SPECTRAL ANALYSIS OF POLYHARMONIC SIGNALS

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ABSTRACT

In this work a method of high precision calculation harmonics magnitudes of polyharmonic signal (errors are less than 0.2%) is analyzed. It is supposed that base frequency of signal is not known and it is measured with finite accuracy.

1. INTRODUCTION

There are many problems of digital processing which require calculating spectrum of signal. Usually typical problem of spectral analyzing is solved by applying algorithm based on discrete Fourier transform (DFT). Structure of standard spectral analyzer is shown on figure 1.

![Figure 1. Structure of standard spectral analyzer](image)

Generally, dismissing method of technical implementation of processing algorithm, the errors of calculating magnitudes are typically 1-10%. They depend on ratio between signal frequency and sampling frequency. To decrease error value, methods to reach an integer repetition factor between sampling and signal frequencies are applied. It provides a minimization of spectral erosion [1, 2] (the effect is known as leakage). The problem of spectral erosion is shown on figure 2 (sampling frequency is 1024 Hz, DFT length is N=512, signal consists of first three harmonics with relative magnitudes: 1, 0.1 and 0.4). Spectral erosion appears by reason that repetition factor is not integer. It means that frequency signal and DFT resolution on frequency are not multiples. The energy of such harmonics is distributed in spectrum additively. In fact, DFT resolution in that example is 2 Hz and, therefore frequency 50 Hz has integer repetition factor and frequency 50,4 Hz does not have.

In this work it is suggested to consider a problem of calculating spectrum of polyharmonic signals with accuracy value below 0.2%. These signals are described by expression:

\[ x(n\Delta t) = \sum_{k=1}^{K} A_k \sin(2\pi f_b n\Delta t + \phi_k) + \eta(n\Delta t) \]

where \( \Delta t \) – sampling rate of signal; \( f_b \) – frequency of base (main) harmonic; \( A_k, \phi_k \) – magnitudes and phases of \( k \)-th harmonic; \( \eta(n\Delta t) \) – noise reduced to ADC output (it includes errors of quantization, noise of analogue signal processing (including from sensors)); \( K \) – number of harmonics.

![Figure 2. Example of spectral erosion](image)
Therefore calculation of a frequency difference ∆ω = 2π f_s / N; N – length of signal selection (DFT length).

Before DFT calculation with finite length (N) it is supposed that a signal is multiplied by a function with rectangular window in time domain. It is well known [3] that Fourier-image of rectangular window is given as:

\[ F(f) = T e^{-j2\pi f T(N-1)} \frac{\sin(\pi f T/N)}{\sin(\pi f/T)} \]  

(1)

where \( T = N/f_s \) – interval of analyzing signal.

To lower the effect of spectral erosion repetition factor should be integer:

\[ f_d = \frac{N f_b}{w} \]  

(2)

where \( f_d \) – sampling frequency of signal; \( N \) – DFT length; \( f_b \) – frequency of base harmonic; \( w \) – any positive integer number.

Therefore, if expression (2) is correct, signal frequency is a multiple of DFT resolution on frequency. On another side, substituting condition (2) in expression (1) and passing to limits give that

\[ F(f) = \frac{N^2}{f_s} \]  

By that reason, rectangular window under that condition does not distort signal spectrum on all harmonics. This is important from precise magnitudes measurement point of view.

Thus, the main idea to lower effect of spectral erosion is the changing sampling frequency (signal re-sampling). This is reached by signal approximation in time domain. Let’s take the number \( w \) as integer so the following expression should be a rule:

\[ f_d = \frac{N f_b}{w} > f_s \]

Further, special algorithm for converting of input sequence (digital signal) with sampling frequency \( f_b \) to a sequence with sampling frequency \( f_d \) is used. Then while accuracy of measuring base frequency is high, the effect of spectral erosion should be eliminated.

From theoretical point of view, precise signal re-sampling with a bandwidth satisfying sampling theorem may be done by applying following expression:

\[ x(n\Delta t_1) = \sum_{k=-\infty}^{\infty} x(k\Delta t) \frac{\sin(n \Delta t_1 - k)}{n \Delta t_1 - k} \]  

(3)

where \( \Delta t \) – sampling rate; \( \Delta t_1 \) – new sampling rate.

However, using (3) in practice is impossible by reason of finite length of signal selection. Therefore, the problem of choice of the least sampling frequency \( f_d \) and of minimizing interpolation order is actual. Moreover, this choice should satisfy to required error value. The interpolation order defines errors values. However, the research has given that interpolation polynomial should not be the best in the sense of signal approximation at interpolation nodes. Let’s assume, in result of interpolation the shape of a signal is distorted considerable. But DFT of such signal has correct spectrum on the actual harmonics and distortion of signal shape is explained by the presence of other spectral components. Then such distortion is allowed and it is permitted to apply interpolation with low order. This is important for signal processing in real time. But this method does not allow calculating errors values analytically. Then numerical calculation of errors values for each harmonic is required.

Let’s denote the value \( R = \frac{f_d}{f_s} = \frac{\Delta t_d}{\Delta t_s} \). Since operations of filtering, interpolation and DFT are linear, the input signal (for simplification of calculation) is presented by one harmonic with frequency \( \omega_p \). Then DFT of length \( N \) will be found as:

\[ X(k\Delta \omega_d) = \sum_{m=-M}^{M} a_m(n) e^{-j(2\pi i k n / N)} \]  

(4)

where \( u = \frac{i \Delta t_d - n \Delta t_s}{\Delta t_s} \) ; \( n = \lceil i R \rceil \) – index of digital sequence with sampling frequency \( f_d \) (operator \( \lceil i R \rceil \) means rounding to zero); \( i \) – index of digital sequence with sampling frequency \( f_d \).

There is the same initial signal in square brackets of (4) but it has another sampling frequency. A nonlinear rule realizes connection between indices. It provides to satisfy the condition of interpolation: the point \( x(i\Delta t_d) \) must be between two nearby interpolation nodes: \( n\Delta t_s \leq i\Delta t_d \leq (n+1)\Delta t_s \). Therefore calculation of a point \( x(i\Delta t_d) \) requires to find integer \( n \), which satisfies condition of resampling \( 0 \leq i R - n < 1 \). It means the nonlinear connection of indices will influence on the spectrum. But operation relatively the signal is linear. Really, permutation of input sequence with nonlinear rule does not change linear property of mathematical operators (interpolation).

To connect different timing frames one creates abstract signal \( y(t) \) with sampling frequency \( 1/\Delta t \). It includes all points of initial signal \( x(n) \) (sampling frequency \( f_d \)) and all points of interpolated signal \( x_d(i) \) (sampling frequency \( f_d \)). This is possible if sampling frequency \( (1/\Delta t) \) of signal \( y \) is the least common multiple of values \( f_d \) and \( f_s \). Then point \( x(n\Delta t_1) \) in signal \( y \) is distant from point \( x(i\Delta t_d) \) on integer number of points \( d\Delta t = i\Delta t_d - n\Delta t_s \) (\( \Delta t_s = d_s \Delta t \) and \( \Delta t_d = d_d \Delta t \) are multiples of \( \Delta t \)). Moreover, spectral composition of these three signals is identical on actual harmonics. Take-
ing into account independence of order summing in (4),
DFT interpolated signal is given as:

\[ X(k\Delta\omega_d) = \sum_{m=-M}^{M} \sum_{i=0}^{d-1} a_m(iR - [iR]) e^{-\frac{2\pi m}{N}} \]

(5)

\[ x[iR + m\Delta t_s] e^{-\frac{2\pi m}{N}} \]

Direct calculation of (5) is not easy because it contains nonlinear operation to calculate indices in time
domain. Taking into account independence of order summing on \( i \), a replacement of index is carried out:
\( i = i_1 d_s + i_2 \). Moreover, increasing DFT length by \( d_s \) times gives at ones increasing DFT resolution:
\( \Delta\omega' = \frac{\Delta\omega}{d_s} \). At the same time spectrum composition
does not change, but actual harmonics run away on \( d_s \)
points: \( k_1 = d_s k \). Next, using scaling and shifting theorems for any \( k \in [0; Nd_s - 1] \) results to:

\[ X\left( \frac{k}{d_s} \Delta\omega_d \right) = \frac{1}{d_s} \sum_{i=0}^{d_s-1} a_m(iR - [iR]) \sum_{m=-M}^{M} \left[ a_m(i_1 R - i_2 R) e^{-\frac{2\pi m}{Nd_s}} \right] \]

(6)

In square brackets of expression (6) the operation
\( (i_2 R - [i_2 R]) \) is a permutation, where \( i_2 \in [0; d_s - 1] \).
Therefore, sum in square bracket (6) may be substituted by a different sum:

\[ a_m(i_2 R - [i_2 R]) e^{-\frac{2\pi m}{Nd_s}} \]

\[ + \sum_{i_2=1}^{d_s-1} a_m(i_2 R - [i_2 R]) = a_m(0) e^{-\frac{2\pi m}{Nd_s}} + \sum_{i_2=1}^{d_s-1} a_m(i_2 R - [i_2 R]) e^{-\frac{2\pi m}{Nd_s}} \]

Next transformations are possible only if interpolation order is chosen. It is clear now, that spectrum of
interpolated signal is a weighted total of spectrum of initial signal. Therefore, in such spectrum for every non
zero harmonic exists no more than \( d_s \) additional spectral components, which magnitudes are lower as interpolation
order is increasing.

3. EXAMPLE

As an example, connection between indices of signal before and after linear interpolation was considered.
Coefficients of linear interpolator are
\( a_{-1}(u) = -\frac{u}{2} \); \( a_0(u) = 1 \); \( a_1(u) = \frac{u}{2} \), where
\( u = (i_2 R - [i_2 R]) \). Then spectrum of interpolated signal is described as:

\[ X(k\Delta\omega_d) = \frac{1}{d_s} \sum_{m=-M}^{M} \left[ a_m(iR - [iR]) \right] e^{-\frac{2\pi m}{Nd_s}} \left[ \frac{n_k}{Nd_s} \right] \]

(7)

The expression (7) defines magnitudes and frequencies of \( d_s \) additional spectral components for every non zero harmonic of initial signal. Figure 3 shows the
result of calculating spectrum of sine signal with frequency 1 kHz and sampling frequency 16 kHz after linear
interpolation.

![Figure 3. Spectrum of sine signal after linear interpolation (frequency range 1002-8000)](image)

Note that the difference between contiguous items of spectral component equals difference of sampling frequency before and after interpolation. This may
be not evident from (7). But it looks better if sine signal is represented by sum of exponents and it is substituted
in (7) and then the functional groups \( \frac{\sin(xd_s)}{d_s} \sin(x) \) are analyzed.

Results of mathematical modeling confirm findings by theoretical results. For example, the result of interpolated signal spectrum calculation (frequency 1 kHz, sampling frequency 16 kHz) is shown on figure 4 (4-th order interpolation of Stirling). The new sampling frequency (after resampling) is 16384 Hz and shift of frequency is
\( \frac{d_s}{d_s} f_d = 384 \) Hz (that agrees with ones
of linear interpolation). By figure 4 it is obvious that additional spectral components appear at frequencies $1000 \pm 384v$ ($v$ – integer). Also spectrum includes reflection of ones by symmetry of real signal spectrum. Note that the ratios of magnitudes of additional spectral components to magnitude of «parent» harmonic are greater than 90 dB (it is defined by selection of sampling frequency and interpolation order).

Generally, the spectrum of signal resampling may be corrected by known frequencies of additional spectral components. But in practice it is a difficult problem because frequencies of additional spectral components depend on base frequency measurement result, which always contains errors. It is evident that error of correction will be higher as frequency of harmonic is greater. Therefore, if frequencies of actual and unwanted harmonic are equal then one should suppose that result includes random additional error. Otherwise, selective filtering of spectrum is possible.

4. ABOUT ERRORS

The choice of interpolation order is defined by the value of sampling frequency and by required accuracy of signal processing. It is evident that the increasing interpolation order permits processing faster signals (with higher frequency spectral components). Therefore, a signal with all spectral components having zero phases provides a worst case of signal processing. It creates faster changes analyzed signal. For example, such signals may have rectangular or triangle shape.

Discussing about analytical estimation of interpolation error, one may use formula from [4] for case of interpolation with 4-th order and sine signal (frequency is $f_b$ and magnitude is 1) with sampling frequency 16 kHz:

$$e(n) < \left| u \right| \left( u^2 - 1 \right) \ldots \left( u^2 - 16 \right) \left[ \frac{10^{-3}}{9!} \frac{16}{16} \right]^9 \max \left| \sin(\theta) \left( 2\pi f_h \right) \right|$$

Assuming $u=0.5$ as the worst case of interpolation on the middle of uncertain interval and ignoring square of value $u$, it is obtained $e(n)<2.5 \cdot 10^{-7}$% with $f_h=1.6$ kHz.

To solve problem of spectrum calculation of polyharmonic signal it is required after ADC to apply resampling by interpolation. It minimizes the influence of spectral erosion. The spectrum extends after resampling. Then the signal may be filtered. Therefore, the structure of algorithm (shown in figure 5) includes corresponding blocks.

![Figure 5. Algorithm structure of signal processing](image)

The block «restoring magnitudes of harmonics» is needed for spectrum correction because errors of frequency measurement exist and effect of spectral erosion always will be present. The main idea of this block is based on Parseval theorem. It is evident that influence of spectral erosion should be low, as measuring of signal frequency is more accurate. Eliminating spectral erosion is a necessary condition to improve the work of this block.

Getting analytical expressions for dependence between magnitudes calculation errors and frequency measurement error is a problem because of expressions are very complex. Moreover, results from [5] show that determined methods of errors estimation for complexity algorithm give too high values of errors and then it is failure in practice. To determine maximum errors, numerical methods with input signals realizing worst case are used. This brings to light needed dependence. In figure 6 results of calculation (frequency of base harmonic is 50 Hz, signal consists of 40 harmonics for example) are given.

![Figure 6. Errors of measuring signal magnitudes](image)
According to figure 6 the maximum error for signal with frequency 1600 Hz is less than 0.22% while absolute error of frequency measuring of signal is less than 1.6 Hz and errors are less than 10^-3% if base frequency is known as well (it conforms to theoretical value).

5. INFLUENCE OF NOISE

An actual question about influence of differentiable Gaussian noise and of rounding errors on results of spectrum calculation was considered.

It is assumed that digital signal on ADC output includes noises of three kinds: signal noise, quantization noise and jitter noise (characterizes a probability of getting sample of signal in interval of clock front). Since those kinds of noise are not correlated then it is assumed that digital signal on ADC output is influenced by wide bandwidth Gaussian noise. The signal after ADC may be filtered by low pass filter. Interpolating a mixture of actual signal and additional noise with restricted bandwidth is not a problem. Really, the linear property of algorithm operation does not restrict kind of signal, but it is required that signal must be continuous and smooth. Therefore the noise will be interpolated as a part of signal. Then additional (noised) spectral components appear in spectrum. There magnitudes are lower as its frequency does not multiple of DFT resolution so spectrum erosion spreads its energy. Applying FIR filtering before DFT calculation provides restriction frequency bandwidth for preventing superposition of spectrum.

Therefore the analyzed signal must have quantization level according to value of SNR. Then noise influencing on spectrum will be minimized. The numerical modeling confirms provided results. Modeling is carried out on a different implementation of the algorithm (the initial data – the same as for figure 6):
- calculation when numbers is represented as floating point (64 bits) – figure 6;
- calculation when input signal is represented as integer (12 bits) – figure 7a;
- calculation when additive noise SNR=40 dB influences – figure 7b.
- calculation when input signal is represented as integer (12 bits) and additive noise (SNR=40 dB) influences simultaneous – figure 7c.

For example, in accordance with given data, the maximum error for signal with frequency 1600 Hz is less than 0.23% (figure 7a) and less than 0.21% (figure 7b) while error of measuring frequency of base harmonic is less than 1.6 Hz. Also it is limited by 0.01% (worst case) if the base frequency is known exactly.

In all cases of modeling, the structure of algorithm includes following blocks: generator of polyharmonic signal, interpolator and block of spectrum calculating. The output of model is presented by error values of measuring magnitude of each harmonics magnitudes.

Figure 7. Errors of harmonics magnitudes measuring
Also note that data on figure 7 is provided by one realization of statistical noise. Modeling algorithm many times with different statistical noise shows a stability of solution and permissible values of errors.
6. COMPARING RESULTS

A comparison of different methods results has meaning while they solve the same problem. Every solving method has positive and negative features from point of view different problems of digital processing.

There are some methods for solving problem of spectral erosion (leakage). Many articles and papers are dedicated to the method, for example [6-10]. The comparison of some is considered in [6]. Data of Table 1 is taken from [6] and it was extended with results the considered in this paper method (with the same parameters as in [6]: \( f_b = 463.87 \text{ Hz}, f_s = 50 \text{ kHz}, N = 1024 \)).

<table>
<thead>
<tr>
<th>Harmonic</th>
<th>Error of estimation amplitudes of polyharmonic signal (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>of algorithm from [6], error of measuring the base freq. is 0.5 Hz</td>
</tr>
<tr>
<td>1</td>
<td>0.184</td>
</tr>
<tr>
<td>3</td>
<td>0.63</td>
</tr>
<tr>
<td>5</td>
<td>1.44</td>
</tr>
<tr>
<td>7</td>
<td>0.08</td>
</tr>
<tr>
<td>9</td>
<td>0.78</td>
</tr>
</tbody>
</table>

Results from Table 1 show that accuracy of processing polyharmonic signal by considered algorithm (with 4-th interpolation order) is significantly higher than for other methods from [6].

7. CONCLUSION

As a result of work an algorithm was developed, which provides precise solution of spectral analyzing problem for polyharmonic signals. Note that while the signal \( (f_b = 50 \text{ Hz} \) with 40 harmonics) with SNR=40 dB is digitized by 12 bits ADC with sampling frequency 16 kHz, the errors of measuring harmonics magnitudes will be less than 0,19% if the base frequency of signal is known with accuracy better than 0,03 Hz. Simultaneously the errors of measuring magnitudes of the same signal by standard algorithm (figure 1) is around 2,7% with known base frequency exactly. Measuring the base frequency with needed accuracy usually is not a problem [11].

The comparison of considered method with known methods shows that while calculating complexity of algorithms and errors of base frequency measurement remain comparable values, the accuracy of method is better.

Developed algorithm was implemented to solve the problem of monitoring quality of electrical energy [12]. Also results of this method of processing signal may be implemented to process signals, sampling frequency that does not satisfy to the condition of sampling theorem.

8. REFERENCES