + m, x). \tag{4.6.9}
\]

Indeed, differentiating \( i \) times, \( i = 1, 2, \ldots, K \), both sides of equality (4.6.2) and putting after that \( x = 1 \) with the help of (4.6.9) we obtain

\[
\frac{d^i}{dx^i} G(x; p_1, p_2, q) |_{x=1} = P_0(p_1, p_2, q) \frac{d^i}{dx^i} F(p_1, p_2, q, x) |_{x=1} =
\]

\[
= P_0(p_1, p_2, q) \prod_{j=0}^{i-1} \frac{(p_1 + j)(p_2 + j)}{q + j} \cdot F(p_1 + i, p_2 + i; q + i, 1) =
\]

\[
= \prod_{j=0}^{i-1} \frac{(p_1 + j)(p_2 + j)}{q + j} P_0(p_1, p_2, q) =
\]

\[
= \prod_{j=0}^{i-1} \frac{(p_1 + j)(p_2 + j)}{q + j} \Gamma(q - p_1 - p_2 - i) \Gamma(q + i) =
\]

\[
= \prod_{j=1}^{i-1} \frac{(p_1 + j)(p_2 + j)}{\rho - j - 2},
\]

where \( \Gamma(x) \) is Euler’s Gamma Function. Here (4.5.11) and the formula \( x \Gamma(x) = \Gamma(x + 1) \) were used.

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Thus, under the assumption (4.6.3)

\[ M_K = \prod_{i=0}^{K-1} \frac{(p_1 + i)(p_2 + i)}{\rho - i - 2}. \] (4.6.10)

Now, let us apply the second way. Differentiating \( K \) times the expression under the integral at the right-hand-side of (4.6.2) by \( x \) and putting after that \( x = 1 \) we obtain

\[ M_i = \frac{p_1(p_1 + 1) \cdots (p_1 + K - 1) \cdot B(p_2 + K, q - p_1 - p_2 - K)}{B(p_2, q - p_1 - p_2)} = \frac{p_1(p_1 + 1) \cdots (p_1 + K - 1) \cdot \Gamma(p_2 + K)\Gamma(q - p_1 - p_2 - K)}{\Gamma(p_2)\Gamma(q - p_1 - p_2)} = \prod_{i=0}^{K-1} \frac{(p_1 + i)(p_2 + i)}{\rho - i - 1}, \]

which coincides with (4.6.10). Here (4.5.8) was used.

In particular, from (4.6.10) we conclude:

\[ m_1 = M_1 = \frac{p_1p_2}{\rho - 2}, \] (4.6.11)

and, due to (4.6.7), (4.6.11), we have

\[ m_2 = M_2 + m_1 = \frac{p_1(p_1 + 1)p_2(p_2 + 1)}{(\rho - 2)(\rho - 3)} + m_1 = m_1 \cdot (1 + \frac{(p_1 + 1)(p_2 + 1)}{\rho - 3}) = m_1 \cdot \frac{q - 1 + p_1p_2}{\rho - 3}. \] (4.6.12)

Putting in (4.6.11) and (4.6.12) \( p_2 = 1, p_1 = p \) (or \( p_1 = 1, p_2 = p \)) we obtain the first and second moments of Waring Distribution (3.9.11) and (3.9.18) with \( q - 1 \) instead of \( q \), respectively.

From (4.6.8), (4.6.11) and (4.6.12) the expression for the variance of Regular Hypergeometric Distribution is derived

\[ \sigma^2(p_1, p_2, q) = m_1(\frac{q - 1 + p_1p_2}{q - p_1 - p_2 - 2} - m_1) = \frac{p_1p_2}{(\rho - 2)(\rho - 3)}(q - 1 - \frac{p_1p_2}{\rho - 2}). \] (4.6.13)

### 4.6.3 The Structure of Hypergeometric Distributions

Gathering all important results on four-parametric family of Hypergeometric Distributions, defined by (4.4.2)-(4.4.3), and on its three-parametric subfamily of Regular Hypergeometric Distributions, defined by formulas (4.5.4)-(4.5.6) and (4.5.9), we may describe the structure of Hypergeometric Distributions.

Theorem 4.3
(a) Hypergeometric Distribution is proper iff either (4.4.5) or (4.4.6) hold.

(b) The family of Hypergeometric Distribution is generated by standard birth-death process with coefficients of form (4.4.8).

(c) Hypergeometric Distribution \(\{\hat{p}_n(p_1, p_2, q, \Theta)\}\) varies regularly iff \(\Theta = 1\).

(d) Regular Hypergeometric Distribution \(\{P_n(p_1, p_2, q)\}\) has exponent \((-\rho)\) of regular variation, where \(\rho = q + 1 - p_1 - p_2 \in (1, +\infty)\), and its right tail as well as this distribution itself exhibit constant slowly varying component (see, formula (4.5.13)).

(e) If \((p_1 - 1)(p_2 - 1) \leq \rho\), then Regular Hypergeometric Distribution \(\{P_n(p_1, p_2, q)\}\) decreases, is downward convex and unimodal. Otherwise, i.e. if \((p_1 - 1)(p_2 - 1) > \rho\), then Regular Hypergeometric Distribution \(\{P_n(p_1, p_2, q)\}\) decreases and is downward convex starting from index \(n_0 = [x]\), where \(x = \frac{(p_1 - 1)(p_2 - 1)}{\rho} - 1 > 0\).

(f) The Generating Function of \(\{P_n(p_1, p_2, q)\}\) is given by (4.6.2).

(g) If \(K + 1 < \rho < +\infty\), where \(K \geq 1\) is an integer, then the \(K\)-th factorial moment of distribution \(\{P_n(p_1, p_2, q)\}\) is given by formula (4.6.10).

In particular, under the assumption \(2 < \rho < +\infty\) the mean value of \(\{P_n(p_1, p_2, q)\}\) is given by formula (4.6.11).

(h) If \(2 < \rho < +\infty\), the distribution \(\{P_n(p_1, p_2, q)\}\) has finite variance, which is evaluated with the help of formula (4.6.13).

4.7 Special Forms of Coefficients

The standard birth-death process with intensities

\[\lambda_n = \lambda_1^* + \lambda_2^* \phi_n, \quad \mu_{n+1} = \mu_1^* + \mu_2^* \phi_{n+1}, \quad n = 0, 1, 2, \ldots,\]  \hspace{1cm} (4.7.1)

where \(\lambda_i^* \in R^+, \mu_i^* \in R^+\) for \(i = 1, 2\), and

\[\phi_0 = 0, \quad \{\phi_n\} \text{ increases, } \lim_{n \to +\infty} \phi_n = +\infty, \lim_{n \to +\infty} \frac{\phi_{n+1}}{\phi_n} = 1,\]  \hspace{1cm} (4.7.2)

in our opinion, may explain the general properties of many large-scale biomolecular systems. In particular, the model (4.7.1)-(4.7.2) suits well to model the process of gene expression in eukaryotic cells which exhibits a strong stochastic component, chaotic movement, and a skewed distribution of the number of events. The strong stochastic component of the evolutionary process of the number of transcripts of a particular gene is presented in terms of parameters \(\lambda_2^*\) and \(\mu_2^*\). The factors \(\lambda_2^* \psi_n\) and \(\mu_2^* \psi_n\) characterize the units intensities at a given epoch. The chaotic movement is presented in terms of parameters
\( \lambda_i \) and \( \mu_i \). So the evolutionary process is formed by two independent local processes. This model in particular case \( \phi_n = n, n = 0, 1, 2, \ldots \), has been considered in [4].

The presented model has been deeply investigated in [16]. We’ll formulate the results from [16], which as we’ll see are particular cases of results from Sections 4.2-4.3.

### 4.7.1 The Results

Here the steady-state condition (4.1.3) has the form

\[
T_0 = \sum_{n \geq 1} \prod_{m=1}^{n} \frac{\lambda_i + \lambda_i \phi_{m-1}}{\mu_i + \mu_i \phi_{m}} < +\infty, \quad \text{or, if we define} \quad e = \frac{\lambda_i}{\mu_i}, \quad \Theta = \frac{\lambda_i}{\mu_i}, \quad f = \frac{\mu_i}{\mu_i}, \quad \text{it can be written as}
\]

\[
T_0 = \sum_{n \geq 1} \prod_{m=1}^{n} \frac{e + \Theta f \cdot \phi_{m-1}}{1 + f \cdot \phi_{m}} < +\infty. \tag{4.7.3}
\]

Introducing the ratio \( c = \frac{e}{\Theta} \), noting that \( \Theta(c - 1) = e - \Theta \), and putting

\[
\psi_{n} = 1 + f \cdot \phi_{n}, \quad n = 0, 1, 2, \ldots, \tag{4.7.4}
\]

we transform (4.7.2) into

\[
\psi_{0} = 1, \quad \{\psi_{n}\} \text{ increases,} \quad \lim_{n \to +\infty} \psi_{n} = +\infty, \quad \lim_{n \to +\infty} \frac{\psi_{n+1}}{\psi_{n}} = 1, \tag{4.7.5}
\]

and the steady-state existence condition into

\[
T_0 = T(\Theta, c) = \sum_{n \geq 1} \Theta^n \cdot \prod_{m=1}^{n} \frac{\psi_{m-1} + c - 1}{\psi_{m}} < +\infty. \tag{4.7.6}
\]

The transform (4.7.4) doesn’t affect only the form of \( T_0 \) but also the form of stationary distribution \( \{p_n\} \). Thus, from the very beginning we may consider the birth-death process with intensities

\[
\lambda_n = \Theta(\psi_{n} + (c - 1)), \quad \mu_{n+1} = \psi_{n+1}, \quad n = 0, 1, 2, \ldots, \tag{4.7.7}
\]

where \( \Theta \in R^+, \quad c \in R^+ \), and constraints (4.7.5) hold.

Defined by (4.7.7) standard birth-death process satisfies assumptions 1.-2. and assumption 3. with \( n_0 = 0, \quad \gamma_n = \frac{\psi_n}{1-c}, \quad n = 0, 1, \ldots, \quad c \neq 1 \), from Section 4.2. Depending on the growth of sequence \( \{\psi_{n}\} \), there are two cases

- **either** \( \sum_{n \geq 1} \frac{1}{\psi_{n}} = +\infty \), \tag{4.7.8}

- **or** \( \sum_{n \geq 1} \frac{1}{\psi_{n}} < +\infty \). \tag{4.7.9}

Now, we may formulate, with the help of Theorem 4.2, the following result (see, also [16]).
Theorem 4.4 Let the conditions (4.7.5) take place.

(a) Let additionally (4.7.8) hold. Then: $T(\Theta, c) < +\infty$ iff

\[
either 0 < \Theta < 1, \ 0 < c < +\infty, \ or \ \Theta = 1, \ 0 < c < 1.
\]

(b) Let additionally (4.7.9) hold. Then: $T(\Theta, c) < +\infty$ iff $0 < \Theta \leq 1, \ 0 < c < +\infty$.

For the so-called etalon linear case

\[
\psi_n = 1 + n, \ n = 0, 1, 2, \cdots,
\]

by (4.7.6), we have the steady-state existence condition

\[
T_0 = T(\Theta, c) = \sum_{n \geq 1} \Theta^n \cdot \prod_{m=1}^{n} \frac{m + c - 1}{m} < +\infty.
\]

Since in etalon linear case (4.7.8) holds, therefore (4.7.11) takes place iff

\[
either 0 < \Theta < 1, \ 0 < c < +\infty, \ or \ \Theta = 1, \ 0 < c < 1.
\]

This result was presented in [4].

4.7.2 Population’s Lifetime

Since natural selection and mutation are presented in the form (4.7.1) (or (4.7.7)) of intensities (coefficients) of special birth-death model considered above, so this model may be interpreted also as a population model. Here $\lambda_2^2\phi_n$ and $\mu_2^2\phi_{n+1}$ illustrate natural selection, as well as $\lambda_1^1$ and $\mu_1^1$ illustrate mutation. Then we have the right to talk about a lifetime $\pi$ of population in this population model.

Now, we are able to formulate, with the help of Theorems 4.1 and 4.4 the following result.

Theorem 4.5 For the mean value of the population lifetime in population model defined by (4.7.1)-(4.7.2) the following formula holds

\[
\lambda_1^* \cdot E\pi = \sum_{m \geq 1} \Theta^m \prod_{i=1}^{m} \frac{c - 1 + \psi_{i-1}}{\psi_i} < +\infty
\]

in following two cases:

(1) If (4.7.8) holds and

\[
either 0 < \Theta < 1, \ 0 < c < +\infty, \ or \ \Theta = 1, \ 0 < c < 1;
\]

(2) If (4.7.9) holds, and

\[
0 < \Theta \leq 1, \ 0 < c < +\infty.
\]

In all other cases $E\pi = +\infty$. 175
Proof. Let us write down (4.1.14) in the form
\[ \lambda_0 \cdot E\pi = T. \] (4.7.15)

Due to (4.7.14) and the form of \( T \) is our case (see, (4.7.6)), according to statements (a) and (b) of Theorem 4.4, we get formulas (4.7.13) and (4.7.14), respectively.

Other cases automatically lead to divergence of the steady-state condition. Here also the equality \( \lambda_0 = \lambda_1^* \) is used. Theorem 4.5 is proved.

For the etalon linear case (4.7.10), by (4.7.15), we come to the following formula for the mean value of the population \( \lambda_1^* \cdot E\pi = \sum_{m \geq 1} \Theta_m \prod_{i=1}^m \frac{i+c-1}{i} < +\infty \) if either \( 0 < \Theta < 1, \ 0 < c < +\infty \), or \( \Theta = 1, \ 0 < c < 1 \). In all other cases \( E\pi = +\infty \).

### 4.7.3 Distributions with Moderate Growth: Particular Cases

The considered model exhibits moderate growth of coefficients given by (4.7.1), so the stationary distributions of this model exhibit moderate growth. Since from this point we consider all class of distributions of moderate growth generated by standard birth-death process, it is more convenient now to return to the model being investigated in Sections 4.2-4.3.

In asymptotic analysis of the class of distributions of moderate growth the parameters \( a_1, a_2, \cdots, a_{n_0} \) are not essential. Therefore we assume that \( n_0 = 0 \). Different values \((-1, 0, +1)\) of the parameter \( \alpha \) give different types of distributions and separate analysis of them is needed. For each value of \( \alpha \) the essential parameters are \( \Theta \) and \( b \).

By Theorem 4.2, the role of \( \Theta \) is clear, \( d \) and \( b \) are scale parameters, linear and non-linear, respectively. This follows from the Remarks in Section 4.3, because for \( n \) large enough we have
\[ \frac{p_n(\alpha)}{p_0(\alpha)} \approx d \cdot \Theta^n \exp(-\alpha \cdot b \cdot \sum_{m=1}^n \frac{1}{\delta_m}). \]

So, the non-linear scale parameter exhibits exponential growth, i.e. is the exponential scale parameter.

Due to (4.2.14), the sequences \( \{\delta_n\} \) and \( \{\varepsilon_n\} \) are asymptotically equivalent and for the time being put \( \delta_n = \varepsilon_n = \psi_n \). In this way we obtain a subclass of distributions from the class of distributions of moderate growth of types \( \{p^+_n\}, \{p^-_n\}, \{p^0_n\} \), where
\[
\begin{align*}
p^\pm_0 &= (1 + d \cdot \sum_{n \geq 1} \Theta^n \prod_{m=1}^{n-1} (1 \pm \frac{b}{\psi_m}))^{-1}, \\
p^\pm_K &= d \cdot p^\pm_0 \Theta^K \prod_{m=1}^{K-1} (1 \pm \frac{b}{\psi_m}), \quad K = 1, 2, \cdots,
\end{align*}
\] (4.7.16)

and
\[
\begin{align*}
0 < \Theta < 1, & \ 0 \leq b < +\infty, \ 0 < d < +\infty \quad \text{for} \ \{p^+_n\}, \\
0 < \Theta \leq 1, & \ 0 \leq b < \psi_1, \ 0 < d < +\infty \quad \text{for} \ \{p^-_n\}.
\end{align*}
\]
The class of distributions \( \{p_0^\pm\} \) has the same form as \( \{p_n^\pm\} \) with \n\[ \Theta = 1, \quad 0 \leq b < +\infty, \quad 0 < d < +\infty. \]

We call the class of stationary distributions generated by standard birth-death process with (4.7.5)-(4.7.7) a class of Generalized Komolgorov-Waring Distributions. This class coincides with the subclass above if we set \( 0 < b < 1 \) for \( \{p_0^\pm\} \), \( d = c \), and put \( b = c - 1 \) for \( \{p_n^+\} \) and \( b = 1 - c \) for \( \{p_n^-\} \).

The class of Generalized Komolgorov-Waring Distributions includes the following distributions that are of interest in modeling biomolecular systems.

1. The family of Komolgorov-Waring Distributions [4], which includes as a subfamily a family of well-known Waring Distributions.

2. The family of Generalized Pareto Distributions is generated by standard birth-death process with intensities

\[
\lambda_n = \Theta((n + 1)^\alpha + c - 1), \quad \mu_{n+1} = (n + 2)^\alpha, \quad n = 0, 1, 2, \ldots, \quad 1 < \alpha < +\infty. (4.7.17)
\]

This is a new three-parametric family of distributions \( \{p_n^\pm\} \) of the form

\[
\begin{align*}
p_0^\pm &= (1 + (1 \pm b) \sum_{n \geq 1} \frac{\Theta^n}{(n+1)^\alpha} \prod_{m=1}^{n-1} (1 \pm \frac{b}{(m+1)^\alpha}))^{-1}, \\
p_K^\pm &= (1 \pm b)p_0^\pm \frac{\Theta^K}{(K+1)^\alpha} \prod_{m=1}^{K-1} (1 \pm \frac{b}{(m+1)^\alpha}), \quad K = 1, 2, \ldots,
\end{align*}
\]

with

\[
0 < \Theta \leq 1, \quad 0 \leq b < +\infty \quad \text{for} \quad \{p_n^+\},
\]

\[
0 < \Theta \leq 1, \quad 0 < b < 1 \quad \text{for} \quad \{p_n^-\}.
\]

4.8 Regularly Varying Frequency Distributions I

The goal of this Section consists in extracting from the class of distribution of moderate growth defined in the previous Section a subclass of skewed distributions which may become regularly varying under some additional assumptions on \( \{\delta_n\} \).

Next goal is to find necessary and sufficient conditions on \( \{\delta_n\} \) for the regular variation of the distributions from extracted subclass.

In this Section we deal with the class of distributions of moderate growth written in the form

\[
\begin{align*}
p_0^\pm &= (1 + d \cdot \sum_{n \geq 1} \frac{\Theta^n}{n^n} \prod_{m=1}^{n-1} (1 \pm \frac{b}{n^m}))^{-1}, \\
p_K^\pm &= d \cdot p_0^\pm \cdot \frac{\Theta^K}{K^\alpha} \prod_{m=1}^{K-1} (1 \pm \frac{b}{m^\alpha}), \quad K = 1, 2, \ldots
\end{align*}
\]

(4.8.1)
with \(-1 < b < +\infty\) for \(\{p_n^+\}\) and \(0 < b < \delta_1\) for \(\{p_n^-\}\).

Distribution \(\{p_n^0\}\) has the form of \(\{p_n^+\}\) with \(\Theta = 1\) and

\[
\sum_{n \geq 1} \frac{1}{\delta_n} < +\infty.
\]  

(4.8.2)

### 4.8.1 Narrowing the Class

Denote \(\bar{A} = \lim_{n \to +\infty} \frac{n}{\delta_n}\).  

(4.8.3)

**Theorem 4.6** Distributions \(\{p_n^\pm\}\) with \(0 < \Theta < 1\) and \(\{p_n^0\}\) with \(\bar{A} = +\infty\) cannot vary regularly at infinity under the assumptions (4.2.13) and (4.2.14).

In proof of **Theorem 4.6** we use the following

**Lemma 4.1** Let us denote for \(s = 2, 3, \ldots\)

\[
\bar{B}(s) = \lim_{n \to +\infty} \sum_{m = n}^{s-1} \frac{1}{\delta_m}.
\]  

(4.8.4)

Then:

\(\bar{A} < +\infty\) implies \(\bar{B}(s) < +\infty\) for all \(s = 2, 3, \ldots\);

\(\bar{B}(s) < +\infty\) for some \(s = 2, 3, \ldots\) implies \(\bar{A} < +\infty\).

**Proof.** Since \(0 < \bar{B}(2) < \bar{B}(3) < \cdots\), therefore if \(\bar{B}(s) < +\infty\) for some \(s\), then \(\bar{B}(l) < +\infty\) for \(l = 2, 3, \ldots, s\). Let us assume that \(\bar{A} < +\infty\). For \(s = 2, 3, \ldots\) and \(n = 1, 2, \ldots\) we have

\[
\frac{(s - 1)n}{\delta_{sn}} < \sum_{m = n}^{s-1} \frac{1}{\delta_m} < \frac{(s - 1)n}{\delta_n},
\]  

(4.8.5)

because \(\{\delta_n\}\) increases. From the second inequality (4.8.5) we obtain

\(\bar{B}(s) < +\infty\) for \(s = 2, 3, \ldots\).

In reverse, let \(\bar{B}(2) < +\infty\). From the first inequality (4.8.5) we get \(\frac{n}{\delta_{2n}} < \sum_{m=n}^{2n-1} \frac{1}{\delta_m}\), or tending \(n \to +\infty\)

\[
\lim_{n \to +\infty} \frac{2n}{\delta_{2n}} < 2 \cdot \bar{B}(2) < +\infty.
\]  

(4.8.6)

Since \(\{\delta_n\}\) increases and \(\lim_{n \to +\infty} \frac{1}{\delta_n} = 0\), so from (4.8.4) we obtain

\[
\lim_{n \to +\infty} \frac{2n + 1}{\delta_{2n+1}} = 2 \cdot \lim_{n \to +\infty} \frac{n}{\delta_{2n+1}} \leq 2 \cdot \lim_{n \to +\infty} \frac{n}{\delta_{2n}} < +\infty,
\]

which together with (4.8.6) leads to \(\bar{A} < +\infty\).

If \(\bar{B}(s) < +\infty\) for some \(s\), then \(\bar{B}(2) < +\infty\), and we come to the previous case.

Now, let us prove **Theorem 4.6**
1. For $s = 2, 3, \ldots$, $n = 1, 2, \ldots$, and $0 < \Theta < 1$ we have
\[
\frac{p_{sn}}{p_n} = \Theta^{(s-1)n} \cdot \frac{\varepsilon_n}{\varepsilon_{sn}} \prod_{m=n}^{sn-1} \left( 1 - \frac{b}{\delta_m} \right), \quad 0 < b < \delta_1.
\]
Since $\lim_{n \to +\infty} \Theta^{(s-1)n} = 0$, $1 - \frac{b}{\delta_m} < 1$ for $m = n, n + 1, \ldots, \varepsilon_{sn} = (\varepsilon_{sn}) (\frac{\delta_{sn}}{\delta_m})$ (the last equality together with (4.2.13)-(4.2.14) implies that for $\varepsilon \in (0, 1)$ starting from some index $n_0$ we have
\[
\frac{\varepsilon_n}{\varepsilon_{sn}} < 1 + \varepsilon, \quad n = n_0, n_0 + 1, \ldots,
\]
therefore
\[
\lim_{n \to +\infty} \frac{p_{sn}}{p_n} = 0 \text{ for } s = 2, 3, \ldots.
\]

2. For $s = 2, 3, \ldots$, $n = 1, 2, \ldots$, and $0 < \Theta < 1$ we have
\[
\frac{p_{sn}^+}{p_n} = \Theta^{(s-1)n} \cdot \frac{\varepsilon_n}{\varepsilon_{sn}} \prod_{m=n}^{sn-1} \left( 1 + \frac{b}{\delta_m} \right), \quad 0 < b < +\infty.
\]
For $\varepsilon \in (0, 1)$ satisfying condition $\Theta(1 + \varepsilon) < 1$ starting from some index $n_0$ inequality (4.8.7) holds and $\frac{b}{\delta_m} < \varepsilon$ for $m = n_0, n_0 + 1, \ldots$ simultaneously. Therefore, for $n = n_0, n_0 + 1, \ldots$ and $s = 2, 3, \ldots$ we have
\[
0 < \frac{p_{sn}^+}{p_n} < (\Theta(1 + \varepsilon))^{(s-1)n} \cdot (1 + \varepsilon) \to 0 \text{ as } n \to +\infty.
\]

3. Let $\Theta = 1$ and $\check{A} = +\infty$ (for $\{p_n^+\}$ and $\{p_n^0\}$). Due to Lemma 4.1, for $s = 2, 3, \ldots$
\[
\lim_{n \to +\infty} \sum_{m=n}^{sn-1} \frac{1}{\delta_m} = \check{B}(s) < \sum_{n \geq 1} \frac{1}{\delta_n},
\]
which excludes the case of $\{p_n^0\}$ for future consideration. So, we deal with $\{p_n^-\}$. For $s = 2, 3, \ldots$; $n = n_0, n_0 + 1, \ldots$ we have
\[
0 < \frac{p_{sn}^-}{p_n} = \frac{\varepsilon_n}{\varepsilon_{sn}} \prod_{m=n}^{sn-1} \left( 1 - \frac{b}{\delta_m} \right) < (1 + \varepsilon) \exp\left( \sum_{m=n}^{sn-1} \ln(1 - \frac{b}{\delta_m}) \right), \quad 0 < b < \delta_1,
\]
where the inequality (4.8.7) was used. By Lemma 4.1, $\check{B}(2) = +\infty$, therefore there is a sequence $\{n_K\}$ of integers, $0 < n_1 < n_2 < \cdots$, such that
\[
\lim_{K \to +\infty} \sum_{m=n_K}^{2n_K-1} \frac{1}{\delta_m} = +\infty.
\]

By (4.8.10)-(4.8.11), $0 \leq \lim_{K \to +\infty} \frac{p_{2n_K}^-}{p_{n_K}} \leq (1 + \varepsilon) \lim_{K \to +\infty} \exp(-b \cdot \sum_{m=n_K}^{2n_K-1} \frac{1}{\delta_m}) = 0$. 

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Theorem 4.6 is proved.

Let us throw out distributions of moderate growth mentioned in Theorem 4.6. The remainder is a subclass of distributions of moderate growth, which is described as follows.

Let \( d \in R^+ \) be a parameter, \( \{\varepsilon_n\} \) and \( \{\delta_n\} \) be sequences of positive numbers such that
\[
\begin{align*}
\{\delta_n\} \text{ increases, } & \lim_{n \to +\infty} \delta_n = +\infty, \lim_{n \to +\infty} \frac{\delta_{n+1}}{\delta_n} = 1, \\
& \lim_{n \to +\infty} \frac{n}{\delta_n} = 1, \ A := \lim_{n \to +\infty} \frac{n}{\delta_n} < +\infty.
\end{align*}
\] (4.8.12)

The remained distributions of moderate growth \( \{p_n\} = \{p_n(d,b)\} \) take the forms
\[
\begin{align*}
p_0 &= (1 + d \cdot \sum_{n \geq 1} \frac{1}{\varepsilon_n} \prod_{m=1}^{n-1} (1 - \frac{b}{\delta_m}))^{-1}, \\
p_K &= \frac{dp_0}{\varepsilon_K} \prod_{m=1}^{K-1} (1 - \frac{b}{\delta_m}), \ K = 1, 2, \ldots,
\end{align*}
\] (4.8.13)
with \( \sum_{n \geq 1} \frac{1}{\delta_n} = +\infty \).

Putting without loss of generality \( \delta_1 = 1 \) we may include the part of distributions of moderate growth of form (4.8.14) with \( -1 < b < +\infty \) in (4.8.13):
\[
\begin{align*}
p_0 &= (1 + d \sum_{n \geq 1} \frac{1}{\varepsilon_n} \prod_{m=1}^{n-1} (1 + \frac{|b|}{\delta_m}))^{-1}, \\
p_K &= \frac{dp_0}{\varepsilon_K} \prod_{m=1}^{K-1} (1 + \frac{|b|}{\delta_m}), \ K = 1, 2, \ldots,
\end{align*}
\] (4.8.14)
with \( \sum_{n \geq 1} \frac{1}{\delta_n} < +\infty \).

Distributions of moderate growth described above are suspected to vary regularly at infinity for some sequences \( \{\delta_n\} \). Thus, after a "negative" result - Theorem 4.6, now, we are going to obtain a "positive" result.

### 4.8.2 The Main Result

Let \( d \in R^+ \) be a parameter, \( \{\varepsilon_n\} \) and \( \{\delta_n\} \) be sequences of positive numbers satisfying (4.8.12). Any collection \( (d, \{\varepsilon_n\}, \{\delta_n\}) \) generates a one-parametric family (with parameter \( b \)) of distributions of moderate growth of type (4.8.13) if \( I = +\infty \) and of type (4.8.14) if \( I < +\infty \), where \( I = \sum_{n \geq 1} \frac{1}{\delta_n} \).

Now, our goal is discovering conditions on \( \{\delta_n\} \) which lead to regular variation of \( \{p_n\} \). In the present Section we solve this problem with additional assumption: the limit exists
\[
0 \leq \lim_{n \to +\infty} \frac{n}{\delta_n} := A < +\infty.
\] (4.8.15)

The result is sound as follows.
Theorem 4.7

1. \( \{p_n\} \) varies regularly iff \( \{\delta_n\} \) varies regularly;

2. If \((-\rho)\) and \(\alpha\) are exponents of \(\{p_n\}\)'s and \(\{\delta_n\}\)'s regular variation, respectively, then

\[
\rho = \alpha + (|b|) \cdot A, \quad \rho \in [1, +\infty), \quad \alpha \in [1, +\infty).
\]  

(4.8.16)

Note that the inclusion \(\alpha \in [1, +\infty)\) (see, (4.8.16)) is a consequence of (4.8.15). Indeed, let us assume the opposite, i.e. \(\alpha \in [0, 1)\). Then \(\delta_n = 1 + n^\alpha \cdot L(n), \ n = 0, 1, 2, \ldots\), and, by known property of regular variation [21], for \(\varepsilon \in (0, 1-\alpha)\) starting from some index \(1 + n^\alpha \cdot L(n) < n^{\alpha + \varepsilon}\).

Therefore, \(A > \lim_{n \to +\infty} \frac{n}{n^\alpha + \varepsilon} = +\infty\), which contradicts (4.8.15).

Remark 4.4 For regularly varying \(\{\delta_n\}\) with exponent \(\alpha\) the inclusion \(\alpha \in [1, +\infty)\) holds even if (4.8.15) doesn’t take place and only (4.8.12) holds.

The proof is similar to above given.

Theorem 4.7 is based on following auxiliary

Lemma 4.2 If (4.8.15) holds then for \(s = 2, 3, \ldots\) the limit exists

\[
B(s) := \lim_{n \to +\infty} \sum_{m=n}^{sn-1} \frac{1}{\delta_m} = A \cdot \ln s.
\]  

(4.8.17)

Proof. Let \(A = 0\). By (4.8.15), \(\frac{1}{\delta_m} = o\left(\frac{1}{m}\right), \ n \to +\infty\). or for \(s = 2, 3, \ldots\)

\[
\sum_{m=n}^{sn-1} \frac{1}{\delta_m} = o\left(\sum_{m=n}^{sn-1} \frac{1}{m}\right), \ n \to +\infty.
\]

Since \(\lim_{n \to +\infty} \sum_{m=n}^{sn-1} \frac{1}{m} = \ln s\) for \(s = 2, 3, \ldots\), therefore, for \(s = 2, 3, \ldots\)

\[
\sum_{m=n}^{sn-1} \frac{1}{\delta_m} = o(1), \ n \to +\infty.
\]

Let \(0 < A < +\infty\). For \(\varepsilon \in (0, 1)\) starting from some index \(n \geq 1\) the inequalities hold

\[
\frac{A(1-\varepsilon)}{m} < \frac{1}{\delta_m} < \frac{A(1+\varepsilon)}{m}, \ m = n, n+1, \ldots.
\]  

(4.8.18)

By (4.8.18), we obtain (4.8.17). Lemma 4.2 is proved.

Theorem 4.7 has a final complete form if \(\bar{A} = 0\) (then \(\bar{A} = A\)).

Corollary 4.1 Let \(\bar{A} = 0\). Then:

1'. \(\{p_n\} \) varies regularly iff \(\{\delta_n\} \) varies regularly; 2'. \(\rho = \alpha \in [1, +\infty)\).
Now, we are ready to prove Theorem 4.7. Note that for \( s = 2, 3, \ldots \)
\[
\lim_{n \to +\infty} \frac{\varepsilon_n}{s_n} = \lim_{n \to +\infty} \frac{\delta_n}{s_n} \quad \text{(if limit exists).}
\]  
(4.8.19)

Indeed, \( \lim_{n \to +\infty} \frac{\varepsilon_n}{s_n} = \lim_{n \to +\infty} \frac{\delta_n}{s_n} \cdot \lim_{n \to +\infty} \frac{\delta_n}{s_n} = \lim_{n \to +\infty} \frac{\delta_n}{s_n} \) for \( s = 2, 3, \ldots \), where the asymptotical equivalency of \( \{\varepsilon_n\} \) and \( \{\delta_n\} \) is used (see, (4.8.12)).

For \( \{p_n\} \) of type (2.4.13) and \( s = 2, 3, \ldots \), due to Lemma 4.2 and (4.8.19),
\[
\lim_{n \to +\infty} \frac{p_{sn}}{p_n} = \lim_{n \to +\infty} \frac{\varepsilon_n}{s_n} \cdot \exp(-b \cdot \lim_{n \to +\infty} \frac{1}{\delta_n} \sum_{m=n}^{sn-1} \frac{1}{\delta_m}) =
\]
\[
= \lim_{n \to +\infty} \frac{\delta_n}{s_n} \cdot \exp(-b \cdot A \cdot \ln n) = \frac{1}{s^{bA}} \cdot \lim_{n \to +\infty} \frac{\delta_n}{s_n}, \quad 0 < b < 1,
\]  
(4.8.20)

if limit exists. From (4.8.20) we conclude that \( \{p_n\} \) varies regularly iff \( \{\delta_n\} \) varies regularly and \( \rho = \alpha + bA \).

For \( \{p_n\} \) of type (4.8.14) and \( s = 2, 3, \ldots \), similarly to the previous case, we obtain
\[
\lim_{n \to +\infty} \frac{p_{sn}}{p_n} = \frac{1}{s^{-bA}} \cdot \lim_{n \to +\infty} \frac{\delta_n}{s_n}, \quad -1 < b < +\infty,
\]  
(4.8.21)

if limit exists. Since \( I < +\infty \) in this case, so \( A = 0 \) and we may replace the multiplier \( (1/s^{-bA}) \) at the right-hand-side of (4.8.21) by \( (1/s^{bA}) \).

Now, as in (4.8.20), formula (4.8.21) proves Theorem 4.7 in this case.

### 4.9 The Special Model

The Special Model has been introduced in Section 4.2. It appears as a standard birth-death process with asymptotically equivalent sequences of intensities (coefficients) \( \{\lambda_n\} \) and \( \{\mu_n\} \) satisfying additional assumption:
\[
\lim_{n \to +\infty} |\mu_n - \lambda_n| := b \in R^+.
\]  
(4.9.1)

The initial value \( \lambda_0 \) in sequence \( \{\lambda_n\} \) is the second parameter \( d \) (together with the first one, namely, \( b \)) of the model. Here the case \( \mu_n = \lambda_n, \ n = 1, 2, \ldots \), is excluded. Now, put for \( n = 1, 2, \ldots \)
\[
\varepsilon_n = \mu_n, \quad \delta_n = \frac{b \cdot \mu_n}{|\mu_n - \lambda_n|}.
\]  
(4.9.2)

Thus, sequences \( \{\varepsilon_n\} \) and \( \{\delta_n\} \), due to (4.9.1)-(4.9.2), are asymptotically equivalent and may be chosen independently.

Further, we assume that conditions (4.8.12) hold. Then, the stationary distributions of Special Model are given by (4.8.13)-(4.8.14), where in case (4.8.13) we put \( 0 < b < 1 \).

The preliminary information on Special Model is done.

We extract the Special Model because more deep results in this case in distinction with the general one is possible to obtain. This is the goal of the present Section.
4.9.1 General Statement

In the Special Model the following very important statement holds.

Lemma 4.3 Denote

\[ q_n' = \sum_{K \geq n} \frac{\epsilon_K}{\delta_K} p_K, \quad n = 1, 2, \cdots, \quad D = q_1', \]  \hspace{1cm} (4.9.3)

where \{p_n\} is a stationary distribution of Special Model. Then:
1. \[ 0 < D < +\infty; \]  \hspace{1cm} (4.9.4)

2. For distributions of form (4.8.13)

\[ \epsilon_n = \frac{dp_0 - bD + bq_n'}{p_n}, \]  \hspace{1cm} (4.9.5)

where \(0 < b < 1, \quad d \in R^+;\)

3. For distributions of form (4.8.14)

\[ \epsilon_n = \frac{dp_0 + bD - bq_n'}{p_n}, \]  \hspace{1cm} (4.9.6)

where \(-1 < b < +\infty, \quad d \in R^+.

Proof. 1. Let us prove (4.9.4). Put

\[ q_n = \sum_{K \geq n} p_K. \]

For \(\epsilon \in (0, 1)\) starting from some index \(n \geq 1\) the inequalities hold

\[ 1 - \epsilon < \frac{\epsilon_K}{\delta_K} < 1 + \epsilon, \quad K = n, n+1, \cdots. \]

Therefore, for \(m = n, n+1, \cdots \)

\[ (1 - \epsilon)q_m < q_m' < (1 + \epsilon)q_m, \]  \hspace{1cm} (4.9.7)

which, due to norming condition \(p_0 + p_1 + p_2 + \cdots = 1\), proves (4.9.4).

2. From (4.8.13) we have

\[ \frac{p_{n+1}}{p_n} = \frac{e_n}{e_{n+1}}(1 - \frac{p}{e_n}), \quad n = 1, 2, \cdots \] with initial condition \(p_1 = d/e_1\), where without loss of generality we put \(e_0 = 1\). Now, if we define \(a_{n+1} = \epsilon_{n+1} \cdot p_{n+1} + a_1 = p_1 e_1\), then

\[ a_{n+1} = a_n - b \cdot \frac{\epsilon_n}{\delta_n} p_n = \cdots = a_1 - b \cdot \sum_{K=1}^{n} \frac{\epsilon_K}{\delta_K} p_K = dp_0 - b \cdot \sum_{K=1}^{n} \frac{\epsilon_K}{\delta_K} \cdot p_K \] for \(n = 1, 2, \cdots \).

Therefore, \(\epsilon_n = \frac{dp_0 - b \sum_{K=1}^{n-1} \frac{\epsilon_K}{\delta_K} \cdot p_K}{p_n}, \quad 0 < b < 1 \) for \(n = 2, 3, \cdots \).

3. Similarly, from (4.8.14) for \(n = 1, 2, \cdots \) we proceed

\[ a_{n+1} = a_n + b_n \cdot \frac{\epsilon_n}{\delta_n} p_n = \cdots = a_1 + b \cdot \sum_{K=1}^{n} \frac{\epsilon_K}{\delta_K} p_K = dp_0 + b \cdot \sum_{K=1}^{n} \frac{\epsilon_K}{\delta_K} \cdot p_K. \]

Therefore, \(\epsilon_n = \frac{dp_0 + b \sum_{K=1}^{n-1} \frac{\epsilon_K}{\delta_K} \cdot p_K}{p_n}, \quad -1 < b < +\infty \) for \(n = 2, 3, \cdots \).

Lemma 4.3 is proved.

We call the relations (4.9.5) and (4.9.6) the reverse equalities for distributions of type (4.8.13) and (4.8.14), respectively.
4.9.2 Probability $p_0$

Let us exclude the case $b = 0$ in (4.8.14), because in this case we already have

$$p_0 = (1 + d \cdot \sum_{n \geq 1} \frac{1}{\varepsilon_n})^{-1}.$$

The following result is unexpected because in particular case $\varepsilon_n = \delta_n$, $n = 1, 2, \ldots$, for distributions of moderate growth of type (4.8.13) it gives simple expression for probability $p_0$. Namely,

$$p_0 = \frac{b}{d}, \quad 0 < b < 1. \quad (4.9.8)$$

This formula follows from

Theorem 4.8

(a) For distributions of moderate growth of type (4.8.13)

$$p_0 = \frac{b \cdot D}{d}, \quad 0 < b < 1. \quad (4.9.9)$$

(b) For distributions of moderate growth of type (4.8.14) denote

$$J := \prod_{n \geq 1} (1 + \frac{b}{\delta_n}), \quad -1 < b < +\infty, \quad b \neq 0. \quad (4.9.10)$$

Then:

$$J \in \begin{cases} (0, 1) & \text{if } -1 < b < 0, \\ (1, +\infty) & \text{if } 0 < b < +\infty, \end{cases} \quad (4.9.11)$$

and

$$p_0 = \frac{b \cdot D}{d \cdot (J - 1)}, \quad -1 < b < +\infty, \quad b \neq 0. \quad (4.9.12)$$

Proof. (a) Forming the ratio $1 = (p_n/p_n)$, where $p_n$ in numerator is taken from (4.9.5), i.e. $p_n = \frac{d p_0 - b D + b q_n'}{\varepsilon_n}$, and $p_n$ in denominator is taken from (4.8.13), for $n = 1, 2, \ldots$ we obtain

$$1 = \frac{d p_0 - b \sum_{K=1}^{n-1} \frac{\varepsilon_K}{\delta_K} p_K}{d p_0 \cdot \prod_{m=1}^{n-1} (1 - \frac{b}{\delta_m})}, \quad 0 < b < 1. \quad (4.9.13)$$

Let us show that

$$\lim_{n \to +\infty} \prod_{m=1}^{n} \left(1 - \frac{b}{\delta_m}\right) = \prod_{n \geq 1} \left(1 - \frac{b}{\delta_n}\right)(= J) = 0. \quad (4.9.14)$$
Indeed, for \( n = 1, 2, \ldots \)

\[
0 < \prod_{m=1}^{n} \left( 1 - \frac{b}{\delta_m} \right) = \exp\left( - \sum_{m=1}^{n} \left| \ln \left( 1 - \frac{b}{\delta_m} \right) \right| \right) < \\
< \exp\left( - \sum_{m=1}^{n} \frac{b}{\delta_m} + \frac{1}{2} \sum_{m=1}^{n} \left( \frac{b}{\delta_m} \right)^2 \right) < \exp\left( \frac{b^2}{2} \cdot \sum_{n \geq 1} \frac{1}{\delta_n^2} \right) \cdot \exp\left( - \sum_{m=1}^{n} \frac{b}{\delta_m} \right).
\] (4.9.15)

Since \( \sum_{n \geq 1} \frac{1}{\delta_n^2} < \sum_{n \geq 2} \frac{1}{n(n-1)} + 1 = 1 + \sum_{n \geq 1} \left( \frac{1}{n} - \frac{1}{n+1} \right) = 2 \), and \( I = +\infty \), therefore tending \( n \to +\infty \) in (4.9.15) we prove (4.9.14).

Now, tending \( n \to +\infty \) in (4.9.13) we conclude:

*Since the limit of denominator at the right-hand side of (4.9.12) as \( n \to +\infty \) equals to zero, and the left-hand side of (4.9.12) is a finite positive number, therefore, necessarily, numerator in the right-hand side of (4.9.12) tends to zero as \( n \to +\infty \).*

Thus, \( \frac{d\nu b}{\nu} = \lim_{n \to +\infty} \sum_{K=1}^{n-1} \frac{\epsilon_K}{\delta_K} p_K = D \), which coincides with (4.9.9).

(b) From (4.9.6) and (4.8.15) for \( n = 1, 2, \ldots \) and \( b \neq 0 \), \(-1 < b < +\infty \) we get the following equality \( d\nu \prod_{m=1}^{n} \left( 1 + \frac{b}{\delta_m} \right) = d\nu_0 + b \cdot \sum_{K=1}^{n} \frac{\epsilon_K}{\delta_K} p_K \), or tending \( n \to +\infty \)

\[
p_0 \cdot (J - 1) = \frac{bD}{d} \left\{ \begin{array}{ll}
> 0 & \text{if } 0 < b < +\infty, \\
< 0 & \text{if } -1 < b < 0.
\end{array} \right.
\] (4.9.16)

Since \( 0 < p_0 < 1 \) and \( 0 < \left| \frac{bD}{d} \right| < +\infty \), so from (4.9.16) we get (4.9.11), where \( J \) is defined by (4.9.10). At the same time, the equality (4.9.16) coincides with (4.9.12).

*Theorem 4.8* is proved.

In particular case \( \epsilon_n = \delta_n \), \( n = 1, 2, \ldots \), the equality (4.9.8) holds for distributions of type (4.8.13). Indeed,

\[
D = \sum_{n \geq 1} \frac{\epsilon_n}{\delta_n} p_n = \sum_{n \geq 1} p_n = 1 \text{ (see, (4.9.3))}
\] (4.9.17)

So, from (4.9.9) we have (4.9.8).

In this particular case for distributions of type (4.8.14), due to (4.9.12) and (4.9.17), we obtain

\[
p_0 = \frac{b}{d(J - 1)}, \quad -1 < b < +\infty, \quad b \neq 0,
\] (4.9.18)
where \( J \) is defined by (4.9.10). This formula seems to be more simple from the computational point of view than known for this case formula (4.8.14) for \( p_0 \):

\[
p_0 = (1 + d \cdot \sum_{n \geq 1} \frac{1}{\delta_n} \prod_{K=1}^{n-1} (1 + \frac{b}{\delta_K}))^{-1}.
\]

### 4.10 Regularly Varying Frequency Distributions II

In the present Section the main result on regular variation of stationary distributions of types (4.8.13) and (4.8.14) is improved. Here a new approach based on reverse equalities being obtained in the previous Section is applied. The improvement has to be made for the case

\[
0 < \bar{A} := \lim_{n \to +\infty} \frac{n}{\delta_n} < +\infty.
\]

#### 4.10.1 The Example

First of all, we must be convinced that there is a sequence \( \{\delta_n\} \) satisfying conditions (4.8.12) and such that

\[
\lim_{n \to +\infty} \frac{n}{\delta_n} < \lim_{n \to +\infty} \frac{n}{\delta_n}(= \bar{A}) < +\infty.
\]

Let us construct the required example. Let us draw a "broken" line whose pieces of straight lines are of two types: with slope \( 1 - \varepsilon \) and with slope \( 1 + \varepsilon \), where \( \varepsilon \in (0, 1) \) is some fixed number. The pieces of two types alternate each other. The curve begins from the point \((0, 1)\) on the plane. Denote by \((t_0, y_0) = (0, 1), (t_1, y_1), (t_2, y_2), \ldots, (t_n, y_n), \ldots\) successive points of the curve’s non-differentiability on the \((t, y)\)-plane. Thus, the pieces of this curve (broken line) \( y = \delta(t) \) in intervals \((t_{2n}, t_{2n+1})\), \( n = 0, 1, 2, \ldots \) have a slope \( 1 - \varepsilon \), and in intervals \((t_{2n-1}, t_{2n})\), \( n = 1, 2, \ldots \) have a slope \( 1 + \varepsilon \). We choose numbers \( \{t_K\} \) satisfying conditions \( \frac{y_{2n-1}}{t} = 1 - \varepsilon \) and \( \frac{y_{2n}}{t} = 1 + \varepsilon \) for \( n = 1, 2, \ldots \).

It is clear that the function \( y = \delta(t) \) defined on \( R^+ \) is positive, \( \delta(0) = 1, \delta(t) \) increases as \( t \) increases, \( \lim_{t \to +\infty} \frac{1}{\delta(t)} = \frac{1}{1+\varepsilon} \) and \( \lim_{t \to +\infty} \frac{1}{\delta(t)} = \frac{1}{1-\varepsilon} \). At the same time \( \lim_{t \to +\infty} \frac{\delta(t+1)}{\delta(t)} = 1 \).

The same properties has the sequence \( \{\delta_n\} \) of positive numbers, where we put

\[
\delta_n = \delta(n), \ n = 0, 1, 2, \ldots.
\]

Note that in our construction necessarily \( t_{n+1} - t_n \to +\infty \) as \( n \to +\infty \).

Thus, we built a sequence \( \{\delta_n\} \) satisfying conditions (4.8.12) and (4.10.2).

#### 4.10.2 The Main Result

The improvement of Theorem 4.7 uses the forward and reverse equalities (4.8.13) and (4.9.5), respectively. Let us make comments. The remained case in Theorem 4.7 which
requires an improvement is related with distributions of moderate growth of type (4.8.13) with \( A \in R^+ \) (in all other cases \( A = 0 \) and in (4.10.2) instead of inequality we have equality of limits). For this case the reverse equalities are of the form (4.9.5).

The result sounds as follows.

**Theorem 4.9** Let us consider distributions of moderate growth of type (4.8.13) with \( A \in R^+ \). Then:

1. \( \{p_n\} \) varies regularly at infinity iff the limit (4.8.15) exists.
2. The exponent \((-\rho)\) of \( \{p_n\} \)'s regular variation equals to \((-1 + b \cdot A)\).

**Proof.** First of all, we have to notice that the existence of limit (4.8.15) with \( A \in R^+ \) implies the regular variation of sequence \( \{\delta_n\} \) with exponent 1. Indeed, for \( s = 2, 3, \cdots \) from (4.8.15) we obtain \( \lim_{n \to +\infty} \frac{\delta_n}{\delta_{sn}} = \lim_{n \to +\infty} \frac{\delta_n}{\delta_{sn}} \cdot \frac{s}{\delta_n} = \frac{A}{A} s = s \).

For \( s = 2, 3, \cdots \) and \( n = 1, 2, \cdots \) from (4.9.5) taking into account notations (4.9.3) we obtain

\[
\frac{p_{sn}}{p_n} = \frac{\varepsilon_n}{\varepsilon_{sn}} \cdot \frac{dp_0 - bD + bq_{sn}}{dp_0 - bD + bq_n} = \frac{\varepsilon_n}{\varepsilon_{sn}} \cdot \frac{q_{sn}}{q_n},
\]

(4.10.3)

where the equality (4.9.8) was used.

From inequalities (4.9.7) it follows that sequences \( \{q_n\} \) and \( \{q_n'\} \) are asymptotically equivalent. Since, at the same time, sequences \( \{\varepsilon_n\} \) and \( \{\delta_n\} \) are asymptotically equivalent too, therefore, (4.10.3) may be written in form of asymptotical equivalency for \( s = 2, 3, \cdots \)

\[
\frac{p_{sn}}{p_n} \approx \frac{\delta_n}{\delta_{sn}} \cdot \frac{q_{sn}}{q_n}, \quad n \to +\infty.
\]

(4.10.4)

The forward equalities (4.8.13) for \( s = 2, 3, \cdots \) and \( n = 1, 2, \cdots \) give

\[
\frac{p_{sn}}{p_n} = \frac{\varepsilon_n}{\varepsilon_{sn}} \exp(\sum_{m=n}^{s-1} \ln(1 - \frac{b}{\delta_m})),
\]

which, due to asymptotical equivalency of sequences \( \{\varepsilon_n\} \) and \( \{\delta_n\} \), in accordance with (4.8.20) implies for \( s = 2, 3, \cdots \)

\[
\frac{p_{sn}}{p_n} \approx \frac{\delta_n}{\delta_{sn}} \exp(-b \cdot \sum_{m=n}^{s-1} \frac{1}{\delta_m}), \quad n \to +\infty.
\]

(4.10.5)

The preparatory work is over. Now let us pass directly to the proof of Theorem 4.9.

**THE NECESSITY**

Let \( \{p_n\} \) vary regularly and its exponent \((-\rho)\) satisfies condition \( \rho \in [1, +\infty) \).

By Theorem 1(a), VIII, 9, [23], the sequence \( \{q_n\} \) varies regularly with exponent \((-\rho + 1)\). Therefore, by (4.10.4), \( s^{-\rho} = \lim_{n \to +\infty} \frac{p_{sn}}{p_n} = \lim_{n \to +\infty} \frac{\delta_n}{\delta_{sn}} \lim_{n \to +\infty} \frac{q_{sn}}{q_n} = s^{-(\rho - 1)} \cdot \lim_{n \to +\infty} \frac{\delta_n}{\delta_{sn}} \).

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or \( \lim_{n \to +\infty} \frac{\delta_n}{s} = s \) for \( s = 2, 3, \cdots \). It means that \( \{ \delta_n \} \) varies regularly at infinity with exponent \( \alpha = 1 \). Next, from (4.10.5) for \( s = 2, 3, \cdots \) we obtain

\[
\lim_{n \to +\infty} \exp\left( -b \cdot \frac{1}{\delta_n} \sum_{m=n}^{sn-1} \frac{1}{\delta_m} \right) = \lim_{n \to +\infty} \frac{p_n}{n} \cdot \lim_{n \to +\infty} \frac{\delta_n}{s} = s^{-(\rho-1)},
\]

or for \( s = 2, 3, \cdots \) the limit (4.8.17) exists: \( B(s) = \lim_{n \to +\infty} \sum_{m=n}^{sn-1} \frac{1}{\delta_m} = \frac{\rho-1}{\delta} \ln s \).

Here \( \rho \in (1, +\infty) \), otherwise, by Lemma 4.1, if \( \rho = 1 \), then \( A = 0 \), which contradicts the assumption \( A \in R^+ \) of Theorem 4.9.

Denote \( A = \frac{\rho-1}{\delta} \), so \( B(s) = A \cdot \ln s \). Further, for \( s = 2, 3, \cdots \) and \( n = 1, 2, \cdots \) the following inequalities hold

\[
\frac{\delta_n}{s} \left( \frac{s+1}{s} \right)^{n-1} \sum_{m=sn}^{(s+1)n-1} \frac{1}{\delta_m} < n \frac{\delta_n}{s} \sum_{m=sn}^{(s+1)n-1} \frac{1}{\delta_m}.
\]

For \( \varepsilon \in (0, 1) \) starting from some index \( K \geq 1 \) the inequalities hold

\[
(1 - \varepsilon)A \cdot \ln \frac{s+1}{s} = (1 - \varepsilon)(B(s + 1) - B(s)) < \sum_{m=sn}^{(s+1)n-1} \frac{1}{\delta_m} < (1 + \varepsilon)(B(s + 1) - B(s)) = (1 + \varepsilon)A \cdot \ln \frac{s+1}{s}
\]

for \( s = 2, 3, \cdots \) and \( n = 1, 2, \cdots \). Here \( K = K(s) \).

Then, for a fixed \( s = 2, 3, \cdots \) and given \( \varepsilon \) the inequalities (4.10.6) may be rewritten in the form \((1 - \varepsilon) \cdot A \cdot \ln \frac{s+1}{s} \cdot \frac{\delta_n}{\delta_s} < n \frac{\delta_n}{\delta_s} < (1 + \varepsilon) \cdot A \cdot \ln \frac{s+1}{s} \cdot \frac{\delta_n}{\delta_s}\) for \( n = K, K + 1, \cdots \). Tending \( n \to +\infty \) in these inequalities we get the following ones

\[
(1 - \varepsilon) \cdot A \cdot \ln \frac{s+1}{s} < \lim_{n \to +\infty} \frac{n}{\delta_n} \leq \lim_{n \to +\infty} \frac{n}{\delta_n} \leq (1 + \varepsilon) \cdot A \cdot \ln \frac{s+1}{s},
\]

for \( s = 2, 3, \cdots \). Tending \( s \to +\infty \) and \( \varepsilon \downarrow 0 \) we prove the necessity of statement 1.

THE SUFFICIENCY

Since the existence of limit (4.8.15) implies regular variation of \( \{ \delta_n \} \), therefore we are in frame of statement 1 of Theorem 4.7, which implies \( \{ p_n \} \)’s regular variation.

Now, statement 2 of Theorem 4.9 follows from statement 2 of Theorem 4.7.

Theorem 4.9 is proved.

4.10.3 Slowly Varying Component

In accordance with Theorems 4.7 and 4.9, we change conditions on \( \{ \delta_n \} \), i.e. the condition (4.8.12). Namely, now we assume

\[
\delta_0 = 1, \ \{ \delta_n \} \text{ increases and varies regularly; the limit exists}:
\]

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\[ 0 \leq \lim_{n \to +\infty} \frac{n}{\delta_n} = A < +\infty; \quad (4.10.7) \]

The condition on positive sequence \( \{\varepsilon_n\} \) is conserved

\[ \lim_{n \to +\infty} (\varepsilon_n / \delta_n) = 1. \quad (4.10.8) \]

Then distributions of \textit{moderate growth} of types (4.8.13)-(4.8.14) together form \textit{some} class of \textit{regularly varying} distributions which we call a \textit{Regular Class}, and distributions from this class we call \textit{Regular Distributions}. \textit{Regular Distributions} are presented in the form

\[ p_n \approx n^{-\rho} \cdot L_1(n), \quad n \to +\infty, \quad (4.10.9) \]

with \( \rho \in [1, +\infty) \), where \((-\rho)\) is the \textit{exponent} of \( \{p_n\} \)'s \textit{regular variation}, and \( \{p_n\} \) exhibits \textit{slowly varying component} \( \{L_1(n)\} \). The form of presentation (4.10.9) gives the possibility to choose \( \{L_1(n)\} \) \textit{suitable} for us, i.e. having simple expression.

Below, some preliminary information on \( \{p_n\} \)'s \textit{slowly varying component} is derived.

Let

\[ \delta_n = 1 + n^\alpha \cdot L(n), \quad n = 0, 1, 2, \cdots \quad (4.10.10) \]

with \textit{exponent} \( \alpha \in [1, +\infty) \) of \( \{\delta_n\} \)'s \textit{regular variation}.

Consider possible cases of \( \{L_1(n)\} \)'s asymptotic behavior as \( n \to +\infty \).

1. For \textit{Regular Distributions} of type (4.8.14), due to (4.9.10) and (4.9.6), we have

\[ p_n \approx \frac{1}{n^\alpha L(n)} (dp_0 + bD - bq_n') \approx n^{-\alpha} \frac{dp_0 + bD}{L(n)}, \quad n \to +\infty, \quad (4.10.11) \]

with \(-1 < b < +\infty\). Here we use \( \lim_{n \to +\infty} q_n' = 0 \) and, due to \textit{Theorem 4.8}, we have \( dp_0 + bD \neq 0 \). Thus, in accordance with (4.9.9) and (4.9.11), in this case the \textit{slowly varying component} may be chosen as

\[ L_1(n) = \frac{dp_0 + bD}{L(n)}, \quad n = 1, 2, \cdots \quad (4.10.12) \]

2. Let us find the \textit{slowly varying component} for \( \{p_n\} \) of type (4.8.13). Let \( A = 0 \). Then, due to \textit{Theorem 4.7}, we have \( \rho = \alpha = 1 \). By using (4.8.13) we obtain

\[ p_n \approx \frac{bD}{nL(n)} \cdot \prod_{m=1}^{n-1} \left(1 - \frac{b}{mL(m)}\right), \quad n \to +\infty. \]

Here also \textit{Theorem 4.8} is used \( (dp_0 = bD) \).
It is clear that in this case the \textit{slowly varying component} may be chosen as

\[ L_1(n) = bD \cdot \prod_{m=1}^{n-1} \left(1 - \frac{b}{mL(m)}\right), \quad 0 < b < 1. \tag{4.10.13} \]

It leads, by the way, to the following result which is of interest for \textit{Theory of Regularly Varying Functions}.

\textbf{Theorem 4.10} If \{\(L(n)\)\} \textit{varies slowly} and \(\lim_{n \to +\infty} L(n) = +\infty\), then for any \(b \in (0, 1)\) the sequence \{\(L_1(n)\)\} of the form (4.10.13) \textit{varies slowly}.

Let us show that it is possible to choose another form of constant slowly varying component. Denote for \(n = 1, 2, \cdots\)

\[ \lambda_n = \frac{\varepsilon_n \cdot p_n}{bD \exp(b \cdot \sum_{m=1}^{n-1} \frac{1}{\delta_m})} = \exp\left(\sum_{m=1}^{n-1} [\ln(1 - \frac{b}{\delta_m}) + \frac{b}{\delta_m}]\right). \tag{4.10.14} \]

For \(0 < b < 1\) the function \(f(x) = \ln(1 - \frac{b}{x}) + \frac{b}{x}, \quad x \in [1, +\infty)\), is positive and increases as \(x\) increases. The positivity follows from the inequality \(f(x) \geq \frac{b^2}{x^2}\) (it is the second term in \((\ln(1 - x))'s\) expansion). Further,

\[ \frac{df(x)}{dx} = \frac{d}{dx} (\ln(x - b) - \ln x + \frac{b}{x}) = \frac{b}{x} \left(\frac{1}{x - b} - \frac{1}{x}\right) > 0. \]

Next, \(\sum_{m=1}^{n-1} f(\delta_m) < \sum_{m=1}^{n-1} \left(\frac{b}{\delta_m}\right)^2 < b^2(1 + \sum_{m=2}^{n} \frac{1}{m(m-1)}) = 2b^2, \quad n = 2, 3, \cdots\)

Therefore, \(0 \leq \lim_{n \to +\infty} \sum_{m=1}^{n} f(\delta_m) = \sum_{n=1}^{\infty} f(\delta_n) := P_b < +\infty\).

It means that, due to (4.10.14), \(\lambda_n = \exp(P_b + \Theta_n)\) for \(n = 1, 2, \cdots\) where \(\lim_{n \to +\infty} \Theta_n = 0\). Therefore, by (4.10.14), \{\(L_1(n)\)\} may be chosen as

\[ L_1(n) = \frac{1}{L(n)} b \cdot D e^{P_b} e^{b \sum_{m=1}^{n-1} \frac{1}{mL(m)}}. \tag{4.10.15} \]

Note that \(\sum_{m=1}^{\infty} \frac{1}{mL(m)} = +\infty\) in (4.10.15).

For \textit{Regular Distribution} of type (4.8.13) with \(0 < A < +\infty\) operating by the same manner we may only claim in general that as \{\(L_1(n)\)\} may be taken

\[ L_1(n) = A \cdot dp_0 n^A \cdot \prod_{m=1}^{n-1} \left(1 - \frac{b}{\delta_m}\right). \tag{4.10.16} \]

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Chapter 5

Dediscretization in Biomolecular Systems

5.1 Object of Study: Description

As we already know the mechanism of biomolecular large-scale systems dynamic often is explained with the help of standard birth-death process with various specific constraints on its coefficients. The stationary distributions of the process, which always have a skew to the right, are used as frequency distributions of different events taking place in large-scale biomolecular systems. In Chapter 4 a huge class of such distributions with moderate growth of the process’ coefficients is obtained. Moreover, this class is verified in order to satisfy empirical facts being well-known in macroevolution of biomolecular systems and being discussed from the mathematical point of view in Chapter 1. As a result of this from the obtained class a subclass is extracted, which under some natural assumptions on the process’ coefficients forms a class of regularly varying distributions, The last class includes all before known and successfully used distributions designed to model the empirical frequency distribution of the number of different type events arising in large-scale biomolecular systems, in particular, of the number of expressed genes in the transcriptome and the number of protein domain occurrences in the prometomes.

We are going to study the particular case of this class with $n_0 = 0$ and special form of parameter $d$, which shows the same qualitative and quantitative even behavior as the general one. We already mentioned that $d$ is not an essential parameter.

5.1.1 Description of the Class

In order to present the above mentioned class denote by $\Lambda$ the class of regularly varying with exponent $\alpha \in [1, +\infty)$ increasing sequences $\{\delta_n\}$ with $\delta_0 = 1$ satisfying following conditions: $\{\delta_n\}$ is: (a) downward convex, i.e. $\delta_{n-1} + \delta_{n+1} > 2\delta_n$ for $n = 1, 2, \cdots$; (b) log-upward convex, i.e. $\delta_n^2 > \delta_{n-1} \cdot \delta_{n+1}$ for $n = 1, 2, \cdots$. 

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The necessity of assumptions (a) and (b) on convexity is substantiated for particular, but wide enough case of the class in order to satisfy empirical facts in [16].

A Special class \( \Lambda_0 \) includes increasing sequences \( \{\delta_n\} \) of the type

\[
\delta_n = 1 + \frac{n}{A}(1 + o(1)), \quad n \to +\infty, \quad A \in \mathbb{R}^+.
\] 

(5.1.1) means that \( \delta_n = n \cdot L(n) + 1, \quad n = 0, 1, 2, \ldots \), where \( \{L(n)\} \) is a slowly varying sequence satisfying condition \( \lim_{n \to +\infty} L(n) = A^{-1} \). Or, we can say that \( \{\delta_n\} \) exhibits constant slowly varying component equal to \((1/A)\).

The sequence \( \{\delta_n^\pm\} \) of type (5.1.1) exhibits quite different behavior. It can be linear \( (\delta_n^\pm = 1 + n) \), downward convex \( (\delta_n^\pm = n + 1 - \ln(n + 1)) \), upward convex \( (\delta_n^\pm = n + 1 + \ln(n + 1)) \), of an oscillating type \( (\delta_n^\pm = (n + 1)(1 - (2)^{-n})) \), etc.

We conserve only the case of sequences of type (5.1.1) with \( o(1) = 0 \) in (5.1.1), or \( L(n) = A^{-1}, \quad n = 0, 1, 2, \ldots \), i.e. so-called linear case which is enough for biomolecular applications if it is possible to choose \( \{\delta_n\} \) from \( \Lambda_0 \).

Now, if \( \{\delta_n\} \in \Lambda \), then the limit exists

\[
\lim_{n \to +\infty} \frac{n}{\delta_n} = 0,
\] 

which follows from Theorem 4.9 in Section 4.10, and in accordance with (5.1.2) two situations arise:

\[
either \sum_{n \geq 1} \frac{1}{\delta_n} = +\infty \text{ (then, denote } \delta_n = \delta_n^-, \quad n = 1, 2, \ldots),
\]

(5.1.3)

\[
or \sum_{n \geq 1} \frac{1}{\delta_n} < +\infty \text{ (then, denote } \delta_n = \delta_n^+, \quad n = 1, 2, \ldots).
\]

(5.1.4)

Let the classes \( \Lambda_+ \) and \( \Lambda_- \) be formed by sequences \( \{\delta_n^+\} \) and \( \{\delta_n^-\} \), respectively. Then \( \Lambda = \Lambda_+ \cup \Lambda_- \) and \( \Lambda_+ \cap \Lambda_- = \emptyset \). For the linear case we are in situation (5.1.3) but the condition (5.1.2) doesn’t hold because, due to (5.1.1) \( \lim_{n \to +\infty} \frac{n}{\delta_n} = A \in \mathbb{R}^+ \).

Introducing asymptotically equivalent to \( \{\delta_n\} \) sequence \( \{\varepsilon_n\} \) from either \( \Lambda \) or \( \Lambda_0 \), i.e.

\[
\lim_{n \to +\infty} \frac{\varepsilon_n}{\delta_n} = 1,
\]

(5.1.5)

we are able to write down the distributions being in the center of our attention in the present Chapter.

Any sequence \( \{\delta_n^-\} \) from \( \Lambda_- \) or from \( \Lambda_0 \) together with \( \{\varepsilon_n^-\} \) generate a family of distributions \( \{p_n^-\} \) of the type (4.8.13) but slightly simplified. Here they are

\[
\left\{ \begin{array}{ll}
p_n^- = (1 + (1 - b) \cdot \sum_{n \geq 1} \frac{1}{\varepsilon_n} \cdot \prod_{m=1}^{n-1}(1 - \frac{b}{\delta_m}))^{-1}, & 0 < b < 1, \\
p_K^- = \frac{p_n^-}{\varepsilon_K} \prod_{m=1}^{K-1}(1 - \frac{b}{\delta_m}), & K = 1, 2, \ldots,
\end{array} \right.
\] 

(5.1.6)
where we put \( \prod_{m=1}^{0} = 1 \). In this case, due to Theorem 4.8, the probability \( p_0^- \) has a simple expression

\[
p_0^- = \frac{b}{1-b} \cdot D,
\]

(5.1.7)

where \( D = \sum_{K \geq 1} \frac{\varepsilon_K}{\delta_K} \cdot p_K^- \in R^+ \), and, in particular, when \( \{\varepsilon_n\} = \{\delta_n^-\} \), we obtain \( D = 1 \) and

\[
p_0^- = \frac{b}{1-b}.
\]

(5.1.8)

Any \( \{\delta_n^+\} \in \Lambda_+ \) together with \( \{\varepsilon_n^+\} \in \Lambda_+ \) (see (5.1.5)) generate a family of distributions of the type (4.8.14) but slightly simplified. Here they are

\[
\begin{cases}
p_0^+ = (1 + (1 + b) \cdot \sum_{n \geq 1} \frac{1}{\varepsilon_n^+} \prod_{m=1}^{n-1} (1 + \frac{b}{\delta_n})^{-1}, & -1 < b < +\infty. \\
p_K^+ = \frac{p_0^+ \cdot (1+b)}{\varepsilon_K^+} \prod_{m=1}^{K-1} (1 + \frac{b}{\delta_n^+}), & K = 1, 2, \cdots,
\end{cases}
\]

(5.1.9)

Distributions \( \{p_n^+\} \) and \( \{p_n^-\} \) vary regularly at infinity with some exponent \( (-\rho) \), where \( \rho \in [1, +\infty) \).

Remark 5.1 Let us consider the standard birth-death process with coefficients

\[
\lambda_0 = 1 - b, \quad \lambda_n = \varepsilon_n^- \cdot (1 + \frac{b}{\delta_n}), \quad \mu_n = \varepsilon_n^- , \quad n = 1, 2, \cdots,
\]

and

\[
\lambda_0 = 1 + b, \quad \lambda_n = \varepsilon_n^+ \cdot (1 + \frac{b}{\delta_n}), \quad \mu_n = \varepsilon_n^+ , \quad n = 1, 2, \cdots,
\]

respectively.

5.1.2 Particular Cases

By choosing different relationships between asymptotically equivalent sequences \( \{\varepsilon_n\} \) and \( \{\delta_n\} \) we may get different particular cases.

The first particular case has been introduced and deeply investigated in [16]. That is:

\[
\varepsilon_n^- = \delta_n^- \text{ for } n = 1, 2, \cdots \text{ in (5.1.6) and } \varepsilon_n^+ = \delta_n^+ \text{ in (5.1.9)}. \tag{5.1.10}
\]

The second particular case is given by equalities: for \( n = 1, 2, \cdots 

\[
\frac{1}{\varepsilon_n^-} = -\frac{1}{b} \ln(1 - \frac{b}{\delta_n^-}) \text{ in (1.5.6)}, \tag{5.1.11}
\]

and

\[
\frac{1}{\varepsilon_n^+} = -\frac{1}{b} \ln(1 + \frac{b}{\delta_n^+}) \text{ in (1.5.9)}. \tag{5.1.12}
\]
The distinction between first and second cases consists in following. In the first case \( \{\varepsilon_n\} \) doesn’t depend on parameter \( b \), in the second one it does.

The third particular case is given by equalities: for \( n = 1, 2, \cdots \)

\[
\begin{align*}
p_0^- &= (1 + (1 - b) \cdot \sum_{n \geq 1} \frac{1}{\delta_n} \exp(-b \cdot \sum_{m=1}^{n-1} \frac{1}{\delta_m}))^{-1}, \\
p_K^- &= \frac{p_0^-(1-b)}{\delta_K} \exp(-b \cdot \sum_{m=1}^{K-1} \frac{1}{\delta_m}), \quad K = 1, 2, \cdots,
\end{align*}
\]

with \( 0 < b < 1 \) and (5.1.3) holds;

\[
\begin{align*}
p_0^+ &= (1 + (1 + b) \cdot \sum_{n \geq 1} \frac{1}{\delta_n^+} \exp(b \cdot \sum_{m=1}^{n-1} \frac{1}{\delta_m^+}))^{-1}, \\
p_K^+ &= \frac{p_0^+(1+b)}{\delta_K^+} \exp(b \cdot \sum_{m=1}^{K-1} \frac{1}{\delta_m^+}), \quad K = 1, 2, \cdots,
\end{align*}
\]

with \( -1 < b < +\infty \) and (5.1.4) holds.

Let us give some explanation. For better understanding note that in order to obtain (5.1.13) and (5.1.14) from (5.1.6) and (5.1.9), respectively, we use equalities

\[
\prod_{m=1}^{n-1} (1 - \frac{b}{\delta_m}) = \exp(\sum_{m=1}^{n-1} \ln(1 - \frac{b}{\delta_m})), \tag{5.1.15}
\]

and

\[
\prod_{m=1}^{n-1} (1 + \frac{b}{\delta_m^+}) = \exp(\sum_{m=1}^{n-1} \ln(1 + \frac{b}{\delta_m^+})), \tag{5.1.16}
\]

respectively. Now, due to the expansion

\[
\ln(1-x) = \sum_{n \geq 1} (-1)^n \cdot \frac{x^n}{n} \quad \text{for} \quad x \in (-1, 1), \tag{5.1.17}
\]

the third case is obtained as follows. We take the first terms of expansions of sums at the right-hand-side of equalities in (5.1.15) and (5.1.16) (see, (5.1.17)) and substitute their exponents into the right-hand-side of (5.1.6) and (5.1.9), respectively. By the L’Hôpital rule

\[
\lim_{x \to +\infty} \frac{\ln(1 \mp (1/x))}{(1/x)} = 1, \tag{5.1.18}
\]

so our \( \{\varepsilon_n\} \) and \( \{\delta_n\} \) are asymptotically equivalent to defined by (5.1.11) and (5.1.12) in second case.

It is more convenient to deal with absolutely continuous distribution functions. That is why later the dediscretization procedure shall be made, which allows to replace the class of regularly varying distributions of types (5.1.6) and (5.1.9) by their, in some sense, absolutely continuous analogs - densities also varying regularly at infinity and conserving ”good” properties of initial frequency distributions. The obtained class in terms of absolutely continuous distribution functions predicts the same features.
At the next step we’ll extract from the obtained class a subclass having the most simple form. Note that such a subclass includes all statistical properties of frequency distributions, and, which is important, can not be obtained from the more ”narrow” class of distributions being suggested in [16] (see, the first particular case). Simpler, the form of distribution function of sequence \(\{\xi_n\}\) in Scheme (3.1.2)-(3.1.3) easier to figure out norming and centring sequences \(\{B_n\}\) and \(\{A_n\}\).

5.2 Dediscretization of Frequency Distributions

We suggest a method of replacement of sums in (5.1.6) and (5.1.9) (we represent \(\prod_n a_n\) with the help of \(\sum_n a_n\)) by integrals, which doesn’t change the qualitative behavior of distributions. This operation simplifies the obtained formulas and allows to suggest new distributions with the same qualitative properties as before for biomolecular applications.

5.2.1 Dediscretization: The Idea and Problems

Given some class of distributions \(\{p_n\}\).

What does, in general, the dediscretization mean for \(\{p_n\}\)?

By our understanding suggested below, this is some procedure being realized on some class of distributions, which leads to a concrete construction of corresponding and ”close” in some sense class of ”smooth” enough (for instance, infinite differentiable) distribution functions.

The constructed class has to satisfy definite restrictions. Necessarily, it must conserve the main qualitative properties of distributions of the original class such as: monotonity; convexity; moments’ existence; regular variation with the same exponent, etc.

What is the reason for dediscretization?

Sometimes the below suggested procedure of dediscretization leads to more simple expressions for distribution functions from the obtained class, which is the result of dediscretization, in comparison with the original class.

The next advantage presented always consists in following. Infinite differentiability, monotony, convexity and other ”good” properties allow to use deeply developed and being well-understandable for everybody tool of Mathematical Analysis.

Finally, once more we have to mention that it is more convenient to deal with continuous functions and suggest interpolations and approximations for them.

The further discretization (the reverse to dediscretization procedure) of the suggested approximation of a class of distribution functions, for instance, might discover new classes of distributions for their usage in biomolecular systems.
In the way of the Dediscretization Approach’s realization, first of all the following problem arises. We must replace the regularly varying sequences \( \{\delta_n\} \) and \( \{\varepsilon_n\} \) in presentations (5.1.6) and (5.1.9) by regularly varying functions, say \( \delta(t) \) and \( \varepsilon(t) \), respectively. In other words, due to representations

\[
\delta_n^\pm = 1 + n^\alpha \cdot L^\pm(n), \quad n = 0, 1, 2, \cdots, \tag{5.2.1}
\]

and

\[
\varepsilon_n^\pm = 1 + n^\alpha \cdot L(n), \quad n = 0, 1, 2, \cdots, \tag{5.2.2}
\]

where \( \lim_{n \to +\infty} (L^\pm(n)/L^\pm(n)) = 1 \) because of asymptotical equivalence of \( \{\delta_n^\pm\} \) and \( \{\varepsilon_n^\pm\} \) (i.e. the sequences \( \{L^\pm(n)\} \) and \( \{L^\pm(n)\} \) are asymptotically equivalent too, are slowly varying and \( \alpha \in [1, +\infty) \)), we must replace the sequences \( \{L^\pm(n)\} \) and \( \{L^\pm(n)\} \) by slowly varying functions, say \( \hat{L}^\pm(t) \) and \( \hat{L}_\pm(t) \), respectively, in order to get, at least, continuous analogs of (5.2.1) and (5.2.2). Namely,

\[
\delta^\pm(t) = 1 + t^\alpha \hat{L}^\pm(t) \quad \text{and} \quad \varepsilon^\pm(t) = 1 + t^\alpha \hat{L}_\pm(t), \quad t \in R^+.
\]

In typical examples known before the limit exists:

\[
either \lim_{n \to +\infty} L^+(n) = c \in R^+, \tag{5.2.4}
or \lim_{n \to +\infty} L^+(n) = +\infty, \tag{5.2.5}
\]

where (see, for instance, [16])

\[
The sequence \( \{L^+(n)\} \) increases and is upward convex, \tag{5.2.6}
\]

i.e. \( L^+(n-1) < L^+(n) \) and \( L^+(n-1) + L^+(n+1) < 2 \cdot L^+(n) \), for \( n = 2, 3, \cdots \).

The "breadth" of the class \( \Lambda \) described in Section 5.1 is so "large" and the frequency distributions in large-scale biomolecular systems possess so many "smoothness" properties that the "narrowing" of \( \Lambda \) by additional assumptions either (2.5.4) or (2.5.5) is quite natural and reasonable.

The interpolation problem is the same for \( \{\delta_n^\pm\} \) and \( \{\varepsilon_n^\pm\} \), so let us consider it for sequence \( \{\varepsilon_n^\pm\} \). For simplicity we omit lower index "\( \pm \)" of slowly varying function \( L^\pm(t) \).

The desired function \( \hat{L}(t) \) has to be built in a constructive way and to satisfy following restrictions (if possible): 1. \( \hat{L}(n) = L(n), \quad n = 1, 2, \cdots \); 2. \( \lim_{n \to +\infty} \frac{L(n)}{L(n)} = 1 \); 3. \( \hat{L}(t) \) is infinite differentiable; 4. \( \hat{L}(t) \) increases (decreases) if \( \{L(n)\} \) increases (decreases); 5. \( \hat{L}(t) \) is convex if \( \{L(n)\} \) is convex, etc.

It is enough to solve the problem for the case (see, (5.2.5))

\[
\lim_{n \to +\infty} L(n) = +\infty. \tag{5.2.7}
\]
Indeed, for the case (see, (5.2.4))

\[
\lim_{n \to +\infty} L(n) = c \in R^+
\]  

(5.2.8)

we consider a slowly varying sequence \( \{L(n) \cdot \ln(n+1)\} \), which, obviously, satisfies condition (5.2.7), i.e. \( \lim_{n \to +\infty} L(n) \cdot \ln(n+1) = c \cdot \lim_{n \to +\infty} \ln(n+1) = +\infty \) because of (5.2.8). It reduces this case to the previous one. If \( \lim_{n \to +\infty} L(n) = 0 \), then we take the sequence \( \{1/L(n)\} \), which satisfies condition (5.2.7).

The way of the \( \hat{L}(t) \) function’s construction may be different.

### 5.2.2 The Dediscretization Procedure

Denote

\[
f^\pm_b(x) = \ln(1 \pm \frac{b}{\delta^\pm(x)}), \quad x \in R^+,
\]

(5.2.9)

where

\[
\delta^\pm(x) = 1 + x^\alpha \cdot L^\pm(x),
\]

\[
\epsilon^\pm(x) = 1 + x^\alpha L^\pm(x)
\]

(5.2.10)

(5.2.11)

and \( L^\pm(x), L^\pm(x) \) are solutions to interpolation problem for sequences \( \{L^\pm(n)\}, \{L^\pm(n)\} \), respectively. This interpolation problem in simple cases is solved in Section 5.3 and in general case in Chapter 6.

Then, \( \delta^\pm(x) \) and \( \epsilon^\pm(x) \) together generate one-parametric family of distribution functions, which shall be a dediscretization of one-parametric family of distributions \( \{p^\pm_n\} = \{p^\pm_n(b)\} \) of types (5.1.6), (5.1.9) generated by sequences \( \{\delta^\pm_n\} \) and \( \{\epsilon^\pm_n\} \).

In order to present the result of dediscretization procedure we need in

**Definition 2.1** We say that the function defined on \([0, +\infty)\)

\[
\hat{F}_\pm(x, b) := \frac{\int_{0-}^{x} \frac{1}{\epsilon^\pm(t)} \exp(\int_{0-}^{t} \frac{f^\pm_u(b)}{\delta^\pm(u)} du) dt}{\int_{0-}^{+\infty} \frac{1}{\epsilon^\pm(t)} \exp(\int_{0-}^{t} \frac{f^\pm_u(b)}{\delta^\pm(u)} du) dt}
\]

(5.2.12)

is a dediscretization of \( \{p^\pm_n(b)\} \) generated by \( \{(\epsilon^\pm_n), \{\delta^\pm_n\}\} \).

**How did we come to functions \( \hat{F}_+(x, b) \) and \( \hat{F}_-(x, b) \)?**

For a given sequence \( \{\delta^\pm_n\} \in \Lambda_- \) denote

\[
f^\pm_n = f^\pm_n(b) = \ln(1 - \frac{b}{\delta^\pm_n}), \quad n = 0, 1, 2, \ldots, \ 0 < b < 1.
\]
The distribution function $F_-(x) = F_-(x,b)$, $x \in [0, +\infty)$, which corresponds to distribution $\{p_n^-(b)\}$ generated by $(\{\varepsilon_n^-\}, \{\delta_n^-\})$, takes the form

$$F_-(x) = F_-(+0) \cdot \sum_{n=0}^{|x|} \exp\left(\sum_{m=0}^{n-1} f_m^-(b)\right),$$

(5.2.13)

where $0 < b < 1$ and $\sum_{m=0}^{n-1} = 0$, $\varepsilon_0 = 0$. Indeed, putting $\prod_{m=1}^{0} = 1$, $\sum_{m=1}^{0} = 0$, due to (5.1.6), for $x \in R^+$ and $0 < b < 1$ we have

$$F_-(x) = p_0^- \cdot (1 + (1 - b) \sum_{n=1}^{[x]} \frac{1}{\varepsilon^n_-} \prod_{m=1}^{n-1} (1 - \frac{b}{\delta^n_-})) =$$

$$F_-(0) \cdot (1 + (1 - b) \sum_{n=1}^{[x]} \frac{1}{\varepsilon^n_-} \exp\left(\sum_{m=0}^{n-1} f_m^-\right)) =$$

$$= F_-(+0)(\frac{1}{\varepsilon^0_-} \exp\left(-\sum_{m=0}^{n-1} f_m^-\right) + \sum_{n=1}^{[x]} \frac{1}{\varepsilon^n_-} \exp(f_0^-) \exp\left(\sum_{m=1}^{n-1} f_m^-\right)) =$$

$$= F_-(+0) \cdot \sum_{n=0}^{[x]} \frac{1}{\varepsilon^n_-} \exp\left(\sum_{m=0}^{n-1} f_m^-\right).$$

By (5.2.13), for $x \in R^+$ a function $g_b^-(x) := \frac{F_-(x)}{F_-(+0)}$, $0 < b < 1$, $g_b^-(0) = 0$, represents a finite, positive, discrete measure, and takes the form

$$g_b^-(x) = \sum_{n=0}^{[x]} \frac{1}{\varepsilon^n_-} \exp\left(\sum_{m=0}^{n-1} f_m^-\right), \ x \in R^+. \quad (5.2.14)$$

Similarly, the distribution function $F_+(x) = F_+(x,b)$, $x \in [0, +\infty)$, which corresponds to distribution $\{p_n^+(b)\}$ generated by $(\{\varepsilon_n^+\}, \{\delta_n^+\})$, takes the form

$$F_+(x) = F_+(+0) \cdot \sum_{n=0}^{[x]} \frac{1}{\varepsilon^n_+} \exp\left(\sum_{m=0}^{n-1} f_m^+\right).$$

(5.2.15)

Note that $F_+(x)$ and $F_-(x)$ have jumps $p_n^+$ and $p_n^-$ at zero, respectively.

In (5.2.15) $f_n^+ = f_n^+(b) = \ln(1 + \frac{b}{\delta_n^+})$, $n = 0, 1, 2, \cdots$, $-1 < b < +\infty$, and $F_+(x)$ generates a finite, positive, discrete measure

$$g_b^+(x) := \frac{F_+(x)}{F_+(+0)} = \sum_{n=0}^{[x]} \frac{1}{\varepsilon^n_+} \exp\left(\sum_{m=0}^{n-1} f_m^+\right), \ x \in R^+, \quad (5.2.16)$$

where $g_b^+(0) = 0$ (compare to (5.2.14)).

The form (5.2.14) and (5.2.16) of $g_b^-$ and $g_b^+$ is suitable for dediscretization procedure. Namely, absolutely continuous on $R^+$ measures

$$\tilde{g}_b^\pm(x) = \int_{0^-}^x \frac{1}{\varepsilon^\pm(t)} \exp\left(\int_0^t f_b^\pm(u)du\right)dt$$

(5.2.17)
with $0 < b < 1$ for sign “−”, and $-1 < b < +\infty$ for sign “+” after normalization give $\hat{F}_+$ and $\hat{F}_-$ of the form (5.2.12).

### 5.3 Interpolation Problem: Examples

If the form of \{L(n)\} in representations of sequences \{δ(n)\} or \{ε(n)\} forming the stationary distributions is known and given by some elementary formula, then replacing discrete argument $n$ by continuous argument $t \in \mathbb{R}^+$ we obtain a function $\hat{L}(t)$ defined on $\mathbb{R}^+$, which, very often, posses all properties we need in. The simpliest example is

$$L(n) = \text{const for } n = 1, 2, \cdots.$$  \hspace{1cm} (5.3.1)

For instance such a constant slowly varying component has the Power Law (1.1.2)-(1.1.3). Then we put $\hat{L}(t) = \text{const for } t \in \mathbb{R}^+$, where constant is the same as in (5.3.1).

For the Pareto Distribution \{p_n\} defined by (1.2.6)-(1.2.7) the constant slowly varying component is $L(n) = n^\rho \cdot p_n = c(\rho, b)(1 + \frac{1}{n})^{-\rho}$, $n = 1, 2, \cdots$. In this case we put $\hat{L}(t) = c(\rho, b)(1 + \frac{t}{\gamma})^{-\rho}$, $t \in \mathbb{R}^+$.

In these examples the conditions (5.2.5), (5.2.6) are not fulfilled.

The examples where the conditions (5.2.5), (5.2.6) are fulfilled and of more interest from the point of view of interpolation problem.

The consideration of two such examples is a content of this Section.

#### 5.3.1 Example 1

Denote $e(0) = 1$, $e(1) = e = \lim_{n \to +\infty}(1 + \frac{1}{n})^n$, $e(K+1) = \exp(e(K))$, $K = 1, 2, \cdots$.

For given $K = 1, 2, \cdots$ let us consider a sequence of positive numbers

$$L_K(n) = \ln \ln \cdots \ln n, \; n > e(K).$$  \hspace{1cm} (5.3.2)

We are going to show that the sequence \{L_K(n)\} for any given $K$ satisfies condition (5.2.5) and: (a) increases; (b) is downward convex, i.e. the condition (5.2.6) holds too.

Then, the sequences being ”correspondingly supplemented” for $n = 1, 2, \cdots$, $e(K)$ (if $K = 1, 2, \cdots$ is fixed) conserve the properties (a), (b) and form slowly varying at infinity sequences.

The fulfillment of the condition (5.2.5) is obvious.

Let us show that the proposition (a) takes place, i.e. for given $K = 1, 2, \cdots$ and for arbitrary integer $n > e(K)$ we have

$$L_K(n + 1) > L_K(n).$$  \hspace{1cm} (5.3.3)
Indeed, for $K = 1, 2, \cdots$ let us introduce a sequence $\{p_n^{(K)}\}$, $n > e^{(K)}$ of positive numbers given with the help of recurrent formula

$$
p_n^{(K)} = \ln(1 + \frac{p_n^{(K-1)}}{\ln \cdots \ln n^K}), \quad n > e^{(K)},
$$

(5.3.4)

with $p_n^{(0)} = 1$ and $\ln \cdots \ln n = n$ for $K = 1$. Some evaluations are needed, where on each step the result of previous step and recurrent formula (5.3.4) are applied.

For $n > e^{(1)}$

$$
L_1(n + 1) = \ln(n + 1) = \ln(n \cdot (1 + \frac{1}{n})) = \ln n + \ln(1 + \frac{n}{n}) = L_1(n) + p_1^{(1)} > L_1(n).
$$

For $n > e^{(2)}$

$$
L_2(n + 1) = \ln \ln(n + 1) = \ln L_1(n + 1) = \ln(L_1(n) + p_1^{(1)}) =
$$

$$
= \ln((\ln n)(1 + \frac{p_1^{(1)}}{\ln n})) = \ln \ln n + \ln(1 + \frac{p_1^{(1)}}{\ln n}) = L_2(n) + p_2^{(2)} > L_2(n).
$$

Continuing this way for arbitrary $K = 2, 3, \cdots$ and for integer $n > e^{(K)}$ we obtain

$$
L_K(n + 1) = \ln L_{K-1}(n + 1) = \ln(L_{K-1}(n) + p_{K-1}^{(K-1)}) =
$$

$$
= \ln((\ln \cdots \ln n^K)(1 + \frac{p_{K-1}^{(K-1)}}{\ln \cdots \ln n^K})) = \ln \ln \cdots \ln n + \ln(1 + \frac{p_{K-1}^{(K-1)}}{\ln \cdots \ln n^K}) =
$$

$$
= L_K(n) + p_K^{(K)} > L_K(n).
$$

Thus, (5.3.3), i.e. the proposition (a) is proved.

Let us show that the proposition (b) takes place, i.e. for given $K = 1, 2, \cdots$ and for arbitrary integer $n > e^{(K)} + 1$ we have

$$
L_K(n - 1) + L_K(n + 1) < 2L_K(n).
$$

(5.3.5)

In order to get (5.3.5) for $K = 1, 2, \cdots$ let us introduce two sequences $\{p_n^{(K)-}\}_{n>e^{(K)}}$ and $\{p_n^{(K)+}\}_{n>e^{(K)}}$ given with the help of recurrent formulas

$$
p_n^{(K)} = \ln(1 + \frac{p_n^{(K-1)}}{\ln \cdots \ln n^K}), \quad n > e^{(K)},
$$

(5.3.6)

with initial $p_0^{(K)} = \pm 1$. It is easy to see that $p_n^{(K)+} = p_n^{(K)}$ (see, (5.3.4)), and

$$
p_n^{(K)+} > 0, \quad p_n^{(K)-} < 0.
$$

(5.3.7)

Moreover, for $K = 1, 2, \cdots$ and for integer $n > e^{(K)}$

$$
p_n^{(K)-} + p_n^{(K)+} < 0.
$$

(5.3.8)
Indeed, let us prove (5.3.8) by induction. For $K = 1$ we have

$$\rho_n^{1-} + \rho_n^{1+} = \ln(1 - \frac{1}{n}) + \ln(1 + \frac{1}{n}) = \ln(1 - \frac{1}{n^2}) < 0 \text{ for } n > e_{(1)}.$$ 

If the proposition (5.3.8) holds for $i = 1, 2, \cdots, K - 1$, then, due to (5.3.6), we obtain

$$\rho_n^{(K-1)-} + \rho_n^{(K-1)+} = \ln(1 + \frac{\rho_n^{(K-1)-}}{\ln \ln \cdots \ln n}) + \ln(1 + \frac{\rho_n^{(K-1)+}}{\ln \ln \cdots \ln n}) =$$

$$= \ln(1 + \frac{\rho_n^{(K-1)-}}{\ln \ln \cdots \ln n} + \frac{\rho_n^{(K-1)+}}{\ln \ln \cdots \ln n}). \quad (5.3.9)$$

Due to (5.3.7) for index $K$ and to \textit{inductional assumption} (5.3.8) for index $K - 1$, from (5.3.9) we conclude that (5.3.8) holds also for index $K$.

Now we are ready to establish inequality (5.3.5). For $K = 1, 2, \cdots$ and for arbitrary integer $n > e_{(K)} + 1$ we have (see, (5.3.4))

$$L_K(n - 1) + L_K(n + 1) = \ln \ln \cdots \ln n - 1 + \ln \ln \cdots \ln n + 1 =$$

$$= \ln \ln \cdots \ln (\ln(n(1 - \frac{1}{n}))) + \ln \ln \cdots \ln (\ln(n(1 + \frac{1}{n}))) =$$

$$= \ln \ln \cdots \ln (\ln n + \rho_n^{(1)-}) + \ln \ln \cdots \ln n + \rho_n^{(1)+}) =$$

$$= \ln \ln \cdots \ln ((\ln n)(1 + \frac{\rho_n^{(1)-}}{\ln n})) + \ln \ln \cdots \ln ((\ln n)(1 + \frac{\rho_n^{(1)+}}{\ln n})) =$$

$$= \ln \ln \cdots \ln (\ln n + \rho_n^{(2)-}) + \ln \ln \cdots \ln (\ln n + \rho_n^{(2)+}) + \ln \ln \cdots \ln (\ln n + \rho_n^{(1)+} + \rho_n^{(1)+}) =$$

Continuing by this way we obtain

$$L_K(n - 1) + L_K(n + 1) = \ln \ln \cdots \ln n + \rho_n^{(K-1)-} + \ln \ln \cdots \ln n + \rho_n^{(K-1)+} =$$

$$2 \cdot \ln \ln \cdots \ln n + (\rho_n^{(K)-} + \rho_n^{(K)+}). \quad (5.3.10)$$

Taking into account (5.3.8) and definition of $L_K(n)$ (see, (5.3.2)), from (5.3.10) we come to (5.3.5). Thus, the proposition (b) is proved.

Now, we get the form (5.2.1) (or (5.2.2)) from (5.3.2) if for $K = 1, 2, \cdots$ we put

$$\hat{L}_K(t) = \ln \ln \cdots \ln t, \ t > e_{(K)}. \quad (5.3.11)$$
These functions vary slowly at infinity, increase and are downward convex (preliminary they are "correspondingly supplemented" for $t \in (0, e(K))$). Indeed, the function

$$
\frac{d}{dt} \hat{L}_K(t) = (t \cdot \prod_{i=1}^{K-1} \ln \ln \cdots \ln t)^{-1}, \quad \prod_{i=1}^{0} = 1,
$$

(5.3.12)

for given $K = 1, 2, \cdots$ and $t > e(K)$ is positive and decreases.

Now, it is clear that for constructed interpolation the restrictions 1.-5. of Section 5.2 are fulfilled.

### 5.3.2 Example 2

For any given $K = 1, 2, \cdots$ let us introduce sequences \( \{L^{(1)}_K(n)\}_{n>e(K)} \) and \( \{L^{(2)}_K(n)\}_{n>e(K)} \) defined by following elementary formulas

$$
L^{(1)}_K(n) = \ln n \cdot \ln \ln n \cdots \ln \ln \cdots \ln K = \prod_{i=1}^{K} \ln n \cdots \ln K = \prod_{i=1}^{K} L_i(n),
$$

(5.3.13)

where \( L_K(n) \) is defined by formula (5.3.2), and

$$
L^{(2)}_K(n) = L^{(1)}_K(n) \cdot \ln \ln \cdots \ln K = L^{(1)}_K(n) \cdot L_K(n).
$$

(5.3.14)

These sequences satisfy condition (5.2.7) and for them the propositions (a), (b) take place. The first fact is obvious, others (propositions (a), (b)) as we just have seen in Example 1 much easier to prove for continuous case by using the tool of Mathematical Analysis instead of Combinatorial Methods. These sequences being "correspondingly supplemented" for $n = 1, 2, \cdots [e(K)]$ vary slowly. It is easily established by induction on $K$ using the forms (5.3.13) and (5.3.14) and by following two obvious facts.

1. If \( \{L'(n)\} \) and \( \{L''(n)\} \) vary slowly, then \( \{L'(n) \cdot L''(n)\} \) varies slowly.

2. If \( \{L(n)\} \) varies slowly and (5.2.7) holds, then \( \{\ln L(n)\} \) varies slowly.

Indeed, for $s = 1, 2, \cdots$ we have

$$
\lim_{n \to +\infty} \frac{\ln L(sn) \cdot L(n)}{\ln L(n)} = \lim_{n \to +\infty} \frac{\ln L(n) \cdot L(sn)}{\ln L(n)} = \lim_{n \to +\infty} \frac{\ln L(n) + \ln((L(sn))/L(n)))}{\ln L(n)} = \lim_{n \to +\infty} \left\{ 1 + \frac{1}{\ln L(n)} \frac{L(sn)}{L(n)} \right\} = 1.
$$

Now, we get the form (5.2.10) or (5.2.11) from (5.3.15) and (5.3.14) if for $K = 1, 2, \cdots$ we put

$$
\hat{L}_K^{(1)}(t) = \prod_{i=1}^{K} \ln \ln \cdots \ln t = \prod_{i=1}^{K} \hat{L}_i(t),
$$

(5.3.15)
where \( \hat{L}_K(t) \) is defined by formula (5.3.11), and
\[
\hat{L}^{(2)}_K(t) = \hat{L}^{(1)}_K(t) \cdot \ln \ln \cdots \ln \frac{t}{K} = \hat{L}^{(1)}_K(t) \cdot \hat{L}_K(t).
\] (5.3.16)

We are going to show that the functions \( \hat{L}^{(1)}_K(t) \) and \( \hat{L}^{(2)}_K(t) \) defined by (5.3.15) and (5.3.16), respectively, for a given \( K = 1, 2, \cdots \): (i) increases; (ii) are upward convex.

First of all, the derivatives of first order for these functions equals to:
\[
\frac{d}{dt} \hat{L}^{(1)}_K(t) = \frac{K}{t} \sum_{j=1}^{K} \frac{d\hat{L}_j(t)}{dt} \cdot \prod_{i=1, i \neq j}^{K} \hat{L}_i(t) = \\
= \sum_{j=1}^{K} \left( \prod_{m=1}^{j-1} \ln \ln \cdots \ln t \right)^{-1} \prod_{i=1, i \neq j}^{K} \ln \ln \cdots \ln t = \frac{1}{t} \sum_{j=1}^{K} \prod_{i=j+1}^{K} \ln \ln \cdots \ln t,
\] (5.3.17)
where (5.3.12) is used. The right-hand-side of (5.3.17) is \textit{positive}, which proves that \( \hat{L}^{(1)}_K(t) \) increases;
\[
\frac{d}{dt} \hat{L}^{(2)}_K(t) = \hat{L}_K(t) \frac{d}{dt} \hat{L}^{(1)}_K(t) + \hat{L}^{(1)}_K(t) \cdot \frac{d}{dt} \hat{L}_K(t) = \\
= \frac{\hat{L}_K(t)}{t} \sum_{j=1}^{K} \prod_{i=j+1}^{K} \ln \ln \cdots \ln t + (t \cdot \prod_{m=1}^{K-1} \ln \ln \cdots \ln t)^{-1} \cdot \hat{L}^{(1)}_K(t) = \\
= \frac{1}{t} \ln \ln \cdots \ln t \cdot \left( 1 + \sum_{j=1}^{K} \prod_{i=j+1}^{K} \ln \ln \cdots \ln t \right),
\] (5.3.18)
where (5.3.12) and (5.3.17) are used. The right-hand-side of (5.3.18) is \textit{positive}, which proves that \( \hat{L}^{(2)}_K(t) \) increases.

Secondly, we evaluate the derivatives of second order for these functions. For \( K = 1, 2, \cdots \) and \( t > e(K) \), due to (5.3.12) and (5.3.17) we have
\[
\frac{d^2}{dt^2} \hat{L}^{(1)}_K(t) = \frac{d}{dt} \left( \frac{1}{t^2} \sum_{j=1}^{K} \prod_{i=j+1}^{K} \ln \ln \cdots \ln t \right) = \\
= -\frac{1}{t^2} \sum_{j=1}^{K} \prod_{i=j+1}^{K} \ln \ln \cdots \ln t + \frac{1}{t^2} \sum_{j=1}^{K} \prod_{i=j+1}^{m} \prod_{i=j+1, i \neq m}^{K} \ln \ln \cdots \ln t = \\
= -\frac{1}{t^2} \sum_{j=1}^{K} \prod_{i=j+1}^{m} \ln \ln \cdots \ln t + \frac{1}{t^2} \sum_{j=1}^{K} \prod_{i=j+1}^{m} \ln \ln \cdots \ln t + \sum_{m=j+1}^{K} (\prod_{s=1}^{m-1} \ln \ln \cdots \ln t)^{-1} = \\
= -\frac{1}{t^2} \sum_{j=1}^{K} \prod_{i=j+1}^{m} \ln \ln \cdots \ln t \cdot (1 - U_{j,K}(t)),
\] (5.3.19)
where
\[
U_{j,K}(t) = \sum_{m=j+1}^{K} (\prod_{s=1}^{m} \ln \ln \cdots \ln t)^{-1} \text{ for } j = 1, 2, \cdots, K; \ K = 1, 2, \cdots
\] (5.3.20)
In order to prove the *upward convexity* of \( \hat{L}_K^{(1)}(t) \) for \( t > e(K) \), due to (5.3.19), we must prove the inequalities

\[
U_{j,K}(t) < 1 \quad \text{for} \quad j = 1, 2, \ldots, K \quad \text{and} \quad K = 1, 2, \ldots. \tag{5.3.21}
\]

First of all, for \( K = 1 \) the inequality (5.3.21) is obvious because \( U_{1,1}(t) = 0 \).

Let \( K > 1 \) be fixed. Then, taking into account the inequalities

\[
0 \leq U_{K,K}(t) < U_{K-1,K}(t) < \cdots < U_{1,K}(t), \tag{5.3.22}
\]

we conclude that it is enough to prove (5.3.21) only for \( j = 1 \) and given fixed \( K > 1 \).

Next, it is clear that for \( t > e(K) \) and \( K > 1 \), due to (5.3.20),

\[
U_{1,K}(t) < U_{1,K}(e(K)) = \frac{1}{e(K-1)e(K-2)} + \frac{1}{e(K-1)e(K-2)e(K-3)} + \cdots + \frac{1}{e(K-1)e(K-2)\cdots e(1)} < \frac{K-1}{e(K-1)} < 1.
\]

The proposition for \( \hat{L}_K^{(1)}(t) \) is proved.

Let us prove the *upward convexity* of function \( \hat{L}_K^{(2)}(t) \) for \( K = 1, 2, \ldots \) and \( t > e(K) \).

If \( K = 1 \), then \( \frac{d}{dt}\hat{L}_1^{(2)}(t) = \frac{2}{t}(1 - \ln t) < 0, \ t > e(1) = e \). Let \( K > 1 \) be fixed and \( t > e(K) \). Due to (5.3.17)-(5.3.18), the following relationship holds

\[
\frac{d}{dt}\hat{L}_K^{(2)}(t) = \frac{1}{t}\ln\ln\cdots\ln t + \ln\ln\cdots\ln t \frac{d}{dt}\hat{L}_K^{(1)}(t). \tag{5.3.23}
\]

The derivative of the first term at the right-hand-side of (5.3.23) is

\[
-\frac{1}{t^2}\ln\ln\cdots\ln t - \left( \prod_{i=1}^{K-1} \ln\ln\cdots\ln t \right)^{-1} < 0.
\]

Therefore, it is enough to prove that the derivative of the second term at the right-hand-side of (5.3.23) is *negative*. This derivative, due to (5.3.17) and (5.3.19), takes the form

\[
-\frac{1}{t^2}\ln\ln\cdots\ln t \cdot \sum_{j=1}^{K} \prod_{i=j+1}^{K} \ln\ln\cdots\ln t \cdot (1 - U_{j,K}(t)) = \tag{5.3.24}
\]

\[
-\frac{1}{t^2}\ln\ln\cdots\ln t \cdot \sum_{j=1}^{K} \prod_{i=j+1}^{K} \ln\ln\cdots\ln t \left\{ 1 - U_{j,K}(t) - (\ln t \cdot \ln\ln\cdots\ln t)^{-1} \right\}.
\]

It is enough to show that the expression at the right-hand-side of (5.3.24) is *negative*. Therefore, if we prove that all expressions in *figures brackets* at the right-hand-side of (5.3.24) are
positive, then the proposition shall be established. So it remains to show that 
\[ U_{1,K}(t) + \frac{(\ln t \cdot \ln \ln t \cdots \ln \ln \cdots \ln t)}{K}^{-1} < 1 \] for fixed \( K > 1 \) and \( t > e(K) \) because of inequalities (5.3.22). This statement follows from inequalities

\[
U_{1,K}(t) + \frac{1}{e(K-1)e(K-2) \cdots e(1)} < 1
\]

Thus, the upward convexity of \( \hat{L}^{(2)}_K(t) \) for \( K = 1, 2, \cdots \) and \( t > e(K) \) is proved.

5.4 Dediscretization: The Asymptotically Linear Case

The structure of stochastic birth-death process leads to comparatively complicate form from the point of view of computations stationary distributions. This structure naturally reflects also on results of dediscretization. Thus, if we change nothing besides of the replacement in some sense of discrete argument by a continuous one, then we can not expect to get any essential simplification in the form in general. Of course, the possibilities to apply the tool of Mathematical Analysis (not only in proofs of statements, but also in order to get simplification in particular cases) arise.

In the present Section we consider a case of asymptotically linear sequences \( \{\varepsilon_n\} \) and \( \{\delta_n\} \), which allows simplification of the result of dediscretization.

5.4.1 Improving The Form of Distributions \( \{p_n\} \)

It is possible to write down the distributions \( \{p_n\} \) in one united symmetric form, which depends on two parameters, say \( p \) and \( q \), satisfying conditions

\[
0 < p < q < +\infty \quad \text{if} \quad \sum_{n \geq 1} \frac{1}{\zeta_n} = +\infty, \quad (5.4.1)
\]

\[
0 < p < +\infty, \quad 0 < q < +\infty \quad \text{if} \quad \sum_{n \geq 1} \frac{1}{\zeta_n} < +\infty. \quad (5.4.2)
\]

The mentioned form uses sequences \( \{\zeta_n\} \) and \( \{\lambda_n\} \) instead of \( \{\delta_n\} \) and \( \{\varepsilon_n\} \), respectively, with the following relations among them

\[
\frac{1}{q} \zeta_n = \delta_n - 1, \quad \frac{1}{q} \lambda_n = \varepsilon_n - 1, \quad n = 0, 1, 2, \cdots. \quad (5.4.3)
\]
Now, distributions \( \{p_n\} \) may be rewritten in terms of \( p, q, \{\zeta_n\} \) and \( \{\lambda_n\} \) as follows

\[
p_0 = (1 + p \cdot \sum_{n \geq 1} \frac{1}{q + \lambda_n} \prod_{m=1}^{n-1} \frac{p + \zeta_m}{q + \zeta_m})^{-1}, \tag{5.4.4}
\]

\[
p_K = \frac{p \cdot p_0}{q + \lambda_K} \prod_{m=1}^{K-1} \frac{p + \zeta_m}{q + \zeta_m}, \quad k = 1, 2, \ldots. \tag{5.4.5}
\]

The asymptotical equivalency of sequences \( \{\delta_n\} \) and \( \{\epsilon_n\} \), due to relations (5.4.3), leads to asymptotical equivalency of sequences \( \{\zeta_n\} \) and \( \{\lambda_n\} \).

The corresponding distribution function of distribution \( \{p_n\} \) of type (5.4.4)-(5.4.5) is

\[
F_{p,q}(x) = F_{p,q}(+0) \cdot \sum_{n=0}^{[x]} \frac{1}{q + \lambda_n} \exp\left(\sum_{m=0}^{n-1} \ln \frac{p + \zeta_m}{q + \zeta_m}\right). \tag{5.4.6}
\]

for \( x \in [0, +\infty) \). The dediscretization of distribution function \( F_{p,q}(x), \ x \in [0, +\infty) \), defined by (5.4.6) takes the form

\[
\hat{F}_{p,q}(x) = c(p, q) \cdot \int_{0-}^{x} \frac{1}{q + \lambda(t)} \exp\left(\int_{0-}^{t} \ln \frac{p + \zeta(u)}{q + \zeta(u)} du\right) dt \tag{5.4.7}
\]

for \( x \in [0, +\infty) \) with \( c(p, q) = \left(\int_{0-}^{+\infty} \frac{1}{q + \lambda(t)} \exp\left(\int_{0-}^{t} \ln \frac{p + \zeta(u)}{q + \zeta(u)} du\right) dt\right)^{-1} \).

Here \( \frac{1}{q} \zeta(t) = \delta(t) - 1, \frac{1}{q} \lambda(t) = \epsilon(t) - 1, \ t \in [0, +\infty) \), where \( \delta(t) \) and \( \epsilon(t) \) are assumed to be already constructed interpolations of sequences \( \{\delta_n\} \) and \( \{\epsilon_n\} \), respectively.

The form (5.4.4)-(5.4.5) with restriction (5.4.1) on parameters \( p \) and \( q \) of distribution \( \{p_n\} \) is chosen by analogy with the linear case. This is exactly the form, for which the etalon linear case \( \lambda_n = \zeta_n = n, \ n = 1, 2, \ldots \) implies the traditional form of well-known family of Waring Distributions

\[
\begin{align*}
p_0 &= (1 + p \cdot \sum_{n \geq 1} \frac{1}{q + n} \prod_{m=1}^{n-1} \frac{p + m}{q + m})^{-1}, & 0 < p < q < +\infty, \tag{5.4.8} \\
p_K &= \frac{p_0}{q + K} \prod_{m=1}^{n-1} \frac{p + m}{q + m}, & K = 1, 2, \ldots,
\end{align*}
\]

Denote

\[
f(t) = \ln\left(\frac{p + \zeta(t)}{q + \zeta(t)}\right), \quad t \in [0, +\infty). \tag{5.4.9}
\]

Then, for \( t \in [0, +\infty) \) we have

\[
\int_{0-}^{t} f(x) dx = t \cdot f(t) - \int_{0-}^{t} x df(x) = \\
= t \cdot \left(\ln\left(\frac{p + \zeta(t)}{q + \zeta(t)}\right) - \int_{0-}^{t} x d\ln(p + \zeta(t)) - \ln(q + \zeta(t))\right) = \\
= t \cdot \ln(p + \zeta(t)) - t \ln(q + \zeta(t)) - \int_{0-}^{t} x \cdot \zeta'(x) dx - \int_{0-}^{t} x \cdot \zeta'(x) dx + \int_{0-}^{t} x \cdot \zeta'(x) dx,
\]

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where \( \zeta'(t) \) denotes the derivative of function \( \zeta(t) \).

If there is a simple relationship between \( \zeta(t) \) and \( \zeta'(t) \), then the linear interpolation as a first formula, therefore we put a family of Waring Distributions defined by (5.4.8). The sequence (5.4.11)

\[
\lambda \rightarrow \lim_{\lambda \to \infty} \frac{1}{\lambda} \int_0^\lambda f(t) dt
\]

is given by simple elementary formula, therefore we put

\[
\zeta(t) = t + 1, \quad t \in [0, +\infty), \quad (5.4.11)
\]

as interpolation of \( \{\zeta_n\} \). Substituting (5.4.11) into (5.4.8) for \( t \in [0, +\infty) \) we obtain

\[
\int_{0}^{t_1} f(x)dx = t \ln(p + t) - t \cdot \ln(q + t) - \int_{0}^{t} \frac{xdx}{p + x} + \int_{0}^{t} \frac{xdx}{q + x} =
\]

\[
= (p + t) \ln(p + t) - (q + t) \ln(q + t) - p \ln p + q \ln q = \ln \left( \frac{(p + t)^{p+t} (q + t)^q}{(q + t)^{q+t} p^p} \right). \quad (5.4.12)
\]

Substituting (5.4.12) into (5.4.7) for \( x \in [0, +\infty) \) we get

\[
\hat{F}_{p,q}(x) = c(p,q) \cdot \int_{0}^{x} \frac{1}{q + \lambda(t)} \exp \left( \ln \left( \frac{(p + t)^{p+t} q^q}{(q + t)^{q+t} p^p} \right) \right) dt =
\]

\[
= \hat{c}(p,q) \cdot \int_{0}^{x} \frac{(p + t)^{p+t}}{(q + t)^{q+t} q + \lambda(t)} dt, \quad (5.4.13)
\]

where the normalization factor \( \hat{c}(p,q) \) takes the form

\[
\hat{c}(p,q) = \left( \int_{0}^{+\infty} \frac{(p + t)^{p+t}}{(q + t)^{q+t} q + \lambda(t)} dt \right)^{-1}.
\]

Note that, due to asymptotical equivalency of functions \( \lambda(t) \) and \( t \), the limit exists

\[
\lim_{t \to +\infty} \frac{\lambda(t)}{t} = 1. \quad (5.4.14)
\]

Let us present two corollaries of the obtained result (5.4.13).

First of all, from (5.4.13) we are able to figure out the asymptotic as \( x \to +\infty \) of the function’s \( \hat{F}_{p,q}(x) \) tail.

\[
1 - \hat{F}_{p,q}(x) = \hat{c}(p,q) \cdot \int_{x}^{+\infty} \frac{(p + t)^{p+t}}{(q + t)^{q+t} q + \lambda(t)} dt = \hat{c}(p,q) \int_{x}^{+\infty} \frac{(p + t)^t}{(q + t)^q (1 + (q/t))^t q + \lambda(t)} dt \approx
\]

\[
\approx \hat{c}(p,q) \exp(- (q - p)) \cdot \int_{x}^{+\infty} \frac{(p + t)^p}{(q + t)^q q + \lambda(t)} dt \approx \hat{c}(p,q) \exp(- (q - p)) \cdot \int_{x}^{+\infty} \frac{1}{q - p} x^{-(q - p)} dt =
\]

\[
= \hat{c}(p,q) e^{-(q - p)} \cdot \frac{1}{q - p} x^{-(q - p)}, \quad x \to +\infty. \quad (5.4.15)
\]

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Here we already used the fact that \( \hat{F}_{p,q}(x) \), as a result of \emph{dediscretization} for distribution \( \{p_n\} \), is a \emph{distribution function}. This fact in general case shall be proved in Section 5.6.

The formula (5.4.15) means that the only tail \( 1 - \hat{F}_{p,q}(x) \), \( x \in R^+ \) (because \( \hat{F}_{p,q}(-x) = 0 \), \( x \in R^+ \)), of distribution function \( \hat{F}_{p,q}(x) \) \emph{varies regularly} with \emph{exponent} \((-q - p))\ and in representation

\[
1 - \hat{F}_{p,q}(x) = x^{-(q-p)} \cdot L(x), \ x \in R^+, \tag{5.4.16}
\]

of the tail the \emph{slowly varying component} \( L(x) \) satisfies condition

\[
\lim_{x \to +\infty} L(x) = \hat{c}(p,q)e^{-(q-p)} \frac{1}{q-p}. \tag{5.4.17}
\]

Due to (5.4.16)-(5.4.17), \( 1 - \hat{F}_{p,q}(x) \) exhibits a \emph{constant slowly varying component}.

Secondly, the evaluations carry to the end for \emph{asymptotically linear} \( \lambda(t) \) of a \emph{special form}.

### 5.4.3 The Asymptotically Linear \( \lambda(t) \)

Put

\[
\lambda(t) = \frac{p - q}{\ln(\frac{p+u}{q+u})}, \ t \in [0, +\infty). \tag{5.4.18}
\]

Let us show that the form (5.4.18) of \( \lambda(t) \) indeed presents an \emph{asymptotically linear} \( \lambda(t) \).

Due to L’Hopital rule \( \lim_{t \to +\infty} \frac{\ln(1 - \frac{p+u}{q+u})}{\frac{p+u}{q+u}} = 1 \). Therefore, from (5.4.18) as \( t \to +\infty \) we have \( \lambda(t) = \frac{p - q}{\ln(1 - \frac{p+u}{q+u})} - q \approx \frac{p - q}{q+u} - q = t, \ t \to +\infty \), which proves the statement.

Now, substituting (5.4.18) into (5.4.7) with \( \zeta(t) = t, \ t \in [0, +\infty) \), we obtain

\[
\hat{F}_{p,q}(x) = c(p,q) \frac{q}{q-p} \int_0^x \mid \ln(\frac{p+t}{q+t}) \mid \exp(-\int_0^t \mid \ln(\frac{p+u}{q+u}) \mid du) dt =
\]

\[
= c(p,q) \frac{q}{q-p} (1 - \exp(-\int_0^x \mid \ln(\frac{p+u}{q+u}) \mid du), \tag{5.4.19}
\]

where we took into account that \( \ln(\frac{p+u}{q+u}) = -\ln(\frac{p+u}{q+u}) \) for \( 0 < p < q < +\infty \) and \( u \in R^+ \).

By (5.4.12), \( \int_0^{+\infty} \mid \ln(\frac{p+u}{q+u}) \mid du = +\infty \), so in (5.4.19) we have \( \frac{c(p,q)}{q-p} = 1 \). It means that (5.4.19) may be rewritten in the form (see, (5.4.12))

\[
\hat{F}_{p,q}(x) = 1 - \exp(\ln(\frac{(p+t)^{p+t} q^q}{(q+t)^{q+t} p^p})) = 1 - \frac{q^q (p+x)^{p+x}}{p^p (q+x)^{q+x}}. \tag{5.4.20}
\]

The expression at the right-hand-side of (5.4.20) gives for the tail asymptotic \emph{final result}. In this case the \emph{slowly varying function} in representation of \( 1 - \hat{F}_{p,q}(x) \) as \( x \to +\infty \) tends to the following constant

\[
\lim_{x \to +\infty} \frac{q^q}{p^p} (p-x)^{p-x} = \frac{q^q}{p^p} \lim_{x \to +\infty} \frac{(1 + (p/x))^{x+p}}{(1 + (q/x))^{x+q}} = \frac{q^q}{p^p} e^{-(q-p)}. \tag{5.4.21}
\]

In this case, due to (5.4.21), for \emph{regularly varying} \( 1 - \hat{F}_{p,q}(x) \) we also come to \emph{constant slowly varying component}.

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5.5 Dediscretization: The Power Type \{ζₙ\}

In the present Section we continue attempts to simplify the form of dediscretization of concrete distributions. Such an attempt has been done in Section 5.4 based on simple relationship between \(ζ(t)\) and its derivative \(ζ'(t)\), where \(ζ(t)\) presents the interpolation of sequence \{\(ζ_n\)\}, which is an essential factor in description of distribution \{\(p_n\)\} of type (5.4.4)-(5.4.5). In this Section we consider the case of power \{\(ζₙ\)\}, i.e.

\[
ζ_n = n^α, \quad n = 0, 1, 2, \cdots, \quad 1 < α < +∞.
\] (5.5.1)

In this case above mentioned idea doesn’t work. That is why here we use a possibility given by asymptotical equivalency of \{\(ζₙ\)\} and \{\(λₙ\)\} in (5.4.4)-(5.4.5).

5.5.1 The Preliminary Analysis

By (5.4.10), taking \(ζ(t) = t\), \(t ∈ [0, +∞)\), \(1 < α < +∞\), which presents the interpolation of given by elementary manner \{\(ζₙ\)\} (see, (5.5.1)), for \(t ∈ [0, +∞)\) we proceed

\[
0 < \int_{0^-}^{t} f(x)dx = t \cdot \ln(p + t^α) - t \cdot \ln(q + t^α) - α \cdot \int_{0^-}^{t} \frac{x^αdx}{p + x^α} + α \int_{0^-}^{t} \frac{x^αdx}{q + x^α} =
\ln((p + t^α)q + t^α) + α \int_{0^-}^{t} \frac{dx}{p + x^α} - αq \cdot \int_{0^-}^{t} \frac{dx}{q + x^α} < +∞.
\] (5.5.3)

The last inequality is clear because the integrals at the right-hand-side of (5.5.3) converges as \(t → +∞\). That is why they are finite for finite \(t ∈ R^+\).

Substituting the last expansion into (5.4.13) we find out a distribution function

\[
FP,q(x) = \hat{c}(p,q) \cdot \int_{0^-}^{x} \frac{(p + t^α)^t}{q + t^α} \cdot \frac{1}{q + λ(t)} \cdot \exp(αp \cdot \int_{0^-}^{t} \frac{dx}{p + x^α} - αq \int_{0^-}^{t} \frac{dx}{q + x^α})dt,
\] (5.5.4)

\(0 ≤ x < +∞\), with the corresponding normalization factor \(\hat{c}(p,q)\).

Due to the asymptotical equivalency of functions \(λ(t)\) and \(t^α\), the limit exists

\[
\lim_{t → +∞} \frac{λ(t)}{t^α} = 1.
\] (5.5.5)

Let us present two corollaries of formula (5.5.4).

From (5.5.4) we may figure out the tail’s asymptotic. Denote

\[
I_α(z) = α \cdot z \cdot \int_{0^-}^{+∞} \frac{dx}{z + x^α} \quad \text{for} \quad 1 < α < +∞, \quad 0 < z < +∞.
\] (5.5.6)
Using the notation (5.5.6) as \( x \to +\infty \) we have

\[
1 - \hat{F}_{p,q}(x) = \hat{c}(p,q) \cdot \left( \int_{x}^{+\infty} \frac{(p + t^\alpha)}{q + t^\alpha} \cdot \frac{1}{q + \lambda(t)} \exp(\alpha p \cdot \int_{0}^{t} \frac{dx}{p + x^\alpha} - \alpha q \cdot \int_{0}^{t} \frac{dx}{q + x^\alpha}) dt \right) \\
\approx \hat{c}(p,q) \exp(I_{\alpha}(p) - I_{\alpha}(q)) \cdot \left( \int_{x}^{+\infty} \frac{(1 + (p/t^\alpha))}{(1 + (q/t^\alpha))} \cdot \frac{1}{q + \lambda(t)} dt \right) \\
\approx \hat{c}(p,q) \exp(I_{\alpha}(p) - I_{\alpha}(q)) \cdot \int_{x}^{\infty} \frac{dt}{q + \lambda(t)} \approx \hat{c}(p,q) \exp(I_{\alpha}(p) - I_{\alpha}(q)) \cdot \int_{x}^{+\infty} \frac{dt}{q + t^\alpha} =
\]

\[
= \hat{c}(p,q) e^{I_{\alpha}(p) - I_{\alpha}(q)} \cdot \frac{1}{\alpha - 1} (x + q)^{-(\alpha - 1)} \approx \hat{c}(p,q) e^{I_{\alpha}(p) - I_{\alpha}(q)} \cdot \frac{1}{\alpha - 1} \cdot \frac{1}{x^{\alpha-1}}, \ x \to +\infty (5.5.7)
\]

The expression at the right-hand-side of (5.5.7) says that the tail \( 1 - \hat{F}_{p,q}(x), \ x \in R^+ \), of distribution function \( \hat{F}_{p,q} \) varies regularly at infinity with exponent \( -(\alpha - 1) \) and in representation \( 1 - \hat{F}_{p,q}(x) = x^{-(\alpha - 1)} L_1(x), \ x \in R^+ \), of the tail the slowly varying component \( L_1(x) \) satisfies condition

\[
\lim_{x \to +\infty} L_1(x) = \hat{c}(p,q) e^{I_{\alpha}(p) - I_{\alpha}(q)} \cdot \frac{1}{\alpha - 1}. \quad (5.5.8)
\]

Due to (5.5.8), \( 1 - \hat{F}_{p,q}(x) \) exhibits constant slowly varying component.

### 5.5.2 Simplification

The second idea being before used for etalon linear case works also in the present case.

Indeed, let

\[
\lambda(t) = \frac{p - q}{\ln(p + t^\alpha)} - q, \ t \in [0, +\infty). \quad (5.5.9)
\]

Due to L’Hopital rule, \( \lim_{t \to +\infty} \frac{\ln(1 - \frac{p - q}{p + t^\alpha})}{\ln(\frac{p + t^\alpha}{q + t^\alpha})} = 1 \). Therefore, similarly to the etalon linear case, we conclude that \( \lambda(t) \approx t^{-\alpha}, \ t \to +\infty \). Substituting (5.5.9) into (5.4.7) with \( \zeta(t) \) of the form (5.5.2) for \( x \in [0, +\infty) \) we obtain

\[
\hat{F}_{p,q}(x) = \frac{\hat{c}(p,q)}{p - q} \int_{0}^{x} \ln(p + u^\alpha) \cdot \exp(\int_{0}^{u} \ln(\frac{p + u^\alpha}{q + u^\alpha}) du) dt =
\]

\[
= \frac{\hat{c}(p,q)}{p - q} \left( \exp\left( \int_{0}^{x} \ln(\frac{p + u^\alpha}{q + u^\alpha}) du \right) - 1 \right). \quad (5.5.10)
\]

Note that \( \frac{\hat{c}(p,q)}{p - q} > 0 \) if \( 0 < q < p < +\infty \) and \( \frac{\hat{c}(p,q)}{p - q} < 0 \) if \( 0 < p < q < +\infty \).

By (5.5.3), the integral

\[
\int_{0}^{x} \ln(\frac{p + t^\alpha}{q + t^\alpha}) dt = x \cdot (\ln(1 + \frac{p}{x^\alpha}) - \ln(1 + \frac{q}{x^\alpha})) + \alpha p \cdot \int_{0}^{x} \frac{dt}{p + t^\alpha} - \alpha q \cdot \int_{0}^{x} \frac{dt}{q + t^\alpha}
\]

for any \( x \in R^+ \) is positive if \( 0 < q < p < +\infty \) and is negative if \( 0 < p < q < +\infty \). Since \( \lim_{x \to +\infty} x \cdot (\ln(1 + \frac{p}{x^\alpha}) - \ln(1 + \frac{q}{x^\alpha})) = 0 \), therefore \( \int_{0}^{+\infty} \ln(\frac{p + t^\alpha}{q + t^\alpha}) dt = I_{\alpha}(p) - I_{\alpha}(q), \) where the function \( I_{\alpha}(z), \ 0 < \alpha < +\infty, \ 0 < z < +\infty, \) is defined by formula (5.5.6). It means that

\[
\frac{\hat{c}(p,q)}{p - q} \left( \exp\left( \int_{0}^{+\infty} \ln(\frac{p + u^\alpha}{q + u^\alpha}) du \right) - 1 \right) = \hat{F}_{p,q}(+\infty) = 1 \quad (5.5.11)
\]

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because \( \hat{F}_{p,q} \) is a distribution function (it shall be proved in general case in Section 5.6), and, as a result of this, from (5.5.11) we obtain

\[
\frac{c(p,q)}{p-q} = (\exp(I_\alpha(p) - I_\alpha(q)) - 1)^{-1},
\]

(5.5.12)

It is of interest that the right-hand-side of (5.5.12) may be expressed in terms of Beta Function \( B(x,y) \). Indeed, due to 3.241.2, p.292, [20], for \( 0 < \mu \leq \nu < +\infty \)

\[
\int_{0-}^{\infty} \frac{x^\mu-1}{1 + x^\nu} \, dx = \frac{1}{\nu} B\left(\frac{\mu}{\nu}, \frac{\nu - \mu}{\nu}\right) = \frac{\pi}{\nu} \csc \frac{\mu\pi}{\nu}.
\]

(5.5.13)

From (5.5.13) and (5.5.6) we come to the following equality

\[
I_\alpha(z) = \alpha \cdot z \cdot \int_{0-}^{+\infty} \frac{dx}{z + x^\alpha} = \alpha z^{1/\alpha} \cdot \int_{0-}^{+\infty} \frac{dx/z^{1/\alpha}}{1 + (x^\alpha/z)} = \alpha z^{1/\alpha} \cdot \int_{0-}^{+\infty} \frac{dt}{1 + t^\alpha} =
\]

which implies

\[
\frac{c(p,q)}{p-q} = (\exp((p^{1/\alpha} - q^{1/\alpha})B\left(\frac{1}{\alpha}, \frac{\alpha - 1}{\alpha}\right)) - 1) = (\exp((p^{1/\alpha} - q^{1/\alpha})\pi \csc \frac{\pi}{\alpha}) - 1). \quad (5.5.14)
\]

The right-hand-side expression in (5.5.14) gives a form of normalization factor for distribution function \( \hat{F}_{p,q}(x) \) of form (5.5.10).

### 5.6 Dediscretization: The Ground’s Substantiation

In Section 5.2 the dediscretization of stationary distributions \( \{p_\pm^n\} \) of birth-death special process given by formulas (5.1.6) and (5.1.9) allows to construct a class of functions (5.2.12) defined on \([0, +\infty)\) with jumps at zero. In the present Section we prove that any representative of this class is a distribution function.

#### 5.6.1 The Result

**Theorem 5.1** The functions \( \hat{F}_+(x,b) \) and \( \hat{F}_-(x,b) \) defined by (5.2.12) are distribution functions of some non-negative random variables.

**Proof.** Due to (5.2.12), we have to prove

\[
0 < g_b^\pm(+\infty) < +\infty,
\]

(5.6.1)

where measures \( g_b^\pm(x) \) defined on \([0, +\infty)\) are given by formula (5.2.17).

The first inequality in (5.6.1) is obvious. The fulfillment of the second one is necessary in order to get after normalization of \( g_b^\pm(x) \) by \( g_b^\pm(+\infty) \) distribution functions.
The cases: (a) with sign ’+’ and (b) with sign ’−’ are considered separately.

By the limit (5.1.5) and limit equality \( \lim_{x \to +\infty} (\delta_{n+1}^+ / \delta_n^+) = 1 \), for \( \varepsilon \in (0, 1) \) starting from some index, say \( n_0 \), simultaneously for \( n = n_0, n_0 + 1, \cdots \)

\[
\frac{1 - \varepsilon}{\delta_n^+} < \frac{1}{\varepsilon_n^+} < \frac{1 + \varepsilon}{\delta_n^+}, \quad \frac{1 - \varepsilon}{\delta_{n+1}^+} < \frac{1}{\varepsilon_n^+} < \frac{1 + \varepsilon}{\delta_{n+1}^+}.
\]  

(5.6.2)

(a) First of all, let us notice that here there is a simple subcase

\[
\hat{F}_+(x, 0) = \left( \int_{0-}^{x} \frac{dt}{\varepsilon^+(t)} / \int_{0-}^{+\infty} \frac{dt}{\varepsilon^+(t)} \right), \quad x \in [0, +\infty).
\]

By (5.1.4) and (5.6.2),

\[
0 < \int_{0-}^{+\infty} \frac{dt}{\varepsilon^+(t)} = \sum_{n \geq n_0} \int_{n}^{n+1} \frac{dt}{\varepsilon^+(t)} < \sum_{n \geq n_0} \frac{1}{\varepsilon_n^+} < (1 + \varepsilon) \sum_{n \geq n_0} \frac{1}{\delta_n^+} < +\infty.
\]

Thus, \( g_b^+(+\infty) < +\infty \). If \( -1 < b < 0 \), then \( g_b^+(+\infty) < g_b^+(+\infty) < +\infty \).

Let \( 0 < b < +\infty \). In accordance with the inequalities (5.6.2) and (see, (5.2.9))

\[
\frac{b}{\delta^+(t)} \leq \ln(1 + \frac{b}{\delta^+(t)}) = f_b^+(t), \quad t \in [0, +\infty),
\]

we proceed

\[
\hat{g}_b^+(+\infty) - \hat{g}_b^-(n_0) = \sum_{n \geq n_0} \int_{n}^{n+1} \frac{1}{\varepsilon^+(t)} \exp \left( \int_{0-}^{t} f_b^+(u) du \right) dt < \sum_{n \geq n_0} \frac{1 + \varepsilon}{\delta_{n+1}^+} \int_{n}^{n+1} \exp \left( \int_{0-}^{t} f_b^+(u) du \right) dt < (1 + \varepsilon)^2 \cdot \sum_{n \geq n_0} \frac{1}{\delta_{n+1}^+} \int_{n}^{n+1} \exp \left( \int_{0-}^{t} f_b^+(u) du \right) dt < (1 + \varepsilon)^2 \cdot \sum_{n \geq n_0} \frac{1}{\delta_{n+1}^+} \exp \left( \int_{0-}^{t} f_b^+(u) du \right) dt = \frac{(1 + \varepsilon)^2}{b} \left( \exp \left( \int_{0-}^{t} f_b^+(u) du \right) - \exp \left( \int_{0-}^{n_0} f_b^+(u) du \right) \right).
\]

(5.6.4)

By L’Hospital rule \( \lim_{x \to +\infty} ((f_x^{+\infty} f_b^+(u) du) / (a \cdot f_x^{+\infty} du / \delta^+(u))) = \lim_{x \to +\infty} f_b^+(x) / (a \cdot \delta^+(x)) = 1 \). But, by (5.1.4),

\[
0 \leq \int_{0-}^{+\infty} \frac{du}{\delta^+(u)} = \sum_{n \geq 0} \int_{n}^{n+1} \frac{du}{\delta^+(u)} < 1 + \sum_{n \geq 1} \frac{1}{\delta_n^+} < +\infty
\]

and, due to (5.6.4), the second inequality (5.6.1) holds in this case.

(b) In accordance with the inequalities (5.6.2) and (see, (5.2.9))

\[
f_b^-(t) \leq -\frac{b}{\delta^-(t)} + \frac{1}{2} \left( \frac{b}{\delta^-(t)} \right)^2, \quad 0 < b < 1, \quad t \in [0, +\infty),
\]

(5.6.5)

similarly to (5.6.4) we proceed

\[
\hat{g}_b^+(+\infty) - \hat{g}_b^-(n_0) < (1 + \varepsilon)^2 \cdot \sum_{n \geq n_0} \int_{n}^{n+1} \frac{1}{\delta^-(t)} \exp \left( \int_{0-}^{t} f_b^-(u) du \right) dt < (1 + \varepsilon)^2 \cdot c(b) \cdot \int_{n_0}^{+\infty} \frac{1}{\delta^-(t)} \exp(-b \cdot \int_{0-}^{t} \frac{du}{\delta^-(u)}) dt = \frac{(1 + \varepsilon)^2 \cdot c(b)}{b} \left( \exp(- \int_{0-}^{+\infty} \frac{du}{\delta^-(u)}) + \exp(- \int_{0-}^{n_0} \frac{du}{\delta^-(u)}) \right).
\]

(5.6.6)
where
\[ c(b) = \exp \left( \frac{k^2}{2} \int_{0}^{+\infty} \frac{du}{(\delta(u))^2} \right). \tag{5.6.7} \]

Let us show that $c(b)$ is \textit{finite}. It is enough to prove that $\int_{0}^{+\infty} \frac{du}{(\delta(u))^2} < +\infty$.

We have
\[ \int_{0-}^{+\infty} \frac{du}{(\delta(u))^2} = \sum_{n \geq 0} \int_{n}^{n+1} \frac{du}{(\delta(u))^2} < \sum_{n \geq 0} \frac{1}{(\delta_{n+1})^2}. \tag{5.6.8} \]

Taking into account that for any $\{\delta_n\} \in \Lambda_-$, due to $\delta_0 = 1$ and $\{\delta_n\}$ is downward convex with $\lim_{n \to +\infty} \frac{n}{\delta_n} = 0$, we have
\[ \frac{1}{\delta_n} < \frac{1}{n}, \quad n = 1, 2, \ldots. \tag{5.6.9} \]

Now, with the help of (5.1.3) and (5.6.8)-(5.6.9) we obtain
\[
\begin{cases}
\int_{0-}^{+\infty} \frac{du}{\delta(u)} = \sum_{n \geq 0} \int_{n}^{n+1} \frac{du}{\delta(u)} > \sum_{n \geq 1} \frac{1}{\delta_n^2} = +\infty, \\
0 \leq \int_{0-}^{+\infty} \frac{du}{(\delta(u))^2} < \sum_{n \geq 1} \frac{1}{n^2} < +\infty.
\end{cases} \tag{5.6.10}
\]

Therefore, from (5.6.6)-(5.6.7) because of estimations (5.6.10) we conclude that (5.6.1) holds in this case too. \textit{Theorem 5.1} is proved.

### 5.6.2 Nature of Inequalities (5.6.3) and (5.6.5)

It is important to explain the \textit{nature} of inequalities (5.6.3) and (5.6.5) from general positions.

Let $\{a_n\}$ be a sequence of positive numbers such that
\[ \lim_{n \to +\infty} a_n = 0. \tag{5.6.11} \]

Let us consider two series
\[ S^+ = \sum_{n \geq 1} a_n \quad \text{and} \quad S^- = \sum_{n \geq 1} (-1)^{n+1} \cdot a_n \quad \text{(if exists)}, \tag{5.6.12} \]

and for any integer $n = 1, 2, \ldots$ form partial sums
\[ S^+_n = \sum_{K=1}^{n} a_K \quad \text{and} \quad S^-_n = \sum_{K=1}^{n} (-1)^{K+1} \cdot a_K. \tag{5.6.13} \]

Then, for the partial sums $S^+_n$, $n = 1, 2, \ldots$, defined by (5.6.13) and $S^+$ (see, (5.6.12)), we have inequalities
\[ 0 < S^+_1 < S^+_2 < \cdots < S^+. \tag{5.6.14} \]
Note that $S^+$ may be either finite or infinite. The condition (5.6.11) is only a necessary condition for the convergence of the first series in (5.6.12).

If $\{a_n\}$ is monotone (then, due to (5.6.12), $\{a_n\}$ decreases), then the well-known Leibnitz Theorem on sign-alternating series holds. Namely, the second series in (5.6.12) converges, and for partial sums $S_n^-$, $n = 1, 2, \cdots$, defined by (5.6.13), the inequalities hold

$$ \begin{cases} 
S_{2K}^- < S^- < S_{2K-1}^-, & K = 1, 2, \cdots, \\
S_2^- < \cdots < S_{2K}^- < \cdots, & S_1^- > S_3^- > \cdots > S_{2K-1}^- > \cdots.
\end{cases} $$

(5.6.15)

The series

$$ \ln(1 + x) = \sum_{n \geq 1} \frac{n^n}{n} $$

(5.6.16)

and

$$ -\ln(1 - x) = \sum_{n \geq 1} (-1)^{n+1} \cdot \frac{x^n}{n} $$

(5.6.17)

for $0 < x < 1$ are convergent and are of types $S^+$ and $S^-$, respectively (see, (5.6.12)). In particular, for functions $f_b^\pm(x)$ given by formula (5.2.9), due to (5.6.16)-(5.6.17), the following expansions may be written down

$$ f_b^+(x) = |f_b^+(x)| = \sum_{n \geq 1} \frac{1}{n} (\frac{b}{\delta^+(x)})^n, $$

(5.6.18)

$$ f_b^-(x) = -|f_b^-(x)| = \sum_{n \geq 1} (-1)^n \cdot \frac{1}{n} (\frac{b}{\delta^-(x)})^n $$

(5.6.19)

with $x \in \mathbb{R}^+$, $0 < b < 1$ for $f_b^+(x)$ and $x \in \mathbb{R}^+$, $-1 < b < +\infty$ for $f_b^-(x)$.

The partial sums of the right-hand-side expansions in (5.6.18) and (5.6.19) denote by

$$ [f_b^-(x)]_n = \sum_{K=1}^{n} (-1)^K (\frac{b}{\delta^-(x)})^n \cdot \frac{1}{n}, \quad -1 < b < +\infty, $$

(5.6.20)

and

$$ [f_b^+(x)]_n = \sum_{K=1}^{n} \frac{1}{n} \cdot (\frac{b}{\delta^+(x)})^n, \quad 0 < b < 1, $$

(5.6.21)

for $n = 1, 2, \cdots$. Now, in accordance with inequalities (5.6.15) we get the following inequalities

$$ [f_b^+(x)]_1 < [f_b^+(x)]_3 < \cdots < [f_b^+(x)]_{2K+1} < \cdots < f_b^+(x) < \cdots < [f_b^+(x)]_{2K} < \cdots < [f_b^+(x)]_4 < [f_b^+(x)]_2 $$

(5.6.22)

for sign "$+$" with $0 < b < 1$ and for sign "$-$" with $-1 < b < 0$.

Similarly for sign "$+$" with $0 < b < +\infty$ the following inequalities hold

$$ [f_b^+(x)]_1 < [f_b^+(x)]_2 < \cdots < [f_b^+(x)]_K < \cdots < f_b^+(x). $$

(5.6.23)
Easily seen that

\[ [f_b^+(x)]_1 = \pm \frac{b}{\delta^+(x)}, \quad (5.6.24) \]

\[ [f_b^-(x)]_2 = \pm \frac{b}{\delta^-(x)} + \frac{1}{2} \left( \frac{b}{\delta^-(x)} \right)^2. \quad (5.6.25) \]

The inequalities (5.6.3) and (5.6.5) follow from (5.6.22)-(5.6.25).

5.7 Extracting Distribution Functions with Regularly Varying Tails

In this Section the result of dediscreization of discovered class of frequency distributions, which generated by standard birth-death process, - class of distribution functions (5.2.12) reduced by additional assumptions in order to guarantee regular variation for their only (right) tails.

5.7.1 The Class of Distribution Functions

Denote by \( \Omega \) the class of regularly varying at infinity with exponent \( \alpha \in [1, +\infty) \) increasing, infinite differentiable on \( R^+ \) functions \( \delta(t) \) with \( \delta(0) = 1 \), satisfying following condition: the limit exists

\[ 0 \leq A := \lim_{t \to +\infty} \frac{t}{\delta(t)} < +\infty. \quad (5.7.1) \]

This class is divided on non-intersected classes \( \Omega_0 \) and \( \Omega_1 \): \( \Omega = \Omega_0 \cup \Omega_1 \), \( \Omega_0 \cap \Omega_1 = \emptyset \).

A Special class \( \Omega_0 \) includes increasing functions \( \delta(t), t \in [0, +\infty) \), of the type

\[ \delta(t) = 1 + \frac{t}{A}(1 + o(1)), \quad t \to +\infty, \quad A \in R^+. \quad (5.7.2) \]

Definition 22 The function defined on \( R^+ \) and being asymptotically equivalent to linear function \( \frac{t}{A} \) with some \( A \in R^+ \) is called asymptotically linear.

Thus, (5.7.2) is a representation of asymptotically linear function \( \delta(t) \).

We consider only the linear case, i.e. in (5.7.2) \( o(1) = 0 \).

If \( \delta(t) \in \Omega_1 \), then

\[ (A =) \lim_{t \to +\infty} \frac{t}{\delta(t)} = 0, \quad (5.7.3) \]
and in accordance with (5.7.3) two situations arise:

\[
either \quad \int_{0^+}^{+\infty} \frac{dt}{\delta(t)} = +\infty \quad \text{(then, we write } \delta(t) = \delta^+(t) \text{ for } t \in R^+) \tag{5.7.4}
\]

or \[
\int_{0^+}^{+\infty} \frac{dt}{\delta(t)} < +\infty \quad \text{(then, we write } \delta(t) = \delta^-(t) \text{ for } t \in R^+) \tag{5.7.5}
\]

In both situations (5.7.4) and (5.7.5), due to (5.7.3), we have

\[
0 < c_n := \int_{0^+}^{n} \frac{dt}{\delta(t)} < +\infty, \quad n = 2, 3, \cdots. \tag{5.7.6}
\]

For simplicity we assume that \((\delta(t))'s\) slowly varying component is a non-decreasing function of argument \(t\).

For the linear (also for the asymptotically linear) case we are in situation (5.7.4), but the condition (5.7.3) doesn’t hold.

Introducing asymptotically equivalent to any \(\delta(t) \in \Omega\) function \(\varepsilon(t) \in \Omega_0\), or \(\varepsilon(t) \in \Omega_1\) (then, \(\delta(t) \in \Omega_0\), or \(\delta(t) \in \Omega_1\), respectively), i.e. \(\lim_{t \to +\infty} (\varepsilon(t)/\delta(t)) = 1\), below we describe one class of distribution functions. Without loss of generality we additionally assume that the slowly varying component of \(\varepsilon(t)\) is non-decreasing.

Any pair of functions \((\delta^-(t), \varepsilon^-(t))\) and \((\delta^+(t), \varepsilon^+(t))\) defined on \([0, +\infty)\) generates a family of distribution functions defined on \([0, +\infty)\)

\[
\hat{F}^-_-(x, b) = \frac{\int_{0^-}^{x} \frac{1}{\varepsilon^-(t)} \exp\left(\int_{0^-}^{t} f_b^-(u)du\right)dt}{\int_{0^-}^{+\infty} \frac{1}{\varepsilon^-(t)} \exp\left(\int_{0^-}^{t} f_b^-(u)du\right)dt} \tag{5.7.7}
\]

with \(0 < b < 1\), and

\[
\hat{F}^+_+(x, b) = \frac{\int_{0^-}^{x} \frac{1}{\varepsilon^+(t)} \exp\left(\int_{0^-}^{t} f_b^+(u)du\right)dt}{\int_{0^-}^{+\infty} \frac{1}{\varepsilon^+(t)} \exp\left(\int_{0^-}^{t} f_b^+(u)du\right)dt} \tag{5.7.8}
\]

with \(-1 < b < +\infty\), respectively. Here for \(x \in R^+\)

\[
f_b^\pm(x) = \ln(1 \pm \frac{b}{\delta^\pm(x)}), \tag{5.7.9}
\]

and \(\varepsilon(t) = \varepsilon^-(t)\) if (5.7.3) holds, \(\varepsilon(t) = \varepsilon^+(t)\) if (5.7.4) holds.

From this point, let us forget how the class of distribution functions \(\{\hat{F}^-_-(x, b)\}\) was obtained in Section 5.2.
5.7.2 Regular Variation

Theorem 5.2

1. The functions

\[ 1 - \hat{F}_\pm(x, b) \]  

(5.7.10)

defined by formulas (5.7.7) and (5.7.8) vary regularly at infinity iff \( \delta_\pm(x) \) (then, also \( \varepsilon_\pm(x) \) does) varies regularly at infinity.

2. If \( (-\rho - 1) \) and \( \alpha \) are exponents of regularly varying functions (5.7.10) and \( \delta_\pm(x) \), respectively, then

\[ \rho = \alpha + (|b| \cdot A), \rho \in [1, +\infty), \alpha \in [1, +\infty) \]  

(5.7.11)

where constant \( A \in [1, +\infty) \) is defined by (5.7.1).

Note that the inclusion \( \alpha \in [1, +\infty) \) in (5.7.11) is a consequence of (5.7.1). Indeed, let us assume the opposite, i.e. \( \alpha \in [0, 1) \). Then, \( \delta_\pm(t) = 1 + t^\alpha \cdot L_\pm(t), t \in [0, +\infty) \), and by known property of regular variation [21], for \( \varepsilon \in (0, 1 - \alpha) \) starting from some \( t_0 \in \mathbb{R}^+ \) we have the inequality \( 1 + t^\alpha \cdot L_\pm(t) < t^{\alpha + \varepsilon} \). Therefore, \( A > \lim_{t \to +\infty} \frac{1}{t^{\alpha + \varepsilon}} = +\infty \), which contradicts (5.7.1).

It is possible to give direct long proof of Theorem 5.2. But the proof may be simplified if we notice that distribution function \( \hat{F}_\pm(x, b) \), \( x \in [0, +\infty) \), has density

\[ \hat{\varphi}_\pm(x, b) = \frac{d\hat{F}_\pm(x, b)}{dx} = \frac{\hat{c}_\pm(b)}{\varepsilon^\pm(x)} \exp\left(\int_{0-}^{x} f_b^\pm(u)du\right) \]  

(5.7.12)

(see, (5.7.7) and (5.7.8)), where the normalization factor takes the form

\[ \hat{c}_\pm(b) = \left(\int_{0-}^{+\infty} \frac{1}{\varepsilon^\pm(t)} \exp(\int_{0-}^{t} f_b^\pm(u)du)dt\right)^{-1}. \]

Taking into account that the function \( \hat{\varphi}_\pm(x, b) \) is a continuous analog of distribution \( \{p_n^\pm(b)\} \) defined by formulas (5.1.6) and (5.1.9) we’ll use the continuous analog of the proof of regular variation from Section 4.10. It is only necessary to mention that, due to representation \( 1 - \hat{F}_\pm(x, b) = \int_{x}^{+\infty} \hat{\varphi}_\pm(u, b)du, x \in \mathbb{R}^+ \), Theorem 1(a) from VIII, 9, p.281, [23] may be applied, which states in our case the following fact:

For the regular variation of function (5.7.10) with exponent \( (-\rho - 1) \), where \( \rho \in [1, +\infty) \), the regular variation of function \( \hat{\varphi}_\pm(x, b) \) is necessary and sufficient.

Now, Theorem 5.2 may be reformulated in terms of densities.
Theorem 5.2*

1*. \( \hat{\varphi}_\pm(x, b) \) varies regularly at infinity iff \( \delta^\pm(x) \) varies regularly at infinity.

2*. If \((-\rho)\) and \(\alpha\) are exponents of regular variation of functions \( \hat{\varphi}_\pm(x, b) \) and \( \delta(x) \), respectively, then (5.7.11) holds.

Proof of Theorem 5.2*. First of all, let us establish the following fact.

If (5.7.1) holds, then for \( s \in (1, +\infty) \) the limit exists

\[
B(s) = \lim_{t \to +\infty} \int_t^{st} \frac{1}{\delta(u)} du = A \cdot \ln s, 
\]

where for simplicity the upper index of \( \delta(t) \) is omitted.

Let \( A = 0 \), i.e. (5.7.3) holds. Then, \( \frac{1}{\delta(t)} = o(\frac{1}{t}) \), \( t \to +\infty \), or \( \int_t^{st} \frac{1}{\delta(u)} du = o(\int_t^{st} \frac{1}{u} du) \), \( t \to +\infty \) for \( s \in (1, +\infty) \).

Since for \( s \in (1, +\infty) \)

\[
\lim_{t \to +\infty} \int_t^{st} \frac{1}{u} du = \lim_{t \to +\infty} (\ln(st) - \ln t) = \ln s, 
\]

therefore \( \int_t^{st} \frac{1}{u} du = o(\ln s) = o(1) \), \( t \to +\infty \) for \( s \in (1, +\infty) \).

Let \( 0 < A < +\infty \), i.e. (5.7.2) holds. For \( \varepsilon \in (0, 1) \) starting from some number \( v \in R^+ \) the inequalities hold

\[
\frac{A \cdot (1 - \varepsilon)}{t} < \frac{1}{\delta(t)} < \frac{A \cdot (1 + \varepsilon)}{t}, \ t \in (v, +\infty). 
\]

Due to (5.7.15), for \( t \in (v, +\infty) \) we get two-sides inequality

\[
A(1 - \varepsilon) \ln s = A \cdot (1 - \varepsilon) \int_t^{st} \frac{1}{u} du < \int_t^{st} \frac{1}{\delta(u)} du < A(1 + \varepsilon) \int_t^{st} \frac{1}{u} du = A(1 + \varepsilon) \ln s. 
\]

Tending \( t \to +\infty \) and after that \( \varepsilon \downarrow 0 \) we obtain (5.7.13) in this case.

Let us pass directly to the proof of Theorem 5.2*. Note that for \( s \in (1, +\infty) \)

\[
\lim_{t \to +\infty} \left( \frac{\varepsilon^\pm(t)}{\varepsilon^\pm(st)} \right) = \lim_{t \to +\infty} \left( \frac{\delta^\pm(t)}{\delta^\pm(st)} \right) \text{ (if limits exist)}. 
\]

Indeed, for \( s \in (1, +\infty) \)

\[
\lim_{t \to +\infty} \frac{\varepsilon^\pm(t)}{\varepsilon^\pm(st)} = \lim_{t \to +\infty} \frac{\varepsilon^\pm(t)}{\delta^\pm(t)} \cdot \lim_{t \to +\infty} \frac{\delta^\pm(st)}{\varepsilon^\pm(st)} \cdot \lim_{t \to +\infty} \frac{\delta^\pm(t)}{\delta^\pm(st)} = \lim_{t \to +\infty} \frac{\delta^\pm(t)}{\delta^\pm(st)}, 
\]

where the asymptotical equivalency of \( \delta^\pm(t) \) and \( \varepsilon^\pm(t) \) was used.
For \( \hat{\varphi}_-(x, b) \) and any \( s \in (1, +\infty) \), due to (5.7.13) and (5.7.16), we have

\[
\lim_{x \to +\infty} \frac{\hat{\varphi}_-(sx, b)}{\hat{\varphi}_-(x, b)} = \lim_{x \to +\infty} \frac{\varepsilon^-(x)}{\delta^-(sx)} \exp(-b \cdot \lim_{x \to +\infty} \int_x^{sx} \frac{1}{\delta^-(u)} du) = \frac{1}{s^{A}} \lim_{x \to +\infty} \frac{\delta^-(x)}{\delta^-(sx)}, \quad 0 < b < 1,
\]

if limits exist. In order to get (5.7.17) we take into account that for \( s \in (1, +\infty) \) the functions \( f_b^{\pm}(u)du \) and \( \pm b \cdot \int_x^{sx} \frac{du}{\delta^\pm(u)} \) are asymptotically equivalent as \( x \to +\infty \). Indeed, due to L’Hospital rule, for \( s \in (1, +\infty) \)

\[
\lim_{x \to +\infty} \left( (\int_x^{sx} f_b^{\pm}(u)du)/(\pm b \cdot \int_x^{sx} \frac{du}{\delta^\pm(u)}) \right) = \lim_{x \to +\infty} \left( (f_b^{\pm}(sx) - f_b^{\pm}(x))/(\pm b(\frac{1}{\delta^\pm(sx)} - \frac{1}{\delta^\pm(x)})) \right) = \lim_{x \to +\infty} \ln(1 + \frac{b}{\delta^\pm(sx)}(1 - \frac{\delta^\pm(sx)}{\delta^\pm(x)}) + \frac{1}{\delta^\pm(sx)} \cdot \frac{1}{\delta^\pm(x)})
\]

\[= \lim_{x \to +\infty} \frac{\ln(1 + \frac{b}{\delta^\pm(sx)}(1 - \frac{\delta^\pm(sx)}{\delta^\pm(x)}) + o(\frac{1}{\delta^\pm(sx)}))}{\pm b \cdot \frac{1}{\delta^\pm(sx)}(1 - \frac{\delta^\pm(sx)}{\delta^\pm(x)})} = 1.\]

Here the condition \( \lim_{x \to +\infty} \delta^\pm(x) = +\infty \) was used.

From (5.7.17) we conclude that \( \hat{\varphi}_-(x, b) \) varies regularly at infinity iff \( \delta^-(x) \) varies regularly at infinity. Moreover, we obtain \( \rho = a + b \cdot A \).

For \( \hat{\varphi}_+(x, b) \) and any \( s \in (1, +\infty) \), similarly to the previous case, we obtain

\[
\lim_{x \to +\infty} \frac{\hat{\varphi}_+(sx, b)}{\hat{\varphi}_+(x, b)} = \frac{1}{s^{A}} \lim_{x \to +\infty} \frac{\delta^+(x)}{\delta^+(sx)}, \quad -1 < b < +\infty,
\]

if limits exist. Since (5.7.6) takes place in this case, so \( A = 0 \), and in (5.7.18) at the right-hand-side we may replace the multiplier \( (1 = \frac{1}{s^{A}}) \) by \( \frac{1}{sx^\epsilon} (= 1) \). Now, as in (5.7.17), formula (5.7.18) proves Theorem 5.2* in this case.

### 5.8 Convexity of Distribution Functions

In this Section the convexity of regularly varying distribution functions \( \tilde{F}_\pm(x, b) \) is established. Here even the increase of functions \( \delta^\pm(t) \) and \( \varepsilon^\pm(t) \) is enough to make some conclusions on upward/downward convexity of distribution functions \( \tilde{F}_\pm(x, b) \) on \( R^+ \).

This type of conclusions are discovered by the following

**Theorem 5.3**

1. The functions \( \tilde{F}_-(x, b) \) and \( \tilde{F}_+(x, b) \) with \( 0 < b < 1 \) and \( -1 < b \leq 0 \) defined by formulas (5.7.7) and (5.7.8), respectively, are upward convex on \( R^+ \).
2. For the function \( \hat{F}_+(x, b) \) with \( 0 < b < +\infty \) defined by formula (5.7.8) there is a point \( x_0 \in R^+ \) such that \( \hat{F}_+(x, b) \) is: downward convex in \((0, x_0)\); upward convex in \((x_0, +\infty)\).

**Proof.** The distribution function \( \hat{F}_+ \) has a density on \( R^+ \) defined by formula (5.7.12). For \( x \in R^+ \), due to (5.7.12), we have

\[
\frac{d}{dx} \hat{\varphi}_\pm(x, b) = \frac{\hat{\varphi}_\pm(b)}{\varepsilon^\pm(x)} \cdot \left\{ \int_{0^-}^x f_b^\pm(u) du \right\} \cdot \left\{ f_b^\pm(x) - \frac{1}{\varepsilon^\pm(x)} \cdot \frac{d}{dx} \varepsilon^\pm(x) \right\}. \tag{5.8.1}
\]

For sign "." with \( 0 < b < 1 \) and for sign "+" with \(-1 < b < 0 \) we have \( f_b^\pm(x) < 0, x \in R^+ \).

Therefore, because of (5.8.1) the inequality holds \( \frac{d}{dx} \hat{\varphi}_\pm(x, b) < 0 \) for any \( x \in R^+ \), which proves the statement 1. of Theorem 5.3 with \( b \neq 0 \), i.e. for \( \hat{F}_- \) with \( 0 < b < 1 \) and for \( \hat{F}_+ \) with \(-1 < b < 0 \). The case \( b = 0 \) corresponds to formula (5.7.12). For \( \hat{F}_+ \) it remains to substantiate the limit relationship

\[
\lim_{x \to +\infty} \frac{d}{dx} \varepsilon^+(x) = 1, \tag{5.8.2}
\]

and the limit exists

\[
\lim_{x \to +\infty} \frac{d}{dx} \varepsilon^+(x) = +\infty. \tag{5.8.3}
\]

Taking into account (5.8.2) and (5.8.3) and, due to asymptotical equivalency of functions \( \varepsilon^+(x) \) and \( \delta^+(x) \), continuity of terms at the right-hand-side in the last multiplier of (5.8.1), we conclude that there is a number \( x_0 \in R^+ \) such that

\[
\frac{d}{dx} \hat{\varphi}_+(x, b) \begin{cases} 
> 0 & \text{for } x \in (0, x_0), \\
< 0 & \text{for } x \in (x_0, +\infty). 
\end{cases} \tag{5.8.4}
\]

The inequality (5.8.4) proves the statement 2. of Theorem 5.3.

In order to finish the prove of Theorem 5.3 it remains to substantiate the limit relationship (5.8.3). According to this purpose we use the representation of regularly varying at infinity with exponent \( \alpha \in [1, +\infty) \) function \( \varepsilon^+(t) \): \( \varepsilon^+(t) = 1 + t^\alpha L_+(t) \) (see, (5.2.11)), where \( L_+(t) \) is non-decreasing (see, addition assumption on \( L_{\pm}(t) \) in 5.7.1 of Section 5.7). Then the limit exists

\[
L_+ = \lim_{t \to +\infty} L_+(t) \in R^+ \quad \text{and} \quad \frac{dL_+(t)}{dt} \geq 0 \quad \text{for } t \in R^+. \tag{5.8.5}
\]

As a result of this,

\[
\inf_{t \in R^+} \frac{dL_+(t)}{dt} \geq 0. \tag{5.8.6}
\]
Note that
\[ \frac{d}{dt} \varepsilon^+(t) = t^{\alpha-1} \cdot L_+(t) + t^\alpha \cdot \frac{dL_+(t)}{dt}, \quad t \in R^+. \]  
(5.8.7)

For \( \alpha \in (1, +\infty) \) regarding to (5.8.5)-(5.8.7) the limit relationship (5.8.3) takes place. If \( \alpha = 1 \), then, due to (5.7.6),
\[ \int_0^{+\infty} \frac{dt}{1 + t \cdot L_+(t)} = \int_0^{+\infty} \frac{dt}{\varepsilon^+(t)} \sim \int_0^{+\infty} \frac{dt}{\delta^+(t)} < +\infty, \]
which implies that in limit (5.8.5) we have \( L_+ = +\infty \). Now
\[ \frac{d}{dt} \varepsilon^+(t) = L_+(t) + t \frac{dL_+(t)}{dt} > L_+(t) \quad \text{for} \quad t \in R^+ \]
(5.8.8)
where (5.8.6) was used. Tending \( t \to +\infty \) in (5.8.8) we prove limit relation (5.8.3) also for this case. Theorem 5.3 is completely proved.

Let us make one remark. The following subclass of the considered class of distribution functions (particular case) is of interest: for \( t \in [0, +\infty) \)
\[ \varepsilon^+(t) = \delta^+(t) \quad \text{and} \quad \varepsilon^-(t) = \delta^-(t). \]
(5.8.9)
A deep investigation of a class of stationary distributions generated by standard birth-death process with specific restrictions on the process’ coefficients has been done in [16]. The dediscretization of this class leads exactly to the above mentioned particular case (5.8.9). The example \( \varepsilon^-(t) = \delta^-(t) = 1 + t \) (the so-called linear case), discovers distribution function (5.4.13).

### 5.9 Enlarging the Class of Distribution Functions

In the present Section we enlarge the described in 5.7.1 of Section 5.7 class of distribution functions and come to a class having the same qualitative properties as the original one.

Next, we extract a subclass of distribution functions from the introduced class with simple representation form being suitable for future stable approximation. This class seems to be not included in the original class from 5.7.1 of Section 5.7 and is only in some sense its linear approximation by parameters. This class, in particular, includes parametric non-stationary exponential distribution function.

Now, a discretization of non-stationary exponential distribution function presents new empirical frequency distributions which we hope shall be useful for biomolecular applications.

#### 5.9.1 Auxiliary Statements

Below we formulate several statements which are used in construction of the above mentioned enlarged class of distribution functions. The proofs shall be done in the next Section. For \( x \in R^+ \) let us consider the following equation
\[ (1 + x) \ln(1 + x) = x. \]
(5.9.1)
Lemma 5.1  There is no roots of equation (5.9.1) in $R^+$.

Corollary 5.1  The function

$$ y = \frac{x}{\ln(1 + x)} \quad (5.9.2) $$

increases in $R^+$ as $x$ increases.

Denote by $\hat{\Omega}^+_c$ the class of regularly varying at infinity with exponent $\alpha \in [1, +\infty)$, increasing, infinite differentiable on $R^+$ functions $\hat{\delta}^+_c(t)$ with

$$ \hat{\delta}^+_c(0) = c \in (1, +\infty), \quad (5.9.3) $$

satisfying following conditions: the limit exists

$$ \lim_{t \to +\infty} \frac{t}{\delta^+_c(t)} = 0, \quad (5.9.4) $$

and

$$ \int_{0^+}^{+\infty} \frac{1}{\delta^+_c(t)} \, dt < +\infty. \quad (5.9.5) $$

Lemma 5.2  For any given $b \in R^+$ the inclusion takes place

$$ (\hat{\delta}^+_c(t) =) \frac{b}{\ln(1 + \frac{b}{\delta^+_c(t)})} \in \hat{\Omega}^+_c \quad (5.9.6) $$

with

$$ c = \frac{b}{\ln(1 + b)}. \quad (5.9.7) $$

Let for given $c \in (1, +\infty)$ the inclusion takes place $\hat{\delta}^+_c(t) \in \hat{\Omega}^+_c$. Representing $\hat{\delta}^+_c(t)$ in the form for $t \in [0, +\infty)$

$$ \hat{\delta}^+_c(t) = \frac{b(c)}{\ln(1 + \frac{b(c)}{x(t)})} \quad (5.9.8) $$

we solve the equation (5.9.8) with respect to unknown function $x(t)$. We have

$$ \ln(1 + \frac{b(c)}{x(t)}) = \frac{b(c)}{\hat{\delta}^+_c(t)}, \quad \text{or} \quad \exp(\frac{b(c)}{\hat{\delta}^+_c(t)}) = 1 + \frac{b(c)}{x(t)}, \quad \text{or} $$

$$ x(t) = \frac{b(c)}{1 - \exp(\frac{b(c)}{\hat{\delta}^+_c(t)})} \quad \text{for} \ t \in [0, +\infty). \quad (5.9.9) $$

Lemma 5.3
1. \( b(c) \) with \( c \in (1, +\infty) \) is in \( R^+ \) the only root of the equation
\[
c = \frac{x}{\ln(1 + x)}.
\] (5.9.10)

2. The inclusion holds
\[
\delta^+(t) := x(t) \in \Omega_1 \text{ with } b \in R^+,
\] (5.9.11)
where \( \Omega \) is defined in 5.7.1 of Section 5.7.

Similarly, denote by \( \hat{\Omega}_c^- \) the class of regularly varying at infinity with exponent \( \alpha \in [1, +\infty) \), increasing, infinite differentiable on \( R^+ \) functions \( \hat{\Theta}_c^-(t) \) with
\[
\hat{\Theta}_c^-(0) = c \in (1, +\infty),
\] (5.9.12)
satisfying following conditions: the limit exists
\[
\lim_{t \to +\infty} \frac{t}{\hat{\Theta}_c^-(t)} = 0,
\] (5.9.13)
and
\[
\int_{0+}^{+\infty} \frac{1}{\hat{\Theta}_c^-(t)} dt = +\infty.
\] (5.9.14)

Lemma 5.4 For any given \( b \in (0, 1) \) the inclusion takes place
\[
(\hat{\Theta}_c^-(t) =) \frac{-b}{\ln(1 - \frac{b}{x(t)})} \in \hat{\Omega}_c^-
\] (5.9.15)
with
\[
c = \frac{-b}{\ln(1 - b)}.
\] (5.9.16)

Here \( \delta^\pm(t) \) belong to \( \Omega_1 \) defined in 5.7.1 of Section 5.7 (so, (5.7.4) and (5.7.5) for \( \delta^+(t) \) and \( \delta^-(t) \) holds, respectively), and the set \( \hat{\Omega}_c^+ \) is defined above.

Let for a given \( c \in (1, +\infty) \) the inclusion takes place \( \hat{\Theta}_c^+(t) \in \hat{\Omega}_c^+ \). Representing \( \hat{\Theta}_c^+(t) \) for \( t \in [0, +\infty) \) in the form
\[
\hat{\Theta}_c^+(t) = \frac{-b}{\ln(1 - \frac{b}{x(t)})}
\] (5.9.17)
we solve the equation (5.9.17) with respect to unknown function \( x(t) \). We have
\[
x(t) = \frac{\hat{b}(c)}{1 - \exp(-\frac{b}{x(t)})} \text{ for } t \in [0, +\infty).
\] (5.9.18)

Lemma 5.5

1. \( \hat{b}(c) \) with \( c \in (1, +\infty) \) is the only root in \( R^+ \) of the equation
\[
c = \frac{-x}{\ln(1 - x)}.
\] (5.9.19)

2. The inclusion holds \( \delta^\pm(t) := x(t) \in \Omega_1 \) with \( 0 < b < 1 \).
5.9.2 Enlarging the Class

In order to transform the class of distribution functions of types (5.7.7) and (5.7.8) conserving at the same time its main properties (regular variation of the tails, convexity, etc.) we realize the following chain of actions with the form of equalities (5.7.7) and (5.7.8).

Let us begin from distribution function of type (5.7.8) with given fixed \( b \in \mathbb{R}^+ \).

1. Find the value \( c = \frac{\ln(1+b)}{b} \) (see, (5.9.7)).

2. Find the value \( b(c) \) as only solution of equation \( c = \frac{\ln(1+b)}{b} \) (see, statement 1. of Lemma 5.3).

3. Replace the function \( f^+(t) \) by \( b(c)^+ \delta(t) + c(t) \), where \( \delta(t) \) is defined by (5.9.8) and by Lemma 5.2 the inclusion (5.9.6) holds.

4. Given \( c \in (1, +\infty) \) replace the function \( \varepsilon^+(t) \) by

\[
\hat{\nu}^+(t) = c \cdot \varepsilon^+(t/c^{1/\alpha}) \in \hat{\Omega}^+.
\] (5.9.20)

5. Make dependent parameters \( b \) and \( c \) independent enlarging by this operation the class we are going to obtain.

6. Without loss of generality take instead of \( \hat{\delta}^+(t) \) and \( \hat{\nu}^+(t) \) functions \( \hat{\delta}^+_1(t) = \delta^+(t) \), \( \hat{\nu}^+_1(t) = \varepsilon^+(t) \).

7. Put \( \lambda^+(t) = \frac{1}{\varepsilon^+(t)} \), \( \mu^+(t) = \frac{1}{\delta^+(t)} \).

Then we get a distribution function

\[
G_b^+(x) = \frac{\int_0^x \lambda^+(t) \exp(b \cdot \int_0^x \mu^+(u)du)dt}{\int_0^\infty \lambda^+(t) \exp(b \cdot \int_0^x \mu^+(u)du)dt}
\] (5.9.21)

with \( 0 < b < +\infty \). The functions \( \lambda^+(t) \) and \( \mu^+(t) \) vary regularly with exponent \(-\alpha\), \( \alpha \in [1, +\infty) \), decreases, are infinite differentiable on \( \mathbb{R}^+ \), \( \lambda^+(0) = \mu^+(0) = 1 \), the limits exist

\[
\lim_{t \to +\infty} t \cdot \lambda^+(t) = 0,
\] (5.9.22)

\[
\lim_{t \to +\infty} \lambda^+(t) = 1,
\] (5.9.23)

and

\[
\int_0^\infty \lambda^+(t)dt < +\infty.
\] (5.9.24)

The same programm of actions leads to transformation and to enlargement of the class of distribution functions (5.7.7) and (5.7.8) with \( b \in (0, 1) \) and \( b \in (-1, 0) \), respectively. Finally, we come to distribution functions of types

\[
G_b^-(x) = \frac{\int_0^x \lambda^-(t) \exp(-b \cdot \int_0^x \mu^-(u)du)dt}{\int_0^\infty \lambda^-(t) \exp(-b \cdot \int_0^x \mu^-(u)du)dt}, \quad 0 < b < +\infty,
\] (5.9.25)
and
\[ G_b^+(x) = \int_0^x \lambda^+(t) \exp(-b \cdot \int_0^t \mu^+(u)du)dt = \frac{\exp(b \cdot \int_0^x \lambda^+(u)du) - 1}{\exp(b \cdot \int_0^\infty \lambda^+(u)du) - 1}, \]
0 < b < +\infty. \quad (5.9.26)

The functions \( \lambda^-(t) \) and \( \mu^-(t) \) vary regularly with exponent \((-\alpha)\), \( \alpha \in [1, +\infty) \), decrease, are infinite differentiable on \( R^+ \), \( \lambda^-(0) = \mu^-(0) = 1 \), the limits exist
\[ \lim_{t \to +\infty} t \cdot \lambda^-(t) = 0, \]
\[ \lim_{t \to +\infty} \lambda^-(t) = 1, \]
and
\[ \int_{0^+}^{+\infty} \lambda^-(t)dt = +\infty. \quad (5.9.29) \]

The functions \( \lambda^+(t) \) and \( \mu^+(t) \) in (5.9.26) are the same as in (5.9.21).

### 5.9.3 Particular Case

In particular case \( \lambda^+(t) = \mu^+(t) \) and \( \lambda^-(t) = \mu^-(t) \) for \( t \in [0, +\infty) \) the distribution functions \( G_b^+(x) \) and \( G_b^-(x) \) have very simple form. Indeed, for \( x \in [0, +\infty] \)
\[ f_b^+(x) := \int_0^x \lambda^+(t) \exp(b \cdot \int_0^t \lambda^+(u)du)dt = \frac{1}{b} \int_0^x dt \left( \exp(b \cdot \int_0^t \lambda^+(u)du) - 1 \right), \]
\[ f_b^-(x) := \int_0^x \lambda^+(t) \exp(-b \cdot \int_0^t \lambda^+(u)du)dt = -\frac{1}{b} \exp(-b \cdot \int_0^x \lambda^+(u)du) - 1. \]

Moreover, \( \hat{f}_b^-(+\infty) = \frac{1}{b} \). Taking all this into account for \( x \in R^+ \) and \( b \in R^+ \) we obtain
\[ G_b^+(x) = \frac{\exp(b \cdot \int_0^x \lambda^+(u)du) - 1}{\exp(b \cdot \int_0^\infty \lambda^+(u)du) - 1}, \]
\[ G_b^-(x) = 1 - \exp(-b \cdot \int_0^x \lambda^-(u)du), \]
\[ \hat{G}_b^+(x) = \frac{1 - \exp(-b \cdot \int_0^x \lambda^+(u)du)}{1 - \exp(-b \cdot \int_0^\infty \lambda^+(u)du)}. \]

The distribution function of type (5.9.31) is a so-called parametric non-stationary exponential distribution function.

Now, a simple discretization leads to a desired class of frequency distributions.

For instance, the distribution function (5.9.31) generates a following distribution function
\[ F_b^-(x) = \sum_{n \leq x} p_n^-(b) = 1 - \exp(-b \cdot \sum_{m=0}^{[x]} \lambda^-(m)), \quad x \in R^+ \], where \{\lambda^-\} varies regularly with exponent \((-\alpha)\), \( \alpha \in (1, +\infty) \), decreases \( \lambda_0 = 1 \), \( \lim_{n \to +\infty} n \cdot \lambda^- = 0 \), \( \sum_{n \geq 0} \lambda^- = +\infty \).

As a result of this, \( p_n^-(b) = \exp(-b \cdot \sum_{m=0}^{n} \lambda^- m) - \exp(-b \cdot \sum_{m=0}^{n-1} \lambda^- m), \quad n = 1, 2, \ldots, \) and \( p_0^-(b) = 1 - \exp(-b) \).

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5.10 Appendix

In this Section we prove auxiliary statements formulated in Section 5.9 and give some explanations on procedure of distribution functions class’ enlargement.

Proof of Lemma 5.1. Let us use graphical method. We take into account that the second function increases and is downward convex. Indeed, its derivatives of first and second orders are positive:
\[ 1 + \ln(1 + x) > 0 \] and \[ 1 + \frac{x}{x + 1} > 0. \]

Let us assume that the opposite, i.e. equation (5.9.1) has a root \( \beta \in \mathbb{R}^+ \). For \( x = 1 \) we have \( (1 + x) \ln(1 + x) = 2 \ln 2 > 1 \).
Therefore the only root \( b_0 \) of equation (5.9.1) has to be less than 1, i.e. \( b_0 \in (0, 1) \).

But, for \( x \in (0, b_0) \),
\[
x - (1 + x) \ln(1 + x) > 0. \tag{5.10.1}
\]
At the same time, using for \( x \in (0, 1) \), the following inequality \( \ln(1 + x) > x \) we obtain
\[
(1 + x) \ln(1 + x) = \ln(1 + x) + x \ln(1 + x) > x + x \ln(1 + x) > x,
\]
which contradicts the inequality (5.10.1).

For \( x \in \mathbb{R}^+ \) the inequality holds
\[
x - (1 + x) \ln(1 + x) < 0. \tag{5.10.2}
\]

As a result of this, \( \frac{d}{dx} \left( \frac{\ln(1+x)}{x} \right) = \frac{1}{x^2(1+x)} (x - (1 + x) \ln(1 + x)) < 0 \) for \( x \in \mathbb{R}^+ \) because of (5.10.2). It means that the function \( \frac{\ln(1+x)}{x} \) decreases in \( \mathbb{R}^+ \) as \( x \) increases which proves Corollary 5.1.

Proof of Lemma 5.2. The parameter \( c \) in (5.9.7), as a function of parameter \( b \), increases as \( b \) increases covering the interval \( (1, +\infty) \). Indeed, the extreme values of function (5.9.7) are \[ \lim_{b \to 0} \frac{b}{\ln(1+b)} = 1 \] and \[ \lim_{b \to +\infty} \frac{b}{\ln(1+b)} = +\infty. \]
The function in (5.9.2), due to properties of function \( \delta^+(t) \), increases, is infinite differentiable, satisfies condition (5.9.3) with \( c = \frac{b}{\ln(1+\frac{b}{\delta^+(t)})} = \frac{b}{\ln(1+b)} \).

Next, the function in (5.9.6) is asymptotically equivalent to \( \delta^+(t) \). Indeed, by L’Hopital rule, \[ \lim_{t \to +\infty} \frac{\ln(1+b(\delta^+(t)))}{(b/\delta^+(t))} = 1. \]
This fact implies the fulfillment of conditions (5.9.4) and (5.9.5) because the same conditions has also the function \( \delta^+(t) \) (see, (5.7.3) and (5.7.5)).

Proof of Lemma 5.3. Let us draw the graphs of functions
\[
y = c \quad \text{and} \quad y = \frac{x}{\ln(1 + x)} \quad \text{(see, Figure 9)}.
\]

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Since the second function increases and its limit as \( x \to +\infty \) equals to \(+\infty\), therefore there is only one intersection point of these functions, which proves the uniqueness of the solution of equation (5.9.10). Using the same arguments as above we conclude that all conditions for the inclusion (5.9.11) are fulfilled. In particular, we talk on conditions (5.7.3) and (5.7.5).

Lemma 5.3 says that any function \( \delta^+(t) \) with the help of parameter \( b \in \mathbb{R}^+ \) generates a family of functions \( \left\{ \hat{\delta}^+_c(t) : 1 < c < +\infty \right\} \) from \( \hat{\Omega}^+_c \). In reverse, for any function \( \hat{\delta}^+_c(t) \in \hat{\Omega}^+_c \) with a given \( c \in (1, +\infty) \), there is only one function \( \delta^+(t) \in \Omega_1 \), for which \( \hat{\delta}^+_c(t) \) belongs to a family generated by \( \delta^+(t) \) with the help of parameter \( b \in \mathbb{R}^+ \).

Proof of Lemma 5.4. The parameter \( c \) in (5.9.15), as a function of parameter \( b \), increases as \( b \) increases (\( b \in (0, 1) \)) covering the interval \( (1, +\infty) \). Indeed, for \( b \in (0, 1) \)

\[
\frac{d}{db} \left( -\frac{\ln(1-b)}{b} \right) = \frac{1}{b^2} \ln(1-b) - \frac{1}{b(1-b)} < 0.
\]

It means that the function \( -\frac{\ln(1-b)}{b} \) decreases as \( b \) increases. As a result of this, the function
asymptotically equivalent that the inclusion (5.9.20) takes place. At the same time, the function defined by (5.9.20) is asymptotically equivalent to function \(\hat{\Theta}_c^\pm(t)\), with the help of parameter \(c\) : 1\(b\)/ln(1\(b\)) with \(0 < b < +\infty\), and \(-b/ln(1\(b\))\) with \(0 < b < 1\), generate functions from the set \(\hat{\Omega}_c^\pm\) with corresponding values of \(c\): \(c = b/ln(1\(b\))\) and \(c = -b/ln(1\(-b\))\), respectively.

Lemma 5.5 is proved similarly to Lemma 5.3.

Thus, any function \(\delta^\pm(t)\) with the help of parameter \(b \in (0, 1)\), generates a family of functions \(\{\hat{\Theta}_c^\pm(t) : 1 < c < +\infty\}\) from \(\hat{\Omega}_c^\pm\). In reverse, for any function \(\hat{\Theta}_c^\pm(t) \in \hat{\Omega}_c^\pm\) with a given \(c \in (1, +\infty)\), there is only one function \(\delta^\pm(t) \in \Omega_1\), for which \(\hat{\Theta}_c^\pm(t)\) belongs to a family generated by \(\delta^\pm(t)\) with the help of parameter \(b \in (0, 1)\).

Finally, comments on action 4. in 5.9.2 of Section 5.9 shall be done. For given regularly varying with exponent \(\alpha \in [1, +\infty)\) function \(\varepsilon^\pm(t) \in \Omega_1\) and any constant \(d \in R^+\) the limit exists

\[
\lim_{t \to +\infty} \frac{1}{d^\alpha} \frac{\varepsilon^\pm(dt)}{\varepsilon^\pm(t)} = 1.
\]

(5.10.3)

The limit relationship (5.10.3) means that for any \(d \in R^+\) the functions \(\varepsilon^\pm(t)\) and \(d^{-\alpha} \cdot \varepsilon^\pm(dt)\) are asymptotically equivalent.

For a given \(c \in (1, +\infty)\) in case \(\alpha \in (1, +\infty)\) from the equality \(d^{-\alpha} = c\) we obtain the value \(d = (1/c)^{1/\alpha}\). Then, taking into account that \(\varepsilon^\pm(dt)/t=0 = 1\) for any \(d \in R^+\), we conclude that the inclusion (5.9.20) takes place. At the same time, the function defined by (5.9.20) is asymptotically equivalent to function \(\varepsilon^\pm(t)\) and, obviously, \(\hat{\nu}_c^\pm(0) = c\).

Since for \(\delta^\pm(t)\) and \(\varepsilon^\pm(t)\) presented in the expressions of distribution functions (5.7.7)-(5.7.8) the limit exists \(\lim_{t \to +\infty} \frac{\varepsilon^\pm(t)}{\delta^\pm(t)} = 1\), therefore for functions \(\hat{\nu}_c^\pm(t)\) and \(\hat{\Theta}_c^\pm(t)\), \(\hat{\delta}_c^\pm(t)\) defined by (5.9.20) and (5.9.15), (5.9.6), respectively, the limits exist

\[
\lim_{t \to +\infty} \frac{\hat{\nu}_c^\pm(t)}{\hat{\Theta}_c^\pm(t)} = 1 \quad \text{and} \quad \lim_{t \to +\infty} \frac{\hat{\nu}_c^\pm(t)}{\hat{\delta}_c^\pm(t)} = 1,
\]

i.e. these functions are asymptotically equivalent.
Chapter 6

Regularly Varying Functions: The Selected Topics

6.1 Problems on Regularly Varying Functions

6.1.1 Regular Variation

A Theory of Regularly Varying Function of one real variable has been developed by J. Karamata in period 1930-1960. Already there are several monographs devoted to foundation and initial problems of this theory [21], [65]-[67].

The class of regularly varying functions in generated by and includes, as a subclass, a class of power functions \( \{x^\rho : \rho \in R^1 \in (-\infty, +\infty)\} \). It completely fills a gap between any two power functions at infinity in a sense of growth order.

We already are familiar with contemporary definition of a regularly varying function of one real variable, given in 1.3.1 of Section 1.3.

A measurable on \( R^+ \) function \( R(t) > 0 \) varies regularly at infinity if for any \( x > 0 \) the limit exists

\[
\varphi(x) = \lim_{t \to +\infty} \frac{R(xt)}{R(t)},
\]

(6.1.1)

and

\[
0 < \varphi(x) < +\infty.
\]

(6.1.2)

It follows from definition that \( \varphi(x) = x^\rho, \rho \in R^1, x \in R^+ \), and the convergence in (6.1.1) is uniform in any \( [a,b], 0 < a < b < +\infty \). Here \( \rho \) is called an exponent of \( R(t) \)'s regular variation.

The representation \( R(t) = t^\rho \cdot L(t), t \in R^+, \) of regularly varying with exponent \( \rho \) function \( R(t) \) allows to transform the properties of slowly varying function \( L(t) \) to regularly varying \( R(t) \).

Remind that the definition of slow variation is included in (6.1.1)-(6.1.2) in case \( \rho = 0 \).

The notion of slowly varying function has been introduced by R. Schmidt even earlier than the notion of regular variation (about 1952).
The fundamental facts of this theory are related with the name of J. Karamata. These are:

1. **Karamata Representation Theorem.**

The function $L(t) > 0$ defined on $\mathbb{R}^+$ varies slowly at infinity iff there are constant $A > 0$ and $c > 0$, and functions $a(t)$ and $b(t)$ satisfying conditions

\[
\lim_{t \to +\infty} a(t) = 1, \quad \lim_{t \to +\infty} b(t) = 0
\]

whereas $a(t)$ is measurable on $[A, +\infty)$, $b(t)$ is continuous on $[A, +\infty)$, such that the equality holds

\[
L(t) = c \cdot a(t) \cdot \exp \left( \int_A^t b(x) d\ln x \right), \quad t \in [A, +\infty).
\]

2. Let $U$ be a measure concentrated on $[0, +\infty)$ and such that its Laplace-Stieltjes Transform

\[
\omega(s) = \int_0^{+\infty} e^{-sx} dU(x)
\]

exists for $\lambda > 0$.

**Tauberian Theorem of Karamata.**

If $L$ varies slowly at infinity and $0 \leq \rho < +\infty$, then each of the relations

\[
\omega(s) \sim s^{-\rho} \cdot L(1/s), \quad s \downarrow 0,
\]

and

\[
U(t) \sim \frac{1}{\Gamma(\rho + 1)} t^{\rho} \cdot L(t), \quad t \to +\infty,
\]

implies the other. Here $\Gamma(x)$ is the *Euler’s Gamma Function*.

This theorem has an interesting history. In a famous paper Hardy and Littlewood treated the case $\omega(s) \sim s^{-\rho}$, $s \downarrow 0$, by difficult calculations. Karamata simplified the proof and introduced regular variation into the formulation of this statement. The most simple proof has been done by Feller in [29].

The following supplement to Tauberian Theorem of Karamata serves as a bridge to a variety of Tauberian Theorems for distribution functions with regularly varying tails (see, XIII, 5, p.423, [29]).

The **Supplement.** Let $0 < \rho < +\infty$, and $U$ has an ultimately monotone derivative $u$. Then $\omega(s) \sim S^{-\rho} \cdot L(1/s), \quad s \downarrow 0$, iff

\[
u(t) \sim \frac{1}{\Gamma(\rho)} t^{\rho-1} L(t), \quad t \to +\infty
\]

(compare to (6.1.7)).

Other topics of interest in Theory of Regularly Varying Functions are Interpolation Theorems which are related with results similar to Karamata Representation Theorem, Criteria on regular variation which may be of different types (based either on definition of regular variation or on Karamata Representation Theorem), Equivalent Regularly Varying Functions’ Construction, etc.
6.1.2 Applications in Probability Theory

The importance of the notion of regular variation for Probability Theory, in particular, for Limit Theorems for sums of independent random variables was pointed out in monograph [68] by B. Gnedenko and A. Kolmogorov. But, its penetration into Probability Theory began only after monograph [29] by W. Feller. We already know from Chapter 3 that for correspondingly centered and normed sums of independent identically distributed random variables with distribution function \( F \) limit distribution exists only if the sum of tails \( 1 - F(x) + F(-x), \ x \in R^+ \), varies regularly at infinity (excluding the case of convergence to Normal Law).

Limit Stable Law, as we have already seen in Section 2.7, have regularly varying sums of tails which exhibits constant slowly varying component.

Above mentioned facts once more confirm the importance of monotone regularly varying functions for Probability Theory.

Everybody knows how widely a Method of Characteristic Functions (or more broad Method of Integral Transforms, in particular, laplace Transform, Generating Function) is used in Probability Theory. Thus, the Tauberian Theorems with regular variation for Integral Transforms find out many applications in Probability Theory. For us in this investigation Tauberian Theorems for Laplace Transform and for Generating Function of distribution function and distribution, respectively, with regularly varying only (right) tails are of interest. That is why let us talk about such theorems.

The Supplement to Tauberian Theorem of Karamata has the following Corollary.

Let \( F \) be a distribution function concentrated on \([0, +\infty)\), \( F \) is strictly increasing, and \( \varphi(s) = \int_0^{+\infty} e^{-sx}dF(x), \ s \in [0, +\infty) \), be its Laplace-Stieltjes Transform. Then, \( 1 - \varphi(s) = \int_0^{+\infty} e^{-sx} \cdot (1 - F(x))dx \).

Consider a measure \( U(t) = \int_0^t (1 - F(x))dx, \ t \in [0, +\infty) \) concentrated on \([0, +\infty)\).

Then, \( U(t) \) has the ultimately decreasing derivative \( u(t) = 1 - F(t), \ t \in [0, +\infty) \), and \( \omega(s) = \int_0^{+\infty} e^{-sx}dU(x) = \int_0^{+\infty} (1 - F(x))e^{-sx}dx = \frac{1 - \varphi(s)}{s}, \ s \in R^+ \).

Corollary 6.1 Let \( 0 < \rho < 1 \). Then each of the relations

\[
1 - \varphi(s) \sim s^\rho \cdot L(s), \ s \to 0, \tag{6.1.9}
\]

and

\[
1 - F(t) \sim \frac{1}{\Gamma(1 - \rho)} t^{-\rho} \cdot L(t), \ t \to +\infty, \tag{6.1.10}
\]

implies the other.

Thus, here we get a simplest Tauberian Theorem for tails of distribution functions.

As a rule, the Tauberian Theorems for tails of distribution functions varying regularly at infinity are proved in literature with the help of limit theorems on convergence of sums of independent identically distributed random variables to Stable Laws [33].
Finally, let us describe in general the role of Tauberian Theorems which are related with Integral Transforms. Given the Integral Transform of distribution function, for instance, the Laplace-Stieltjes Transform, its behavior near the origin uniquely determines the asymptotic behavior of distribution function for large values of argument and vice versa.

### 6.1.3 Interpolation Theorems

Let us explain the main idea concerning Interpolation Theorems on the basis of Karamata Representation Theorem. The last theorem says that

\[ L(t) = L_1(t) \cdot L_2(t), \quad t \in [A, +\infty), \quad (6.1.11) \]

where (without loss of generality we put \( c = 1 \)) \( L_1(t) = a(t) \), \( L_2(t) = \exp(\int_A^t b(x)d\ln x) \) may be treated as slowly varying at infinity functions.

It is well-known that in representation (6.1.11) there is a wide arbitrariness while choosing the functions \( L_1 \) and \( b \) in \( L_2 \). That is why there are many proof of this theorem. It is reasonable to add "good" properties to \( L_1 \) at the expense of "worsening" the properties of \( L_0 \), using the choice of \( L_2 \) (i.e. \( b \) in \( L_1 \)). For instance, applying this theorem to \( L_2 \) we get \( L_2(t) = L_3(t) \cdot L_4(t) \) or \( L(t) = (L_1(t)L_3(t)) \cdot L_4(t) \), where \( L_4(t) \) is now continuously differentiable. So, applying the theorem \( n \) times we get some \( (n-1) \) times differentiable asymptotically as \( t \to +\infty \) equivalent to \( L_1 \) slowly varying function.

In this way in [22] the following Interpolation Theorem was established.

Let the sequence \( \{t_n\} \) satisfy conditions: \( 0 < t_1 < t_2 < \cdots \) and \( \lim_{n \to +\infty} t_n = +\infty \).

**Theorem 6.1** For a slowly varying function \( L(t) \) there are slowly varying functions \( L_1(t) \) and \( L_2(t) \) such that: 1. \( \lim_{t \to +\infty} L_1(t) = 1 \); 2. \( L_2 \) is infinite differentiable; 3. \( L(t) = L_1(t) \cdot L_2(t) \) for \( t \in [t_1, +\infty) \); 4. \( L(t_n) = L_2(t_n) \) for \( n = 1, 2, \cdots \); 5. If \( L \) doesn’t decrease (increase), then \( L_2 \) doesn’t decrease (increase).

**Theorem 6.1** generalizes known Adamovie and Seneta Theorems (see, [69] and [21]) concerning the sequences \( \{n\} \) and \( \{\exp(n)\} \), respectively. Moreover, in these theorems the monotonity of \( L_2 \) is considered only at infinity.

### 6.2 Tauberian Theorem for The Laplace Transform

In this Section by using Corollary 6.1 we suggest a direct proof for a Tauberian Theorem with regular variation in terms of Laplace-Stieltjes Transform for distribution functions concentrated on \([0, +\infty)\). This theorem before has been proved non-directly with the help of Limit Theorems on convergence to Stable Laws [33].
6.2.1 The Tauberian Theorem

Let \( E(x) \) be a distribution function concentrated on \([0, +\infty)\) and

\[
e(s) = \int_{0}^{+\infty} e^{-sx} dE(x), \quad s \in [0, +\infty).
\]

For \( n = 1, 2, \cdots \) and \( s \in R^+ \) let us denote \( e_n = \int_{0}^{+\infty} x^n dE(x) \) (the \( n \)-th moment of integer order of distribution function \( E \) ),\( e^{[n]}(s) = e(s) - \sum_{i=0}^{n} \frac{1}{n!} e_i \) (the last notion is meaningful only if \( e_n < +\infty \) ), and for \( \rho \in (n, n+1) \) let us introduce the following constant \( C_{\rho,n} := \frac{\Gamma(\rho-n)\Gamma(n+1-\rho)}{\Gamma(\rho)\sin(\pi\rho)} \), where \( \Gamma(x) \) denotes the Euler’s Gamma Function \( \Gamma(x) = \int_{0}^{\infty} u^{x-1} e^{-u} du, \quad x \in R^+ \).

Applying the well-known formula \( \Gamma(x) \cdot \Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \), we may rewrite formula for constant \( C_{\rho,n} \) in the form

\[
C_{\rho,n} = (-1)^n \frac{\Gamma(\rho)\sin(\pi\rho)}{\Gamma(1-\rho)\sin(\pi\rho)}.
\]

Now, let us formulate the Tauberian Theorem:

Theorem 6.2 For \( \rho \in (n, n+1) \) with \( n = 0, 1, 2, \cdots \) each of relations

\[
1 - E(x) \sim x^{-\rho} \cdot \frac{1}{L(x)}, \quad x \to +\infty,
\]

and

\[
e^{[n]}(s) \sim (-1)^{n+1} C_{\rho,n} \cdot s^\rho \cdot \frac{1}{L(1/s)}, \quad s \downarrow 0,
\]

implies the other.

We have to notice that the particular case \( n = 0 \) of Theorem 6.2 coincides with Corollary 6.1, where we put \( F = E, \varphi = e \), and instead of \( L \) is taken a slowly varying \( 1/L \).

Introducing regularly varying function

\[
R(x) = x^\rho \cdot L(x), \quad x \in R^+,
\]

we may write (6.2.1) in the form

\[
\lim_{x \to +\infty} R(x) \cdot (1 - E(x)) = 1.
\]

Indeed, due to (6.2.1), (6.2.3), (6.2.4) we have \( 1 - E(x) \sim (1/R(x)) \sim x^{-\rho} \cdot (1/L(x)), \quad x \to +\infty \).

The proof of Theorem 6.2 is divided on three Lemmas.

6.2.2 The Proof

Let for \( x \in [0, +\infty) \), \( K = 1, 2, \cdots, n \), where \( n \) is given in Theorem 6.2, and \( m = 1, 2, \cdots \)

\[
E_1(x) = E(x), \quad E_{K+1}(x) = \frac{1}{e_{K+1}} \int_{0}^{x} (1 - E_K(u)) du, \quad e_{K,m} = \int_{0}^{+\infty} x^m dE_K(x).
\]

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In order to explain the last notions we have to mention that if $F$ is a distribution function on $[0, +\infty)$ with finite mean value $\varphi_1 = \int_0^{+\infty} x dF(x) = \int_0^{+\infty} (1 - F(x))dx < +\infty$ (here integration by parts was made), and with Laplace-Stieltjes Transform

$$
\varphi(s) = \int_{0-}^{+\infty} e^{-sx} dF(x) \quad \text{for } s \in [0, +\infty),
$$

then

$$
\frac{1 - \varphi(s)}{s\varphi_1} = \int_{0-}^{+\infty} e^{-sx} d_x (\frac{1}{\varphi_1} \int_0^x (1 - F(u))du).
$$

Indeed, integration by parts in (6.2.5) leads to $\int_0^{+\infty} e^{-sx} F(x)dx = \frac{\varphi(s)}{s}$ or $\int_0^{+\infty} e^{-sx}(1 - F(x))dx = \frac{1 - \varphi(s)}{s}$, which implies (6.2.6).

The function under the differential at the right-hand-side of (6.2.6)

$$
G(x) = \frac{1}{\varphi_1} \int_{0-}^x (1 - F(u))du, \quad x \in [0, +\infty),
$$

is a distribution function concentrated on $[0, +\infty)$. Indeed, $G(x)$ increases as $x$ increases, is absolutely continuous (even with monotone density $1 - F(x)$), $G(-0) = 0$, $G(+\infty) = 1$.

**Lemma 6.1** If $e_{1,n} < +\infty$, then the constants $e_{K,1}$ are finite,

$$
e_{K,1} = \frac{e_{1,K}}{K! \cdot e_{1,1}},
$$

and $E_{K+1}(x)$ is a distribution function concentrated on $[0, +\infty)$ for $K = 1, 2, \ldots, n$.

**Proof.** For $K = 1, 2, \ldots, n$ and $s \in [0, +\infty)$ we have

$$
e_{K+1}(s) = \int_{0-}^{+\infty} e^{-sx} dE_{K+1}(x) = \frac{1 - e_K(s)}{s \cdot e_{1,1}}.
$$

For $m = 1, 2, \ldots$, due to equalities $e_{2,m} = (-1)^m (\frac{d^m}{ds^m} e_2(s))|_{s=0}$, by using Leibnitz formula on differentiation of functions’ product, we obtain $e_{2,m} = \frac{e_{1,m+1}}{(m+1)e_{1,1}}$, $m = 1, 2, \ldots, n - 1$.

It implies the equivalency of conditions $e_{1,n} < +\infty$ and $e_{2,n-1} < +\infty$.

Since $e_{2,1} < +\infty$ under the condition $e_{1,n} < +\infty$, therefore, $E_2(+\infty) = 1$, i.e. $E_2$ is a distribution function concentrated on $[0, +\infty)$.

Similarly, step by step for $K = 2, 3, \ldots, n$ we establish the equivalency of condition $e_{1,n} < +\infty$ and $e_{K,n-K+1} < +\infty$, and also an equality

$$
e_{K,m} = \frac{e_{K-1,m+1}}{(m+1) \cdot e_{K-1,1}} \quad \text{for } m = 1, 2, \ldots, n - K + 1.
$$

From (6.2.8) we obtain (6.2.7). Since $e_{K,1} < +\infty$ and $E_K$ is a distribution function, therefore, $E_{K+1}$ is a distribution function concentrated on $[0, +\infty)$. **Lemma 6.1** is proved.
Lemma 6.2 For $\rho \in (n, n+1)$ with $n = 0, 1, 2, \cdots$ each of relations

$$e_1^{[n]}(s) \sim (-1)^{n+1} \cdot s^{\rho} \frac{1}{L(1/s)}, \ s \downarrow 0,$$

(6.2.9)

and

$$1 - e_{n+1}(s) \sim \frac{n! e_{1,1}^{[n]} e_{1,n}}{e_{1,n}^{[n-1]}} \cdot s^{\rho-n} \frac{1}{L(1/s)}, \ s \downarrow 0$$

(6.2.10)

implies the other.

Proof. From (6.2.9) it follows $e_2^{[n-1]}(s) \sim (-1)^n \frac{1}{e_{1,1}^{[n]}} s^{\rho-K} \frac{1}{L(1/s)}, \ s \downarrow 0$, and vice versa. Similarly, step by step for $K = 2, 3, \cdots, n$ we established the equivalency of relations

$$e_{K+1}^{[n-K]}(s) \sim (-1)^{n-K+1} \frac{1}{e_{1,1}^{[n-K+1]}} \cdot e_{1,1} \cdots e_{K_1} \cdot s^{\rho-K} \frac{1}{L(1/s)}, \ s \downarrow 0,$$

and (6.2.9). The obtained expressions in case $K = n$ coincides with (6.2.10) because, due to (6.2.8), the following formula holds $e_{1,1} \cdot e_{2,1} \cdots e_{n,1} = \frac{e_{1,n}}{n! e_{1,1}}$. Lemma 6.2 is proved.

Lemma 6.3 For $\rho \in (n, n+1)$ with $n = 0, 1, 2, \cdots$ each of relations (6.2.1) and

$$1 - E_{n+1}(x) \sim x^{n-\rho} \frac{e_{1,n}}{n! e_{1,1}} \cdot \prod_{K=1}^{n} (\rho - K) \frac{1}{L(x)}, \ x \to +\infty,$$

(6.2.11)

implies the other.

The relation (6.2.11) may be rewritten in the form (see, (6.2.3)-(6.2.4))

$$\lim_{x \to +\infty} x^{-n} R(x) \cdot (1 - E_{n+1}(x)) = \frac{e_{1,n}}{n! e_{1,1}} \cdot \prod_{K=1}^{n} (\rho - K).$$

Proof. From VIII, 9, Theorem 1, [23] we conclude that for $\rho \in (n, n+1)$ with $n = 1, 2, \cdots$ the relations $\lim_{x \to +\infty} x^{-K} R(x) \cdot (1 - E_{K+1}(x)) = b_{K+1} > 0$, $K = 0, 1, \cdots, n$, are equivalent under following relationships among constants $b_K$:

$$b_{K+1} = (\rho - K) \cdot e_{K,1} \cdot b_K = \cdots = b_1 \cdot \frac{e_{1,n}}{n! e_{1,1}} \cdot \prod_{K=1}^{n} (\rho - K).$$

Putting $b_1 = 1$ we finish the proof of Lemma 6.3.

Let us prove Theorem 6.2. Let $\rho \in (n, n+1)$ with $n = 1, 2, \cdots$ and (6.2.1) holds. By Lemma 3, we conclude that (6.2.11) takes place, which is equivalent to the following equivalency

$$1 - e_{n+1}(s) \sim \Gamma(n+1-\rho) \cdot \frac{n! e_{1,1}^{[n]} (\prod_{K=1}^{n} (\rho - K))^{-1}}{e_{1,n}^{[n-1]}}, \ s \downarrow 0.$$

Since $\Gamma(n+1-\rho) = C_{p,n} (\rho - 1) (\rho - 2) \cdots (\rho - n)$, therefore from the last expansion we get (6.2.2).

Conversely, if (6.2.2) holds, then with the help of Lemmas 2-3 we obtain (6.2.1).

Theorem 6.2 is proved.
6.2.3 Typical Example

Let us consider the one-parametric family of Right-side Stable Laws with the Laplace-Stieltjes Transform \( \rho_\alpha(s) = \int_0^\infty e^{-sx}dS_\alpha(x) = \exp(-sa), \ s \in [0, +\infty), \ \alpha \in (0, 1) \), which, as we know, are concentrated on \([0, +\infty)\) (see, (2.7.1)). Then, \( 1 - \rho_\alpha(s) \sim s^\alpha, \ s \downarrow 0 \).

Due to Corollary 6.1, or Theorem 6.2 with \( n = 0 \), where \( L(t) = 1 \), we conclude \( 1 - S_\alpha(x) \sim x^{-\alpha} \cdot (1/\Gamma(1-\alpha)) \), which is equivalent to (2.7.2).

6.3 Tauberian Theorem for Generating Function

In this Section a new probabilistic method suggested in Section 6.2 for Tauberian Theorems for distribution functions with regularly varying tails' establishment is developed for the case of Generating Function. The reason consists in following. If we deal with distribution function of non-negative discrete random variable, then it is more convenient to use Generating Function instead of Laplace-Stieltjes Transform.

6.3.1 Analog of Tauberian Theorem of Karamata

Let \( \{q_n\} \) be a sequence of positive numbers. If the power series

\[
Q(z) = \sum_{n \geq 0} q_n \cdot z^n
\]

(6.3.1)

converges for \( z \in [0, z_0) \), where \( z_0 \in R^+ \), then \( Q(z) \) is called the Generating Function of the sequence \( \{a_n\} \). Now, \( U(t) = q_0 + q_1 + \cdots + q_n \) for \( t \in [n, n + 1) \), \( n = 0, 1, 2, \cdots \) is a discrete measure concentrated in \([0, +\infty)\). Instead of its Laplace-Stieltjes Transform

\[
\omega(s) = \int_0^\infty e^{-st}dU(t) = \sum_{n \geq 0} e^{-sn} \cdot q_n, \ s \in [0, +\infty),
\]

it is more convenient to consider the Generating Function (6.3.1).

Let us suppose that the series (6.3.1) converges for \( 0 \leq z < 1 \). The analog of Tauberian Theorem of Karamata in discrete case is formulated as follows (see, Theorem 5, XIII, 5, [23]):

If \( L \) varies slowly at infinity and \( 0 \leq \rho < +\infty \) then each of two relations

\[
Q(z) \sim \frac{1}{(1 - z)^\rho} L\left(\frac{1}{1 - z}\right), \ z \uparrow 1,
\]

(6.3.2)

and

\[
q_0 + q_1 + \cdots + q_n \sim \frac{1}{\Gamma(\rho + 1)} n^\rho \cdot L(n), \ n \to +\infty,
\]

(6.3.3)

implies the other.

For us the following Supplement to this statement is important.

If the sequence \( \{q_n\} \) is monotone and \( 0 < \rho < +\infty \), then (6.3.2) is
equivalent to
\[ q_n \sim \frac{1}{\Gamma(\rho)} n^{\rho-1} L(n), \quad n \to +\infty. \]  \hspace{1cm} (6.3.4)

Note that \( \{q_n\} \) is the analog of density \( u(t) \) (see, Section 6.1).

Let \( \{p_n\} \) be a proper distribution, then its Generating Function
\[ P(z) = \sum_{n \geq 0} p_n \cdot z^n \]  \hspace{1cm} (6.3.5)
exists, at least, for \( z \in [0, 1] \). Indeed, \( 0 \leq P(z) \leq \sum_{n \geq 0} p_n = 1 \) for all \( z \in [0, 1] \). Denote
\[ q_K = p_K + p_{K+1} + \cdots, \quad K = 0, 1, 2, \ldots \quad (q_0 = 1), \]  \hspace{1cm} (6.3.6)
and let (6.3.1) be the Generating Function of the sequence \( \{q_n\} \).

There is a relationship between functions \( P(z) \) and \( Q(z) \)
\[ Q(z) = \frac{1 - zP(z)}{1 - z}, \quad 0 \leq z < 1. \]  \hspace{1cm} (6.3.7)
Indeed, due to (6.3.4)-(6.3.6), for \( 0 \leq z < 1 \) we proceed
\[ Q(z) = \sum_{n \geq 0} \left( \sum_{K \geq n} p_K \right) z^n = \sum_{K \geq 0} p_K \sum_{n=0}^{K} z^n = \sum_{K \geq 0} p_K \cdot \frac{1 - z^{K+1}}{1 - z} = \frac{P(1) - z \cdot P(z)}{1 - z} = \frac{1 - zP(z)}{1 - z}, \]
which proves the equality (6.3.7).

Now, from the Supplement and (6.3.7) we obtain the following

Corollary 6.2 Let \( 0 < \rho < 1 \). Then each of the relations
\[ q_n \sim n^{-\rho} \frac{1}{\Gamma(1-\rho)} L(n), \quad n \to +\infty, \]  \hspace{1cm} (6.3.8)
and
\[ 1 - P(z) \sim (1 - z)^{-\rho} L\left( \frac{1}{1 - z} \right), \quad z \uparrow 1 \]  \hspace{1cm} (6.3.9)
implies the other.

6.3.2 Tauberian Theorem

For proper distribution \( \{p_n\} \) denote by \( M_K = \sum_{n \geq K} n(n-1) \cdot \cdots \cdot (n-K+1)p_K, \ K = 1, 2, \ldots \) the \( K \)-th factorial moment, \( M_0 = 1 \), and for \( 0 \leq z \leq 1 \) consider the Generating Function (6.3.5) and the tail (6.3.6) of distribution \( \{p_n\} \).
Theorem 6.3 For \( \rho \in (n, n + 1) \) with \( n = 0, 1, 2, \cdots \) each of relations

\[
q_n \sim n^{-\rho} \frac{1}{L(n)}, \ n \to +\infty,
\]

and

\[
P(z) = - \sum_{K=0}^{n} \frac{(-1)^K}{K!} (1 - z)^K \cdot M_K \sim (-1)^{n+1} C_{\rho,n} \cdot (1 - z)^\rho \cdot \frac{1}{L(1/z)}, \ z \uparrow 1,
\]

implies the other, where the constant \( C_{\rho,n} \) is defined in Section 6.2.

Note that Corollary 6.2 is included in Theorem 6.3 in case \( n = 0 \).

We are able to prove Theorem 6.3 by two different ways.

Either we formulate and prove analogs of Lemmas 6.1-6.3 is discrete case and obtain Theorem 6.3 based on these analogs; or obtain Theorem 6.3 as a corollary of Theorem 6.2.

For diversity we prefer the second way.

Proof. Let \( E(x) \) and \( \{p_n\} \) be distribution function and distribution, respectively, of the same discrete non-negative random variable. Then \( 1 - E(x) = q_n \) for \( n \leq x < n + 1, \ n = 0, 1, 2, \cdots \), where \( q_n \) is defined by formula (6.3.6). That is why for \( s \in [0, +\infty) \)

\[
1 - e(s) = s \cdot \int_{0}^{+\infty} e^{-sx} \cdot (1 - E(x))dx = \sum_{n \geq 0} s \cdot \int_{n}^{n+1} e^{-sx} \cdot q_n dx = (1 - e^{-s}) \cdot Q(e^{-s}) \quad \text{(6.3.12)}
\]

where \( Q(z), \ 0 \leq z < 1, \) is given by (6.3.1). Now, using (6.3.7), where \( P(z), \ 0 \leq z < 1, \) is a Generating Function of distribution \( \{p_n\} \), from (6.3.12) we have

\[
e(s) = e^{-s} \cdot P(e^{-s}), \ s \in [0, +\infty).
\]

According to (6.3.13), from Theorem 6.2 it follows the equivalency of relations (6.3.10) and

\[
e^{-s} \cdot P(e^{-s}) - \sum_{K=0}^{n} \frac{(-s)^K}{K!} e_K \sim (-1)^{n+1} \cdot C_{\rho,n} \cdot s^\rho \cdot \frac{1}{L(1/s)}, \ s \downarrow 0.
\]

In (6.3.14) let us make a replacement of variable \( z = e^{-s} \) or \( s = -\ln z \). For \( 0 < z < 1 \) applying the expansion

\[
- \ln z = - \ln(1 - (1 - z)) = \sum_{K=1}^{n} \frac{(1 - z)^K}{K} + 0((1 - z)^{n+1}), \ z \uparrow 1, \ n = 1, 2, \cdots,
\]

and, in particular, the limit relation \( \lim_{z \uparrow 1} - \frac{\ln z}{1 - z} = 1 \), first of all, because of the uniform convergence in (6.1.1) in any \( [a, b], \ 0 < a < b < +\infty \), for slowly varying \( L \), we conclude that \( \lim_{z \uparrow 1}(L(\frac{1}{1-z})/L(\frac{1}{z})) = 1 \). Further, due to (6.3.14), using also the last limit relationship we come to the following expansion

\[
z \cdot P(z) - \sum_{K=0}^{n} \frac{(-1)^K}{K!} (1 - z)^K \cdot g_K \sim (-1)^{n+1} \cdot C_{\rho,n} \cdot (1 - z)^\rho \cdot \frac{1}{L(1/z)}, \ z \uparrow 1,
\]

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where $g_K, K = 0, 1, \cdots, n$ are some constants. Finally, taking into account the expansion
\[
\frac{1}{z} = \sum_{K=0}^{n} (1 - z)^K + O((1 - z)^{n+1}), \quad 0 < z < 1,
\]
from (6.3.15) we obtain the following expansion
\[
P(z) - \sum_{K=0}^{n} \frac{(-1)^K}{K!} (1 - z)^K \cdot \nu_K \sim (-1)^{n+1} \cdot C_{p,n} \cdot (1 - z)^\rho \cdot \frac{1}{L(1-z)}, \quad z \uparrow 1,
\]
where $\nu_K, K = 0, 1, \cdots, n$, are some constants. Then, necessarily $\nu_K = M_K, K = 0, 1, \cdots, n$, and the last expansion coincides with (6.3.11). Theorem 6.3 is proved.

6.3.3 The Example

Let us consider three-parametric family of Regular Hypergeometric Distributions $\{p_n = P_n(p_1, p_2, q)\}$: (see, (4.5.5)-(4.5.7), (4.5.10))
\[
p_0 = \frac{\Gamma(q-p_1) \cdot \Gamma(q-p_2)}{\Gamma(\rho-1) \cdot \Gamma(q)} , \tag{6.3.16}
\]
where
\[
\rho = q + 1 - p_1 - p_2 , \tag{6.3.17}
\]
and
\[
p_n = p_0 \cdot \prod_{K=0}^{n-1} \frac{(p_1 + K)(p_2 + K)}{(1 + K)(q + K)} , \quad n = 1, 2, \cdots , \tag{6.3.18}
\]
under constraints
\[
0 < p_1 < +\infty, \quad 0 < p_2 < +\infty, \quad 1 < \rho < +\infty. \tag{6.3.19}
\]

According to (4.5.1) and (4.5.13),
\[
p_n \sim \frac{\Gamma(q-p_1) \Gamma(q-p_2)}{\Gamma(p_1) \Gamma(p_2)} \cdot \frac{1}{\Gamma(\rho-1)} \cdot \frac{1}{n^\rho}, \quad n \to +\infty. \tag{6.3.20}
\]

Formula (6.3.20) means that
\[
q_n = \sum_{m \geq n} p_m \sim \frac{\Gamma(q-p_1) \Gamma(q-p_2)}{\Gamma(p_1) \Gamma(p_2)} \cdot \frac{1}{\Gamma(\rho)} \cdot \frac{1}{n^{\rho-1}}, \quad n \to +\infty, \tag{6.3.21}
\]
where the equality $\Gamma(x + 1) = x \Gamma(x)$ was used.

For distribution $\{p_n\}$ given by formulas (6.3.16)-(6.3.18) under constraints (6.3.19) factorial moments, due to (4.6.10), are equal to
\[
M_K = \prod_{i=0}^{K-1} \frac{(p_1 + i)(p_2 + i)}{\rho - i - 2} \quad \text{if} \quad K = 0, 1, \cdots, [\rho], \quad \prod_{i=0}^{-1} = 1. \tag{6.3.22}
\]
Let $\rho \in (n, n+1)$. Then, applying Theorem 6.3 to distribution $\{p_n\}$ we come to the following expansion for its Generating Function

$$P(z) = \sum_{K=0}^{n} \frac{(-1)^K}{K!} (1-z)^K \cdot \prod_{i=0}^{K-1} \frac{(p_1 + i)(p_2 + i)}{\rho - i - 2} + (-1)^{n+1} \cdot \frac{\Gamma(\rho - n) \cdot \Gamma(n + 1 - \rho) \cdot \Gamma(q - p_1) \Gamma(q - p_2)}{(\Gamma(\rho))^2} \cdot \frac{\Gamma(q - p_1) \Gamma(q - p_2)}{\Gamma(p_1) \Gamma(p_2)} (1-z)^\rho (1 + o(1)), \ z \uparrow 1.$$}

### 6.4 Representation Theorem

In Theory of Regularly Varying Functions there are several Representation Theorems of the type of Karamata Representation Theorem. We may also say that they are of "exponential" type. The application of such theorems leads to Interpolation Theorems, for instance, Adamovie and Seneta Interpolation Theorems. But if the conditions on interpolation becomes more and more complicated, then the application of such theorems is technically embarrassing. In such a situation a representation for slowly varying function being of "linear" type is more preferable.

Below we present one theorem of this type. Let us formulate the result.

**Theorem 6.4** If $L$ varies slowly at infinity, then for any $t_0 \in \mathbb{R}^+$ there are measurable $a(t)$ and continuous $b(t)$ defined on $[t_0, +\infty)$ and satisfying conditions

$$\lim_{t \to +\infty} \frac{a(t)}{L(t)} = 0, \quad \lim_{t \to +\infty} \frac{b(t)}{L(t)} = 0,$$

such that for all $t \geq t_0$

$$L(t) = a(t) + L_1(t), \quad L_1(t) = L(t_0) + \int_{t_0}^{t} b(x) dx. \quad (6.4.2)$$

In reverse, if $a(t) > 0$ and $b(t) > 0$ are defined as above, then $L(t)$ defined by (6.4.2) may be extend on $\mathbb{R}^+$ in such a way that $L$ varies slowly at infinity.

Let us prove the sufficiency of Theorem 6.4, i.e. the reverse statement.

By (6.4.1), for $\varepsilon \in (0, 1)$ there is $t_0 > t_0$ such that for all $t \geq t_0$

$$\frac{a(t)}{L(t)} < \varepsilon, \quad \frac{b(t)}{L(t)} < \varepsilon. \quad (6.4.3)$$

For $\lambda \in (1, +\infty)$, by (6.4.3),

$$\left| \frac{L(\lambda t)}{L(t)} - 1 \right| \leq \left| \frac{a(\lambda t)}{L(\lambda t)} \right| \cdot \left| \frac{L(\lambda t)}{L(t)} - 1 \right| + \left| \frac{a(\lambda t)}{L(\lambda t)} \right| \cdot \left| \frac{a(t)}{L(t)} \right| + \int_{t}^{\lambda t} x^{-1} \cdot \left| \frac{b(x)}{L(x)} \right| \frac{L(x)}{L(t)} dx.$$

Taking into account (6.4.3) and the last inequality, for $t \in [t_0', +\infty)$ we have

$$\left| \frac{L(\lambda t)}{L(t)} - 1 \right| \leq \frac{\varepsilon}{1-\varepsilon} \{2 + (\ln \lambda) \cdot \rho(\lambda(t))\},$$

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where \( \rho_\lambda(t) = \sup_{1 \leq x \leq \lambda} \frac{L(xt)}{L(0)} \). Due to arbitrariness of \( \lambda \), for \( t \in [t_0', +\infty) \) we obtain

\[
0 \leq \rho_\lambda(t) \leq 1 + \frac{\varepsilon}{1 - \varepsilon} \{ 2 + (\ln \lambda) \cdot \rho_\lambda(t) \}.
\]

(6.4.4)

Given \( \lambda \) let for a sequence \( \{t_n\} \), where \( 0 < t_1 < t_2 < \cdots \) and \( \lim_{n \to +\infty} t_n = +\infty \), the limit exists \( \rho_\lambda := \lim_{n \to +\infty} \rho_\lambda(t_n) \). If \( \rho_\lambda = +\infty \), then from (6.4.4) it follows

\[
1 \leq \frac{\varepsilon}{1 - \varepsilon} (\ln \lambda) + (1 + \frac{2\varepsilon}{1 - \varepsilon}) \cdot \lim_{n \to +\infty} \frac{1}{1 - \varepsilon} \rho_\lambda(t_n) = \frac{\varepsilon}{1 - \varepsilon} (\ln \lambda),
\]

which is impossible for \( \varepsilon \in (0, \frac{1}{2(\ln \lambda)}) \). Thus, we conclude that \( \rho_\lambda < +\infty \).

Next, by definition of \( \rho_\lambda \), the case \( 0 \leq \rho_\lambda < 1 \) is impossible.

If \( 1 \leq \rho_\lambda < +\infty \), then, by (6.4.4), \( \rho_\lambda \leq \frac{\varepsilon}{1 - \varepsilon} \{ 2 + (\ln \lambda) \cdot \rho_\lambda \} + 1 \). Tending \( \varepsilon \downarrow 0 \), we obtain \( \rho_\lambda \leq 1 \), which implies \( \rho_\lambda = 1 \). Thus, given \( \lambda \in (1, +\infty) \) the limit exists \( \lim_{t \to +\infty} \frac{L(\lambda t)}{L(t)} = 1 \). The measurability and positivity of \( L \) is obvious.

In order to prove the necessity of Theorem 6.4 we suggest a construction of functions \( a(t) \) and \( b(t) \) with the help of following "generalized" linear interpolation. Let:

a) \( 0 < t_0 < t_1 < \cdots \), \( \lim_{n \to +\infty} t_n = +\infty \); b) \( \sup_n (t_{n+1} - t_n) < +\infty \), \( \inf_n (t_{n+1} - t_n) > 0 \).

Further, let functions \( g_n(t) \), \( n = 0, 1, 2, \cdots \), are continuous on \([0, t_{n+1} - t_n)\) and:

\[
\sup_n \sup_{0 \leq t \leq t_{n+1} - t_n} |g_n(t)| < +\infty,
\]

(6.4.5)

\[
\Theta_{n+1} \cdot g_{n+1}(0) = \theta_n \cdot g_n(t_{n+1} - t), \quad n = 0, 1, 2, \cdots,
\]

\[
\hat{g}_n(0) = 0, \quad \hat{g}_n(t_{n+1} - t_n) = 1,
\]

where \( \Theta_n = L(t_{n+1}) - L(t_n), \ n = 0, 1, 2, \cdots, \ \hat{g}_n(u) = \int_0^u g_n(x) dx, \ u \in [0, t_{n+1} - t_n] \).

For instance, the functions \( g_n(t) = \frac{6}{(t_{n+1} - t_n)^2} \cdot (1 - \frac{t}{t_{n+1} - t_n}), \ t \in [0, t_{n+1} - t_n], \ n = 0, 1, 2, \cdots \) satisfy the above introduced conditions.

Finally, we define \( b(t), t \in [t_0, +\infty) \), by equality

\[
b(t) = g_n(t - t_n) \cdot \Theta_n, \quad t \in [t_n, t_{n+1}), \ n = 0, 1, 2, \cdots,
\]

(6.4.6)

and \( a(t), t \in [t_0, +\infty) \), by equalities

\[
a(t) = L(t) - L_1(t)
\]

(6.4.7)

\[
L_1(t) = L(t_{n+1}) + \hat{g}_n(t - t_n) \cdot \Theta_n, \quad t \in [t_n, t_{n+1}) \quad n = 0, 1, \cdots.
\]

Easily verify that in equalities (6.4.7) \( t = t_{n+1} \) is possible.

Thus, the function \( L_1(t) \) defined by (6.4.7) is continuous.

Let us show that \( L_1(t) \sim L(t), \ t \to +\infty \). The uniform boundedness of \( \{g_n\} \) (see, (6.4.5)) implies the uniform boundedness of \( \{\hat{g}_n(t)\} \). Now, for \( a(t) = L(t) - L_1(t) \) (see (6.4.7)) as \( t \to +\infty \), by Theorem on Uniform Convergence and condition b),

\[
a(t) \frac{L(t)}{L(t)} = 1 - \frac{L(t_n)}{L(t)} = \frac{L(t_{n+1})}{L(t)} - \frac{L(t)}{L(t)} \to 0,
\]

i.e. the first limit relation in (6.4.1) is true.

Next, from (6.4.7) we have \( L_1(t_n) = L_1(t_0) + \sum_{K=0}^{n-1} \int_{t_K}^{t_{K+1}} b(x) dx \). Substituting this equality into (6.4.7) we get (6.4.2). Theorem 6.4 is proved.

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6.5 Interpolation Theorem

Let sequence \( \{t_n\} \) satisfy conditions
\[
0 < t_0 < t_1 < \cdots, \quad \lim_{n \to +\infty} t_n = +\infty, \quad \inf(t_{n+1} - t_n) > 0,
\]
and \( B(t), \ B(+0) = 0, \) be a distribution function with regularly varying tail \( B(x) = 1 - B(x), \ x \in R^+ \). In this Section we prove the following interpolation

**Theorem 6.5** There is a distribution function \( B_1(t), \ B_1(+0) = 0, \) such that: 1. \( B(t) \sim B_1(t)(= 1 - B_1(t)), \ t \to +\infty; \) 2. \( B(t_n) = B_1(t_n), \ n = 0, 1, 2, \cdots; \) 3. \( B_1(t) \) is infinite differentiable; 4. If \( B(t) \) is increasing and upward convex, then \( B_1(t) \) is increasing and upward convex.

**Theorem 6.5** shall be proved with the help of **Theorem 6.4**. By our opinion, it is difficult technically to establish **Theorem 6.5** using Representation Theorems of ”exponential” type.

6.5.1 Auxiliary Lemma

Let \( f(t) \) be a distribution function \( f(+0) = 0, \ f(1) = 1, \delta(f) := \int_0^1 f(t)dt. \)

**Lemma 6.4** For any \( \delta \in (0, 1) \) there is an infinite differentiable, increasing on \([0, 1]\) distribution function such that \( f^{(K)}(0) = f^{(K)}(1) = 0 \) for \( K = 1, 2, \cdots \), where \( f^{(K)} \) is the \( K \)-th derivative of \( f \), and \( \delta(f) = \delta. \)

**Proof.** For \( \tau \in R^+ \) a distribution function
\[
\varphi_\tau(t) = \exp(-\frac{\tau}{t} \exp(-\frac{1}{1-t}))
\]
defined on \([0, 1]\) increases by \( t \) because \( \varphi'_\tau(t) = \varphi_\tau(t) \cdot \frac{\tau}{t} \left( \frac{1}{t} + \frac{1}{(1-t)^2} \right) \exp(-\frac{1}{1-t}) > 0 \) for \( t \in (0, 1) \).

For \( \varepsilon \in (0, 1) \) there is \( \tau \in R^+ \) such that \( \int_0^1 \varphi_\tau(t)dt > 1 - \varepsilon \) \( (\int_0^1 \varphi_\tau(t)dt < \varepsilon) \), because \( \varphi_\tau \to 1 \) as \( \tau \to 0 \) \( (\varphi_\tau \to 0 \) as \( \tau \to +\infty) \) uniformly by \( t \in [\frac{\varepsilon}{1+\varepsilon}, 1] \) (by \( t \in [0, 1 - \frac{\varepsilon}{2}] \)).

Therefore, there is \( \tau_0 \in R^+ \) such that \( \varphi_\tau(t) > 1 - \varepsilon^2 \) for \( 0 < \tau < \tau_0 \) and \( t \in [\frac{\varepsilon}{1+\varepsilon}, 1] \)
\[
(\varphi_\tau(t) < \frac{1}{1-\varepsilon} \quad \text{for} \quad \tau > \tau_0 \quad \text{and} \quad t \in [0, 1 - \frac{\varepsilon}{2}], \quad \int_{1-\frac{\varepsilon}{2}}^{1-\varepsilon} \varphi_\tau(t)dt < \frac{\varepsilon}{2} \quad \text{uniformly} \quad \tau \in R^+).
\]

Thus, there are functions \( \varphi \) and \( \varphi_\tau \) of type (6.5.2) such that \( \bar{\Lambda} := \delta(\bar{\Lambda}) \geq \delta \geq \delta(\varphi) := \Lambda. \)

Then, for some \( p \in (0, 1) \) the equality holds \( \delta = p \cdot \Lambda + (1 - p)\bar{\Lambda}, \) and
\[
f(t) = p \cdot \varphi(t) + (1 - p)\overline{\varphi}(t), \quad t \in [0, 1].
\]

**Lemma 6.4** is proved.
6.5.2 "Generalized" Linear Interpolation

Let us consider interpolation (6.4.5)-(6.4.7). Namely, let for \( \{g_n\} \) conditions (6.4.5) be fulfilled, where \( \Theta_n = B(t_{n+1}) - B(t_n) \), \( n = 0, 1, 2, \cdots \), and \( \{t_n\} \) satisfies conditions (6.5.1) with \( t_0 = 0 \). For \( t \in [t_n, t_{n+1}] \) and \( n = 0, 1, \cdots \) put

\[
B_1(t) = B(t_n) + \hat{g}_n(t - t_n) \cdot \Theta_n, \tag{6.5.3}
\]

where \( \hat{g}_n(u) = \int_0^u g_n(x)dx \), \( u \in [0, t_{n+1} - t_n] \). We require additional conditions on \( \{\hat{g}_n\} \):

\[
\hat{g}_n, n = 0, 1, \cdots, \text{ is infinite differentiable on } [0, t_{n+1} - t_n]; \tag{6.5.4}
\]

\[
\hat{g}_n^{(K)}(0) = \hat{g}_n^{(K)}(t_{n+1} - t_n) \text{ for } K = 2, 3, \cdots, \tag{6.5.5}
\]

where \( \hat{g}_n^{(K)} \) denotes the \( K \)-th derivative of \( \hat{g}_n \).

Next, let \( \{f_n\} \) satisfy Lemma 6.4 with \( \delta(f_n) = 1 - \delta_n \), \( n = 0, 1, 2, \cdots \), \( \delta_n \in (0, 1) \).

For \( n = 0, 1, 2, \cdots \) we build a family of functions

\[
\{\hat{g}_n(t) = \hat{g}_n(t, u_n, v_n, \delta_n) : u_n \in \mathbb{R}^+, v_n \in \mathbb{R}^+, 0 < \delta_n < 1\}
\]

of the type

\[
\hat{g}_n(t) = \frac{u_n}{t_{n+1} - t_n} \cdot \int_0^t (1 - f_n(\frac{x}{t_{n+1} - t_n}))dx + \frac{v_n \cdot t}{t_{n+1} - t_n}, t \in [0, t_{n+1} - t_n]. \tag{6.5.6}
\]

The function \( \hat{g}_n, n = 0, 1, 2, \cdots, \) is non-negative, increases and is upward convex on \([0, t_{n+1} - t_n]\). The sequence \( \{\hat{g}_n\} \) satisfies conditions (6.5.4) and (6.5.5).

This choice of \( \{\hat{g}_n\} \) proves Theorem 6.5 if the following system of equation has a solution:

\[
\begin{cases}
  u_n \delta_n + v_n = 1, n = 0, 1, 2, \cdots, \quad u_{n+1} + v_{n+1} = \lambda_n \cdot v_n, \quad 1 < \lambda_n < +\infty, \\
  0 < u_n < +\infty, \quad 0 < v_n < +\infty, \quad 0 < \delta_n < 1.
\end{cases} \tag{6.5.7}
\]

Here \( \lambda_n = \frac{\Theta_n}{\Theta_{n+1}} \cdot \frac{t_{n+1} - t_n}{t_{n+2} - t_{n+1}} \) for \( n = 0, 1, 2, \cdots \). The system of equations (6.5.7) is nothing else but conditions (6.4.5) in our case.

In order to prove Theorem 6.5 it is enough to figure out one sequence \( \{\delta_n\} \subset (0, 1) \), for which under apriori given sequence \( \{\lambda_n\} \subset (1, +\infty) \) the system (6.5.7) has a solution.

6.5.3 Proof of Theorem 6.5

Excluding \( u_n, n = 0, 1, 2, \cdots, \) from (6.5.7) we obtain

\[
v_{n+1} = \alpha_n - \beta_n \cdot v_n, \quad n = 0, 1, 2, \cdots, \tag{6.5.8}
\]

where \( \alpha_n = \frac{1}{1 - \delta_{n+1}}, \quad \beta_n = \frac{\delta_{n+1}}{1 - \delta_{n+1}} \cdot \lambda_n \). That is why for \( n = 0, 1, 2, \cdots \) we come to equalities

\[
v_{n+1} = \alpha_n - \alpha_{n-1} \beta_n + \cdots + (-1)^n \cdot \alpha_0 \cdot \beta_1 \cdots \beta_n + (-1)^{n+1} \beta_0 \cdots \beta_n \cdot v_0. \tag{6.5.9}
\]
The problem reduces to a choice of \( \{ \delta_n \} \subset (0, 1) \) and \( v_0 \in (0, 1) \) such that (6.5.9) must imply that \( 0 < v_{n+1} < 1 \) for all \( n = 0, 1, 2, \cdots \). For the sequence

\[
y_n = \frac{\alpha_0}{\beta_0} - \frac{\alpha_1}{\beta_0 \beta_1} + \cdots + (-1)^n \frac{\alpha_n}{\beta_0 \cdots \beta_n} + (-1)^{n+1} \frac{1}{\beta_0 \cdots \beta_n},
\]

(6.5.10)

it is easy to verify that

\[
y_{n+1} - y_n = (-1)^n \cdot \frac{1}{\beta_0 \cdots \beta_n} \frac{\lambda_{n+1} - 1}{\lambda_n - 1} \begin{cases} > 0 & \text{if } n = 2K, \\ < 0 & \text{if } n = 2K - 1, \end{cases}
\]

(6.5.11)

Consider the following two cases:

a) \( \frac{\lambda_n - 1}{\lambda_{n-1} \lambda_n - 1} < \delta < 1, \ n = 1, 2, \cdots, \) and b) \( 0 < \delta_n \leq \frac{\lambda_n - 1}{\lambda_{n-1} \lambda_n - 1}, \ n = 1, 2, \cdots. \)

In both cases sequences \( \{ y_{2n} \} \) and \( \{ y_{2n+1} \} \), due to (6.5.11), are monotone.

Namely, in case a) \( \{ y_{2n} \} \) increases, \( \{ y_{2n+1} \} \) decreases; in case b) \( \{ y_{2n} \} \) is non-increasing, \( \{ y_{2n+1} \} \) is non-decreasing. At the same time, by (6.5.11), \( y_{2n+1} > y_{2n} \) for \( n = 0, 1, 2, \cdots. \)

Thus, the limits exist

\[
\lim_{n \to +\infty} y_{2n+K}, \ K = 0, 1, \ a := \lim_{n \to +\infty} (y_{2n} - y_{2n+1}) \geq 0, \ a + \lim_{n \to +\infty} y_{2n+1} = \lim_{n \to +\infty} y_{2n}.
\]

It means that there is \( v_0 \) satisfying conditions

\[
y_{2n+1} < v_0 < y_{2n} \text{ for } n = 1, 2, \cdots, \text{ and } v_0 < y_0.
\]

(6.5.12)

Note that choosing \( v_0 \) from conditions (6.5.12) we have \( v_{n+1} < 1 \) for \( n = 0, 1, 2, \cdots \) in equalities (6.5.9) because from (6.5.10) and (6.5.12) it follows

\[
v_{n+1} < \alpha_n - \alpha_{n+1} \beta_n + \cdots + (-1)^n \cdot \alpha_0 \beta_1 \cdots \beta_0 + (-1)^{n+1} \cdot \beta_0 \cdots \beta_n \cdot y_n = 1.
\]

Any choice of \( v_0 \) from (6.5.12) leads to inequalities \( v_{n+1} > 0 \) for \( n = 0, 1, 2, \cdots \). Indeed, let us assume the opposite, i.e. let \( v_{n_0} \leq 0 \) for some integer \( n_0 > 1 \). Then, (6.5.8) implies \( v_{n_0+1} \geq \alpha_{n_0} > 1 \), which contradicts inequality \( v_{n_0+1} < 1 \). Thus, choosing \( v_0 \) from conditions (6.5.12) we have \( 0 < v_{n+1} < 1 \) for \( n = 0, 1, 2, \cdots \).

In particular, as above, we establish that \( v_0 > 0 \). At the same time,

\[
v_0 < y_0 = \frac{1}{\beta_0} (\alpha_0 - 1) = \frac{1}{\lambda_0} < 1.
\]

Theorem 6.5 is proved.
6.6 Interpolation Problem Arising in Dediscretization Procedure

In Section 5.2 an Interpolation Problem was formulated which arises in the way of dediscretization procedure realization for a class of frequency distributions. This problem is solved in the present Section with the help of Interpolation Theorem - Theorem 6.5.

6.6.1 Reformulation of Theorem 6.5

Theorem 6.5 may be interpreted as Interpolation Theorem for the tail $B(x)$ of distribution function $B$: $B(x) = 1 - B(x) = x^{-\rho} \cdot L(x)$, $x \in \mathbb{R}^+$, $\rho \in [0, +\infty)$. Now, if $\rho = 0$, then $\lim_{x \to +\infty} L(x) = 0$, and the Theorem is formulated for slowly varying function $L$.

Next, considering instead of $L$ a slowly varying function $1/L$ we may reformulate Theorem 6.5 for the case of slowly varying $L_0(t)$ with

$$\lim_{t \to +\infty} L_0(t) = +\infty. \tag{6.6.1}$$

Namely, let $L_0(t)$ be a slowly varying function satisfying condition (6.6.1).

Corollary 6.3 There is a slowly varying function $\hat{L}(t)$ such that: 1. $\hat{L}(t) \sim L_0(t)$, $t \to +\infty$; 2. $\hat{L}(n) = L_0(n)$, $n = 0, 1, 2, \ldots$; 3. $\hat{L}(t)$ is infinite differentiable; 4. If $L_0(t)$ is increasing and upward convex then $\hat{L}(t)$ is increasing and upward convex.

6.6.2 Linear Continuous Analog

If the form of given slowly varying sequence $\{L(n)\}$ in representation of frequency distribution $\{p_n\}$, which is subjected to dediscretization procedure, is not given by elementary formula, then, first of all, we build its linear continuous analog as follows. For the increasing, upward convex sequence $\{L(n)\}$ we draw a "broken" line passing through points $(0, L(0))$, $(1, L(1))$, $(2, L(2))$, $\cdots$, $(n, L(n))$, $\cdots$ on the plane. The "broken" line, say $L_0(t)$ defined on $[0, +\infty)$, satisfies conditions $L_0(n) = L(n)$, $n = 0, 1, 2, \ldots$.

We say that: $L_0(t)$ is a linear continuous analog of the sequence $\{L(n)\}$.

For any $t \in \mathbb{R}^+$ and $n = 1, 2, \cdots$ we have

$$\frac{L(nK)}{L(K + 1)} = \frac{L_0(n \cdot [t])}{L_0([t] + 1)} \leq \frac{L_0(n \cdot [t] + 1)}{L_0(t)} \leq \frac{L_0(nK + n)}{L_0(t)} = \frac{L(nK)}{L(K)} = \frac{L(nK')}{L(K' + 1)} = K = K' - 1 = [t],$$

where $[t]$ denotes the integer (entire) part of the positive number $t$.

Taking into account that $\lim_{K \to +\infty} \frac{L(Kn)}{L(K)} = 1$ for $n = 2, 3, \cdots$ and, as a result of this $\lim_{K \to +\infty} \frac{L(K + 1)}{L(K)} = 1$, we conclude that

$$\lim_{t \to +\infty} \frac{L_0(nt)}{L_0(t)} = 1 \tag{6.6.2}$$

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for $x = n$ with $n = 1, 2, \ldots$. Putting $t = (t'/m)$ with $m = 1, 2, \cdots$ we get (6.6.2) for $x = m^{-1}$ with $m = 1, 2, \cdots$. Combining these two cases we conclude that (6.6.2) holds for $x = \frac{m}{n}$ with $m = 1, 2, \cdots$ and $n = 1, 2, \cdots$, i.e. for all positive rational numbers $x$.

The set of such numbers is everywhere dense in $R^+$. Therefore, by Lemma 1, p.275 [23], we convinst that the continuous analog $L_0(t)$ of $\{L(n)\}$ varies slowly at infinity.

By generalization of Adamovic Interpolation Theorem on slowly varying function with condition $\lim_{t \to +\infty} L_0(t) = +\infty$, namely, Corollary 6.3, there is a constructive method of building of an increasing, convex, infinite differentiable, slowly varying function $\hat{L}(t)$ satisfying restrictions: $\hat{L}(n) = L_0(n), \ n = 1, 2, \cdots$ and $\lim_{t \to +\infty} \frac{\hat{L}(t)}{L_0(t)} = 1$. For this function, as we see, conditions 1.-5. from Section 5.2 are fulfilled. Thus, the interpolation problem being formulated in Section 5.2 is completely solved, and as a result of this, the dediscretization procedure of frequency distributions of types (5.1.6) and (5.1.9) is completely substantiated.
Bibliography


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