Reversible Hadamard Transform

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Abstract — A coding method which reconstructs an original digital image without distortion is called ”reversible coding”. Note that when we use classical block transform coding (Cosine, Hadamard, Haar and etc.) we have to make the number of levels of the transform coefficient very large in order to reconstruct the input signal with no distortion. In this paper we propose reversible Hadamard transform for image coding. We give a recursion method for generation of reversible Hadamard transform matrices of higher order and corresponding fast transform algorithms.

I. INTRODUCTION

It is well known that orthogonal transforms are used widely in signal and image processing, especially in image compression and transform coding in telecommunication [1]-[5]. The Fourier transform and its extensions have historically been the first efficient technique for signal analysis and representation. Since the early 1970s, block transforms with real basis functions, particularly the discrete cosine transform (DCT), have been studied extensively for transform coding applications. The availability of simple fast transform algorithms [6],[7] and good signal coding performance made the DCT the standard signal decomposition technique, particularly for image and video. The international standard image-video coding algorithms, i.e. JPEG, MPEG, and etc., all employ DCT-based transform coding. However, the fast DCT of large order requires a great number of float-point multiplication even in a case of most important in practice, when the input data are integer numbers. The integer-to-integer discrete cosine transform as well as another unitary transforms becomes popular in recent years [8]-[20]. Along with discrete trigonometric transforms such as the fast DCT, other non-sinusoidal fast transforms, e.g. the Walsh-Hadamard (WHT) and Haar transforms (HT), have also developed and have found wide applications in digital communications [21], [22].

An attractive property of WHT is that it’s computation involves no multipliers and requires no floating-point operations if the components of the input signal are all integers (since the WHT matrix consists of s only). Both Haar and Walsh-Hadamard fast transforms require fewer computations than the FFT.

The increasing importance of processing large vectors in many scientific and engineering applications requires new ideas for designing highly efficient algorithms for various transforms. The computation of unitary and invertible transforms is in general a complicated and time consuming task and it would not be possible to use these transforms in signal and image processing applications without effective algorithms for calculating them.

A coding method which reconstructs an original digital image without distortion is called ”reversible coding”. Note that in case that we use classical block transform coding (Cosine, Hadamard, Haar and etc.) we have to make the number of levels of the transform coefficient very large in order to reconstruct the input signal with no distortion. In this paper we propose reversible Hadamard transform for image coding. We give a recursion method for generation of reversible Hadamard transform matrices of higher order and corresponding fast transform algorithms.

II. REVERSIBLE SYLVESTER-HADAMARD TRANSFORM

The Hadamard matrix of order n is the (±1)–matrix H_n of size n × n satisfying the orthogonality condition

$$H_n H_n^T = n I_n,$$

where T is a transposition sign, I_n is an identity matrix of order n.

One of the most known Hadamard matrices is the Sylvester matrix [23], which is probably, the oldest Hadamard matrix of order 2^k, and can be generated recursively as follows [29]

$$H_{2k} = \begin{pmatrix} H_{2k-1} & H_{2k-1} \\ H_{2k-1} & -H_{2k-1} \end{pmatrix}, \quad H_1 = (1), \quad k = 1, 2, \ldots, (1)$$

The forward Sylvester-Hadamard (or Walsh-Hadamard) transform of input column-vector x = (x_0, x_1, . . . , x_{N-1}) (N is the power of 2) is defined as y = H_N x. For example for N = 2 we have

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_0 + x_1 \\ x_0 - x_1 \end{pmatrix}.$$}

In [24], [25] there is paid attention to the fact that x_0 – x_1 is even (or odd), if x_0 + x_1 is even (or odd) and is showed that reconstruction without distortion is possible in the following transform

$$\begin{pmatrix} (x_0 + x_1)/2 \\ x_0 - x_1 \end{pmatrix},$$
where \([c]\) is the largest integer which is not greater than \(c\). Using this idea in [26] reversible Walsh-Hadamard transform of order four is given

\[
\begin{bmatrix}
((x_0 + x_1 + x_2 + x_3)/4 + 0.5) \\
((x_0 + x_1 - x_2 - x_3)/2) \\
((x_0 - x_1 - x_2 + x_3)/2) \\
x_0 - x_1 + x_2 - x_3
\end{bmatrix}.
\]

By analogy with (1) we can define recursively reversible Sylvester-Hadamard transform matrices as

\[
[RH]_{2k+1} = \begin{pmatrix}
\frac{1}{2}[RH]_{2k} & \frac{1}{2}[RH]_{2k} \\
[RH]_{2k} & -[RH]_{2k}
\end{pmatrix},
\]

\[
[RH]_{2k+1}^{-1} = \begin{pmatrix}
[RH]_{2k}^{-1} & \frac{1}{2}[RH]_{2k}^{-1} \\
[RH]_{2k}^{-1} & -\frac{1}{2}[RH]_{2k}^{-1}
\end{pmatrix},
\]

where \([RH]_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1 \end{pmatrix}\), \([RH]_2^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}\).

III. DESIGN OF SEQUENTIAL ORDERED REVERSIBLE WALSH-HADAMARD TRANSFORM

Let \([RH]_N\) and \([RH]_N^{-1}\) be the sequential ordered direct and inverse reversible Walsh-Hadamard matrices, and \(A_i\) is the \(i\)-th column and \(B_i\) is the \(i\)-th reverse column of \([RH]_N\). Then the following matrices

\[
\begin{bmatrix}
Q_1 \otimes A_1, Q_2 \otimes A_2, \ldots, Q_1 \otimes A_{N-1}, Q_2 \otimes A_N
\end{bmatrix},
\]

\[
\begin{pmatrix}
Q_1^{-1} \otimes B_1 \\
-\frac{1}{2}Q_1^{-1} \otimes B_2 \\
\vdots \\
Q_1^{-1} \otimes B_{N-1} \\
-\frac{1}{2}Q_2^{-1} \otimes B_N
\end{pmatrix},
\]

are direct and inverse sequential ordered reversible Walsh-Hadamard matrices of order \(2N\).

The direct and inverse sequential ordered reversible Walsh-Hadamard matrices of order 4 are given below

\[
\begin{pmatrix}
1/4 & 1/4 & 1/4 & 1/4 \\
1/2 & 1/2 & -1/2 & -1/2 \\
1/2 & -1/2 & 1/2 & 1/2 \\
1 & -1 & 1 & -1
\end{pmatrix},
\]

\[
\begin{pmatrix}
1 & 1/2 & 1/2 & 1/4 \\
1 & 1/2 & -1/2 & -1/4 \\
1 & -1/2 & -1/2 & 1/4 \\
1 & -1/2 & 1/2 & -1/4
\end{pmatrix},
\]

Introduce the notations: \(D_k = diag\{x_1, x_2, \ldots, x_k\}\), \(\overline{D}_k = diag\{x_1, x_2, \ldots, x_k\}\). It is clear that above given reversible Walsh-Hadamard transform matrices can be represented as

\[
[RH]_2 = D_2H_2, \quad [RH]_2^{-1} = H_2\overline{D}_2,
\]

\[
P_4 = D_4H_4, \quad P_4^{-1} = H_4\overline{D}_4,
\]

\[
P_8 = D_8H_8, \quad P_8^{-1} = H_8\overline{D}_8,
\]

where \(D_2 = diag\{1/2, 1\}\), \(D_4 = diag\{1/4, 1/2, 2, 1/2, 1\}\), \(D_8 = diag\{1/8, 1/4, 1/4, 1/2, 1/4, 1/2, 1/2, 1/2, 1\}\), \(H_N\) is the Walsh-Hadamard matrix of order \(N\).

In Fig. 1 the base functions corresponding to reversible Walsh-Hadamard matrix \(P_8\) are given.
\[ Y = D_N H_N X, \quad X = H_N \overline{D_N} Y. \]

From (3) it follows that for the forward and inverse reversible Walsh-Hadamard transforms only \( N \log_2 N \) addition operations and \( \frac{N}{2} \log_2 N \) shift operations are needed. Flow graph of the forward reversible Walsh-Hadamard transform is given in Fig. 2.

\[
\begin{align*}
R & := \begin{vmatrix}
1/2 & y_0 \\
0 & y_1 \\
0 & x \\
0 & y \\
\end{vmatrix} \\
x & \left[ \begin{array}{c}
x_0 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
\end{array} \right] \\
y & \left[ \begin{array}{c}
y_0 \\
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
y_6 \\
y_7 \\
\end{array} \right]
\end{align*}
\]

**Fig. 2.** Forward reversible Walsh-Hadamard transform of order 8

### V. Reversible Williamson Array

At first we briefly describe the Williamson's approach to the Hadamard matrices construction.

**Theorem V.1:** (Williamson [27], [32]). Suppose there exist four \((\pm 1)\)-matrices \( A, B, C, D \) of order \( n \) satisfying

\[ PQ^T = QP^T, \quad P, Q \in \{ A, B, C, D \}, \]

\[ AA^T + BB^T + CC^T + DD^T = 4nI_n. \]

Then

\[ W_{4n} = \begin{pmatrix}
A & B & C & D \\
B & -A & -C & -D \\
C & -D & A & -B \\
D & -A & -C & B \\
\end{pmatrix} \]

is Hadamard matrix of order \( 4n \).

The matrices \( A, B, C, D \) with properties (4) are called **Williamson matrices**. The matrix (5) is called the **Williamson-Hadamard** matrix.

Let \( A, B, C, D \) be cyclic symmetric Williamson matrices of order \( n \) with first rows \((a_i), (b_i), (c_i), (d_i)\), respectively. Note that \( a_{n-i} = a_i, \ b_{n-i} = b_i, \ c_{n-i} = c_i, \ d_{n-i} = d_i, \ i = 1, 2, \ldots, n - 1 \). Now Williamson-Hadamard matrix (5) can be represented as block cyclic block symmetric matrix by

\[ W_{4n} = \sum_{i=0}^{n-1} Q_i \otimes U^i, \quad \text{where} \]

\[ Q_i = \begin{pmatrix}
a_i & b_i & c_i & d_i \\
-b_i & a_i & -d_i & c_i \\
-c_i & d_i & a_i & -b_i \\
-d_i & c_i & -b_i & a_i \\
\end{pmatrix}, \]

where \( U \) is the cyclic matrix of order \( n \) with first row \((0, 1, 0, \ldots, 0)\).

From (6) we can see that \( Q_{n-i} = Q_i \), and all the blocks of matrix \( W_{4n} \) are Williamson-Hadamard matrices of order 4. In [31] it was proved that cyclic symmetric Williamson-Hadamard block matrices can be constructed using only 5 different blocks such as

\[ Q_0 = \begin{pmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{pmatrix}, \quad Q_1 = \begin{pmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{pmatrix}, \]

\[ Q_2 = \begin{pmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{pmatrix}, \quad Q_3 = \begin{pmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{pmatrix}, \]

\[ Q_4 = \begin{pmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{pmatrix}. \]

For example, Williamson-Hadamard matrix of order 12 is given by

\[ H_{12} = Q_0 \otimes I_3 + Q_1 \otimes U + Q_4 \otimes U^2. \]

Now we introduce the following parametric matrix of order 4 which we call reversible Williamson (RW) array \((s = a^2 + b^2 + c^2 + d^2)\)

\[ [RW](a, b, c, d) = \frac{1}{2\sqrt{s}} \begin{pmatrix}
a & b & c & d \\
-2b & 2a & 2c & 2d \\
-2c & 2d & 2a & -2b \\
-4d & -4c & -4b & 4a \\
\end{pmatrix}. \]

The inverse reversible Williamson array is given by

\[ [RW]^{-1}(a, b, c, d) = \frac{1}{2\sqrt{s}} \begin{pmatrix}
4a & 2b & -2c & -d \\
4b & 2a & -2d & 2c \\
4c & -2d & 2a & b \\
4d & 2c & -2b & a \\
\end{pmatrix}. \]

Now, the Theorem V.1 for reversible Williamson-Hadamard matrices can be formulated as

**Theorem V.2:** (Generalized Williamson Theorem). Let \( A, B, C, D \) be Williamson matrices of order \( n \). Then the matrix \([RW](A, B, C, D)\) is the reversible Williamson-Hadamard matrix of order \( 4n \). The inverse reversible Williamson-Hadamard transform matrix of order \( 4n \) is \([RW]^{-1}(A^T, B^T, C^T, D^T)\). (In this case \( AA^T + BB^T + CC^T + DD^T = 4nI_n\).

**Example V.1:** The complete set of reversible Williamson-Hadamard matrices of order 4 obtained from (8) and (9) for \( a, b, c, d = \pm 1 \) are given below
VI. Reversible Hadamard Transforms Depending on 8 Parameters

(±1)-matrices $A, B, \ldots, H$ are called 8-Williamson matrices of order $n$ if the following conditions are satisfied

$$PQ^T = QP^T, \quad P, Q \in \{A, B, \ldots, H\},$$

$$AA^T + BB^T + \cdots + HH^T = 8nI_n.$$

Similar to the Theorem V.1 we have [21], [29]

**Theorem VI.1:** Let the matrices $A, B, \ldots, H$ be 8-Williamson matrices of order $n$, then the following matrix is a Hadamard matrix of order $8n$

$$
\begin{pmatrix}
A & B & C & D & E & F & G & H \\
-B & A & D & -C & F & -E & -H & G \\
-C & -D & A & B & G & H & -E & -F \\
-D & C & -B & -A & H & -G & F & -E \\
-E & -F & -G & -H & A & B & C & D \\
-F & E & -H & -G & B & -A & D & -C \\
-G & H & E & -F & -C & A & -B & H \\
-H & -G & F & E & -D & -C & B & A
\end{pmatrix},
$$

Now introduce the following parametric matrix (denote by $[RP](a, b, \ldots, h)$) of order 8 which we call the reversible Plotkin (RP) array $(s = a^2 + b^2 + \cdots + b^2)$

$$\begin{pmatrix}
a & b & c & d & e & f & g & h \\
-2b & 2a & 2d & -2c & 2f & -2e & -2h & 2g \\
-2c & -2d & 2a & 2b & 2g & 2h & -2e & -2f \\
-2d & 2e & -2b & 2a & 2h & -2g & 2f & -2e \\
-4e & -4f & -4g & -4h & 4a & 4b & 4c & 4d \\
-4f & 4e & -4h & 4a & 4b & -4d & 4a & -4b \\
-4g & 4h & -4f & 4e & -4d & 4a & -4b & 8h \\
-8h & -8g & 8f & 8e & -8d & -8e & 8b & 8a
\end{pmatrix}_{8s}.$$

The inverse RP array is given by

$$
\begin{pmatrix}
8a & -4b & -4c & -4d & -2e & -2f & -2g & -h \\
-8b & 4a & -4d & 4c & -2f & 2e & 2h & -g \\
-8c & 4d & 4a & -4b & -2g & -2h & 2e & f \\
8d & -4c & 4b & 4a & -2h & 2g & -2f & e \\
8e & 4f & 4g & 4h & 2a & -2b & -2c & -d \\
8f & -4e & 4h & -4g & 2b & 2a & 2d & -c \\
8g & -4h & -4e & 4f & 2c & -2d & 2a & b \\
8h & 4g & -4f & -4e & 2d & 2c & -2b & a
\end{pmatrix}_{8s}.
$$

Now, the Theorem VI.1 for reversible Hadamard matrices can be formulated as

**Theorem VI.2:** Let $A, B, \ldots, H$ be 8-Williamson matrices of order $n$. Then the matrix $[RP](A, B, \ldots, H)$ is the reversible Hadamard matrix of order $8n$. The inverse reversible Williamson-Hadamard transform matrix of order $4n$ is $[RP]^{-1}(A^T, B^T, \ldots, H^T)$. (Note that in this case $AA^T + BB^T + \cdots + HH^T = 8I_n$).

VII. Design of Parametric Reversible Transforms via Multiplicative Method

Recall some definitions. Hadamard matrix $H_m$ of order $m$ which can be represented as

$$H_m = v_1 \otimes A_1 + v_2 \otimes A_2 + \cdots + v_k \otimes A_k,$$

is called $A(m, k)$-matrix [28]-[30], where $v_i$ are orthogonal $(±1)$-vectors of length $k$, $A_i$ are $(0, ±1)$-matrices of dimension $m \times \frac{m}{k}$ satisfying the conditions:

$$A_i \otimes A_j = 0, \quad i, j = 1, 2, \ldots, k,$$

$$\sum_{i=1}^{k} A_i = (±1)-matrix,$$

$$\sum_{i=1}^{k} A_i A_i^T = \frac{m}{k} I_m,$$

$$A_i^T A_i = 0, \quad i, j = 1, 2, \ldots, k.$$

Note that any Hadamard matrix $H_{4n}$ of order $4n$ can be represented as $H_{4n} = (1 \ 1) \otimes A_1 + (1 \ -1) \otimes A_2$. It can be shown that

$$H^{-1} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes A_1^T + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes A_2^T \right].$$

is reversible Hadamard matrix of order $4n$;

$$H^{-1}(a, b) = \begin{pmatrix} a \\ b \end{pmatrix} \otimes A_1^T + \frac{1}{2} \begin{pmatrix} b \\ -a \end{pmatrix} \otimes A_2^T$$

is parametric reversible Hadamard matrix of order $4n$ ($s = a^2 + b^2$).
(iii) $H(a, b, c, d) = \frac{1}{\sqrt{n}} [P_1 \otimes A_1 + P_2 \otimes A_2],$
\[ H^{-1}(a, b, c, d) = [Q_1 \otimes A_1^T + Q_2 \otimes A_2^T] \]

is parametric reversible Hadamard matrix of order $8n$ ($s = a^2 + b^2 + c^2 + d^2$), where

$$P_1 = \begin{pmatrix} a & b & c & d \\ -b/2 & 2a & -2d & 2c \end{pmatrix},$$

$$P_2 = \begin{pmatrix} -2c & 2d & 2a & -2b \\ -4d & -4c & 4b & 4a \end{pmatrix},$$

$$Q_1^T = \begin{pmatrix} a & b & c & d \\ -b/2 & a/2 & -d/2 & c/2 \end{pmatrix},$$

$$Q_2^T = \begin{pmatrix} -c/2 & d/2 & a/2 & -b/2 \\ -d/4 & -c/4 & b/4 & a/4 \end{pmatrix}. $$

Now give some examples. As an initial matrix $H$ consider the matrix $Q_0$ from (7). In that case we have

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}. $$

Using (10) we obtain the following reversible Hadamard matrix of order 4

$$H_1 = \frac{1}{2} diag\{1/2, 1, 1, 1/2\} Q_0,$$

$$H_1^{-1} = Q_0^T diag\{1/2, 1, 1/2, 1\}. $$

Below there are given forward and inverse parametric reversible Hadamard matrices $H_1(a, b) H_1^{-1}(a, b)$ obtained from (11)

$$\frac{1}{8} \begin{pmatrix} a^2 & b/2 & a/2 & b/2 \\ -b & a & -b & a \\ -a/2 & -b/2 & a/2 & b/2 \end{pmatrix} \begin{pmatrix} a & b/2 & -b/2 & -a \\ b/2 & a & -b/2 & a/2 \\ -b/2 & a/2 & b/2 & a \\ a/2 & -b/2 & -a & b \end{pmatrix}. $$

Using equation (12) we obtain the following parametric forward and inverse reversible Hadamard transform matrices depending on four parameters, respectively

$$\begin{pmatrix} a & b & c & d \\ -b/2 & 2a & -2d & 2c \\ 2c & -2d & 2a & 2b \\ 4d & 4c & -4b & -4a \\ -a & -b & -c & -d \end{pmatrix} \begin{pmatrix} a & b & c & d \\ -b/2 & 2a & -2d & 2c \\ 2c & -2d & 2a & 2b \\ 4d & 4c & -4b & -4a \\ -a & -b & -c & -d \end{pmatrix} \begin{pmatrix} a & b & c & d \\ -b/2 & 2a & -2d & 2c \\ 2c & -2d & 2a & 2b \\ 4d & 4c & -4b & -4a \\ -a & -b & -c & -d \end{pmatrix}. $$

VIII. Block-cyclic Reversible Williamson-Hadamard Matrices

Similar to classical Williamson-Hadamard matrices [28], [29] the reversible Williamson-Hadamard matrices of order $4n$ can be represented by block cyclic form. Let

$$A = \sum_{i=0}^{n-1} a_i U_i, \quad B = \sum_{i=0}^{n-1} b_i U_i, \quad C = \sum_{i=0}^{n-1} c_i U_i, \quad D = \sum_{i=0}^{n-1} d_i U_i$$

are cyclic symmetric Williamson matrices of order $n$, i.e.

$$a_i = a_{n-i}, \quad b_i = b_{n-i}, \quad c_i = c_{n-i}, \quad d_i = d_{n-i}, \quad i = 1, n-1,$$

where $U$ is the cyclic matrix of order $n$ with first row $(0, 1, 0, \ldots, 0)$. Note that $U^0 = U^n = I_n, U_{n-i} = U^{i}$. From Theorem V.2 it follows that the following matrix is a reversible Williamson-Hadamard matrix of order $4n$

$$[RW](A, B, C, D) = \frac{1}{4\sqrt{n}} \begin{pmatrix} A & B & C & D \\ -2B & 2A & -2D & 2C \\ -2C & 2D & -2A & 2B \\ -2D & -2C & 2B & 2A \end{pmatrix}. $$

This matrix can also be represented as a block-cyclic matrix as follows

$$[RW]_{4n} = \sum_{i=0}^{n-1} [RW](a_i, b_i, c_i, d_i) \otimes U^i,$$

where $[RW](a_i, b_i, c_i, d_i)$ has the form (8).

The inverse reversible block-cyclic matrix is given by

$$[RW]_{4n}^{-1} = \sum_{i=0}^{n-1} [RW]^{-1}(a_i, b_i, c_i, d_i) \otimes U^{n-i}.$$

From (8) and (9) we obtain

$$[RW](a, b, c, d) = \frac{1}{2\sqrt{5}} DW(a, b, c, d),$$

$$[RW]^{-1}(a, b, c, d) = \frac{1}{2\sqrt{5}} W^T(a, b, c, d) D_1,$$

where $D = diag\{1, 2, 2, 4\}, \quad D_1 = diag\{4, 2, 2, 1\}$, and $W(a, b, c, d)$ is a Williamson array

$$W(a, b, c, d) = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}. $$

Now we can prove that reversible Williamson-Hadamard block matrices can be constructed using only the following 5 different blocks such as

$$[RW]_0 = [RW](1, 1, 1, 1) = \frac{1}{4} DQ_0,$$

$$[RW]_1 = [RW](1, 1, 1, -1) = \frac{1}{4} DQ_1,$$

$$[RW]_2 = [RW](1, 1, -1, 1) = \frac{1}{4} DQ_2,$$

$$[RW]_3 = [RW](1, -1, 1, 1) = \frac{1}{4} DQ_3,$$

$$[RW]_4 = [RW](1, -1, -1, -1) = \frac{1}{4} DQ_4,$$

where $Q_i$ are from (7).
The inverse reversible Williamson-Hadamard block matrices can be constructed using the following blocks

\[
[RW]^{-1}_0 = [RW]^{-1}(1,1,1,1) = \frac{1}{2}Q^T_1 D_1, \\
[RW]^{-1}_1 = [RW]^{-1}(1,1,1,-1) = \frac{1}{2}Q^T_1 D_1, \\
[RW]^{-1}_2 = [RW]^{-1}(1,-1,1,1) = \frac{1}{2}Q^T_2 D_1, \\
[RW]^{-1}_3 = [RW]^{-1}(1,-1,1,-1) = \frac{1}{2}Q^T_3 D_1.
\]

Note that all of matrices from (14) and (15) can be represented only by sequential ordered reversible Walsh-Hadamard matrix of order 4 which we denote by \(S\).

\[
[RW]_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} S \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
[RW]_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} S \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
[RW]_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} S \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
[RW]_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} S \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
[RW]_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} S \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

**Example VIII.1:** Reversible block cyclic Williamson-Hadamard matrix of order 12.

\[
[RW H]_{12} = \frac{1}{\sqrt{3}} \begin{pmatrix}
[RW]_0 & [RW]_4 & [RW]_4 \\
[RW]_4 & [RW]_0 & [RW]_4 \\
[RW]_4 & [RW]_4 & [RW]_0
\end{pmatrix},
\]

\[
[RW H]^{-1}_{12} = \frac{1}{\sqrt{3}} \begin{pmatrix}
[RW]^{-1}_0 & [RW]^{-1}_4 & [RW]^{-1}_4 \\
[RW]^{-1}_4 & [RW]^{-1}_0 & [RW]^{-1}_4 \\
[RW]^{-1}_4 & [RW]^{-1}_4 & [RW]^{-1}_0
\end{pmatrix}.
\]

Using (14) we can write

\[
[RW H]_{12} = \frac{1}{\sqrt{3}} \text{diag}(D, D, D) H_{12}, \\
[RW H]^{-1}_{12} = \frac{1}{\sqrt{3}} H^T_{12} \text{diag}(D_1, D_1, D_1).
\]

Reversible Williamson-Hadamard matrix of order 12 and its inverse matrix can be decomposed as

\[
[RW H]_{12} = (1 + 1) \oplus A_1 + (1 + 1) \oplus A_2,
\]

\[
[RW H]^{-1}_{12} = (1 + 1) \oplus G_1 + (1 + 1) \oplus G_2,
\]

where

\[
A_1 = \frac{1}{\sqrt{3}} (B_0 \otimes I_3 + B_1 \otimes U + B_1 \otimes U^2),
\]

\[
A_2 = \frac{1}{\sqrt{3}} (C_0 \otimes I_3 + C_1 \otimes U + C_1 \otimes U^2),
\]

\[
G_1 = \frac{1}{\sqrt{3}} (E_0 \otimes I_3 + E_1 \otimes U + E_1 \otimes U^2),
\]

\[
G_2 = \frac{1}{\sqrt{3}} (F_0 \otimes I_3 + F_1 \otimes U + F_1 \otimes U^2),
\]

\[
B_0 = \begin{pmatrix}
1 & 1 \\
0 & 0 \\
0 & 0 \\
-1 & -4
\end{pmatrix},
\]

\[
B_1 = \begin{pmatrix}
0 & 0 \\
0 & 2 \\
4 & 0 \\
4 & -4
\end{pmatrix},
\]

\[
E_0 = \begin{pmatrix}
4 & 0 & 0 & -1 \\
4 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
E_1 = \begin{pmatrix}
0 & 2 & 0 & 1 \\
-4 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0
\end{pmatrix},
\]

\[
F_0 = \begin{pmatrix}
0 & -2 & -2 & 0 \\
0 & -2 & -2 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0
\end{pmatrix},
\]

\[
F_1 = \begin{pmatrix}
4 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}.
\]

**IX. Reversible Hadamard Matrix of Geothals-Seidel Type**

Let \((\pm 1)\)-matrices \(X, Y, Z,\) and \(W\) be a generalized Williamson type matrices of order \(n, i.e.

\[
PQ = QP, PRQ^T = QRPT, P, Q \in \{X, Y, Z, W\},
\]

\[
XX^T + YY^T + ZZ^T + WW^T = 4nI_n,
\]

where \(R\) is back diagonal identity matrix.

According to Geothals-Seidel theorem the following matrix is the Hadamard matrix of order \(4n\)

\[
H_{4n} = \begin{pmatrix}
X & YR & ZR & WR \\
-YR & X & -W^TR & Z^TR \\
-ZR & W^TR & X & -Y^TR \\
-WR & Z^TR & Y^TR & X
\end{pmatrix}.
\]

We can prove that the matrix \(G_{4n}\) is the reversible Hadamard matrix of order \(4n\)

\[
G_{4n} = \text{diag}(\frac{1}{4}I_n, \frac{1}{4}I_n, \frac{1}{4}I_n, \frac{1}{4}I_n) H_{4n},
\]

\[
G_{4n}^{-1} = \frac{1}{n} H_{4n}^T \text{diag}(\frac{1}{4}I_n, \frac{1}{4}I_n, \frac{1}{4}I_n, \frac{1}{4}I_n).
\]

**Example IX.1:** Geothals-Seidel type reversible Hadamard forward \((G_{12})\) and inverse \((G_{12}^{-1})\) transform matrices of order 12 are given below
where conditions are satisfied

\[ X \text{ order } 12 \]

\[ X \]

\[ \text{Example IX.2: Orthonormal integer Geothals-Seidel type matrix of order 12.} \]

Consider the following cyclic matrices of order 3

\[ X = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}, \quad Y = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \]

\[ Z = \frac{1}{3} \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}, \quad W = \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 2 \\ -2 & -2 & 1 \end{pmatrix}. \]

We can check that

\[ XX^T + YY^T + ZZ^T + WW^T = I_3. \]

Now by substitution the matrices \( X, Y, Z, W \) into array \( (??) \) we obtain the following orthonormal integer matrix of order 12

\[ P = v_1 \otimes X_1 + v_2 \otimes X_2 + v_3 \otimes X_3, \quad (16) \]

where

\[ v_1 = (1, 1, 1, 1), \quad v_2 = (-2, 2, -2, 2), \quad v_3 = (-2, 2, -2, -2), \]

\[ v_1 v_1^T = 0, \quad v_1 \neq j, \quad v_1 v_1^T = 4, \]

\[ v_2 v_2^T = v_3 v_3^T = 16. \]

Now using (16) we form the following matrices

\[ P_1 = X_1 - 2X_2 - 2X_3, \]

\[ P_2 = X_1 + 2X_2 + 2X_3, \]

\[ P_3 = X_1 - 2X_2 + 2X_3, \]

\[ P_4 = X_1 + 2X_2 - 2X_3. \]

We can check that the following matrix

\[ G = \frac{1}{2\sqrt{4k-3}} \begin{pmatrix} P_1 & P_2 & P_3 & P_4 \\ P_2 & P_3 & P_4 & P_1 \\ P_3 & P_4 & P_1 & P_2 \\ P_4 & P_1 & P_2 & P_3 \end{pmatrix} \]

is the orthonormal integer transform matrix of Geothals-Seidel type with elements \( \{\pm 1, \pm 2\} \).

Now we consider the parametric matrix

\[ P(a, b, c, d) = v_1 \otimes X_1 + v_2 \otimes X_2 + v_3 \otimes X_3, \]

where

\[ v_1 = (a, b, c, d), \quad v_2 = (-2b, 2a, -2d, 2c), \quad v_3 = (-2c, 2d, 2a, -2b), \]
Now consider again cyclic $T$--matrices $X_1 = I_{k+1}$, $X_2$, $X_3$, and form the following parametric cyclic matrices of order $k + 1$.

\[
\begin{align*}
P_1(a, b, c) &= aX_1 - 2bX_2 - 2cX_3, \\
P_2(a, b, d) &= bX_1 + 2aX_2 + 2dX_3, \\
P_3(a, c, d) &= cX_1 - 2dX_2 + 2aX_3, \\
P_4(b, c, d) &= dX_1 + 2cX_2 - 2bX_3.
\end{align*}
\]

Substituting these matrices into Geothals-Seidel array we can obtain a parametric orthonormal reversible transform matrix of Baumert-Hall type of order $4(k + 1)$.

**Example X.1:** Parametric orthonormal reversible transform matrix of Baumert-Hall type of order 12 can be constructed using the following matrices given below

\[
\begin{align*}
P_1 &= \begin{pmatrix} 2 & -2 & -2 & 2 \\ -2 & 2 & -2 & 2 \\ -2 & 2 & 2 & -2 \end{pmatrix}, \\
P_2 &= \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \\
P_3 &= \begin{pmatrix} 2 & 2 & -2 & -2 \\ 2 & 2 & 2 & 2 \\ -2 & -2 & 2 & 2 \end{pmatrix}, \\
P_4 &= \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.
\end{align*}
\]

**REFERENCES**