2-DIMENSIONAL WARPED DISCRETE FOURIER TRANSFORM

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ABSTRACT
In this paper we introduce the concept of the two-dimensional warped discrete Fourier transform (2D-WDFT), a generalization of the one-dimensional warped discrete Fourier transform. The 2D-WDFT is defined and its properties are discussed. An algorithm for computation of the 2D-WDFT is proposed. An application of this transform in the tuning of 2D digital filters is included.

Index Terms – All-pass, DFT, NDTV, WDTV, spectral transformations

1. INTRODUCTION

The discrete Fourier transform (DFT) is an important tool in many areas of digital signal processing. The $N$-point DFT is given by the $N$ samples of the $z$-transform of a length-$N$ sequence at equally-spaced points on the unit circle. Several modifications to the conventional DFT have been introduced in recent years. A generalization of the DFT is the non-uniform DFT (NDFT) which is given by the samples of the $z$-transform at $N$ distinct points in the $z$-plane [1]. Several applications of this transform are in the design of FIR filters, efficient dual-tone multi-frequency signal decoding and antenna pattern synthesis with prescribed nulls [1]. A special case of the NDFT is the warped DFT which has been applied for efficient spectral synthesis, design of tuneable FIR filters and the design of perfect reconstruction filter bank with non-uniform passbands [2].

In this paper we are concerned with two-dimensional discrete-time sequences. The 2D $z$-transform of a finite-extent 2D sequence $x[n_1,n_2]$, $0 \leq n_1 \leq N_1 -1$, $0 \leq n_2 \leq N_2 -1$, is given by

$$X(z_1,z_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1,n_2] z_1^{n_1} z_2^{n_2}.$$  (1)

The $N_1N_2$-point 2D DFT of the 2D sequence $x[n_1,n_2]$ is defined by the frequency samples $X[k_1,k_2] = X(e^{j2\pi(k_1/N_1)}e^{j2\pi(k_2/N_2)})$ obtained by sampling $X(z_1,z_2)$ at $N_1N_2$ equally-spaced points on a rectangular grid on the unit bi-disk in the $(z_1,z_2)$-space:

$$X[k_1,k_2] = X(z_1,z_2)|_{z_1=\exp(2\pi jk_1/N_1), z_2=\exp(2\pi jk_2/N_2)} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1,n_2] e^{-j2\pi k_1n_1/N_1} e^{-j2\pi k_2n_2/N_2},$$

$$0 \leq k_1 \leq N_1 -1, 0 \leq k_2 \leq N_2 -1.$$  (2)

The $N_1N_2$-point inverse 2D-DFT is given by

$$x[n_1,n_2] = \frac{1}{N_1N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X[k_1,k_2] e^{j2\pi k_1n_1/N_1} e^{j2\pi k_2n_2/N_2},$$

$$0 \leq n_1 \leq N_1 -1, 0 \leq n_2 \leq N_2 -1.$$  (3)

The 2D-DFT finds applications in the processing of 2D signals, such as SAR data, images, etc. For example, it has been used in the design of 2D FIR filters using the windowed Fourier series approach [3]. Another application is in the contrast enhancement of images based on the alpha-rooting method [4] and the transform amplitude sharpening [5].

In this paper we introduce a new type of 2D-DFT, to be called the 2D warped discrete Fourier transform (2D-WDFT), in which the frequency samples of the $z$-transform are given by non-uniformly spaced samples on a rectangular grid on the unit bi-disk. In Section 2 we define the 2D-WDFT and discuss some of its properties. Section 3 outlines a simple application of the 2D-WDFT in the tuning of 2D FIR filters. Finally, concluding remarks are given in Section 4.

2. TWO DIMENSIONAL WARPED DISCRETE FOURIER TRANSFORM

The $N_1N_2$-point 2D Nonuniform DFT (2D-NDFT), a generalization of the more common 2D DFT, of the finite-extent 2D sequence $x[n_1,n_2]$, $0 \leq n_1 \leq N_1 -1$, $0 \leq n_2 \leq N_2 -1$, is given by the $N_1N_2$ samples of $X(z_1,z_2)$:
and 2. In sections to relation of the frequency axes, we result, equally spaced samples of X.

A special case of the X_NDFT[k_1,k_2] when the points z_1,k_1

and z_2,k_2 are placed non-uniformly on the unit bi-disk at the

locations: z_1,k_1 = e^{j \omega_1,k_1} and z_2,k_2 = e^{j \omega_2,k_2}. In particular, the N_1N_2-point 2D WDFST is obtained by sampling a modified z-transform \( \hat{X}(\hat{z}_1,\hat{z}_2) \) at N_1N_2 equally spaced points on a rectangular grid on the unit bi-disk in the \((\hat{z}_1,\hat{z}_2)\)-space. The modified z-transform is obtained by a 2D spectral transformation of the z-transform \( X(z_1,z_2) \):

\[
\hat{X}(\hat{z}_1,\hat{z}_2) = X(z_1,z_2) \bigg|_{z_1 \rightarrow A_1(\hat{z}_1), z_2 \rightarrow A_2(\hat{z}_2)},
\]

where \( A_1(\hat{z}_1) \) and \( A_2(\hat{z}_2) \) are stable 2D first-quadrant allpass functions [4]. The mapping given in the above equation warps the frequency scales of the unit bi-disk, and as a result, equally spaced samples of \( \hat{X}(\hat{z}_1,\hat{z}_2) \) on the unit bi-disk in the \((\hat{z}_1,\hat{z}_2)\)-space are mapped onto nonuniformly spaced samples of \( X(z_1,z_2) \) on the unit bi-disk in the \((z_1,z_2)\)-space.

Since it is difficult to ensure the stability of higher-order 2D transfer functions and develop the corresponding warping relation of the frequency axes, we restrict our attention here to the case of a separable single-variable transform:

\[
z_i^{-1} \rightarrow A_i(\hat{z}_i), \quad z_2^{-1} \rightarrow A_2(\hat{z}_2),
\]

where

\[
A_i(\hat{z}_i) = \frac{\hat{D}_i(\hat{z}_i)}{D_i(z_i)} = \frac{\sum_{\ell=1}^{M_i} d^{(i)}_{\ell} \hat{z}_i^{-\ell}}{1 + \sum_{\ell=1}^{M_i} d^{(i)}_{\ell} \hat{z}_i^{-\ell}}, \quad i = 1,2,
\]

where \( \hat{D}_i(\hat{z}_i) \) denotes the mirror-image polynomial of \( D_i(z_i) \). The above single-variable transformations have been investigated for spectral transformation of 2D recursive digital filters [4]. Applying the above spectral transformations to \( X(z_1,z_2) \), we arrive at

\[
\hat{X}(\hat{z}_1,\hat{z}_2) = \frac{P(\hat{z}_1,\hat{z}_2)}{Q_1(\hat{z}_1)Q_2(\hat{z}_2)},
\]

which is seen to be a rational function in \( \hat{z}_1^{-1} \) and \( \hat{z}_2^{-1} \) with a separable denominator polynomial. The 2D-WDFST of \( x[n_1,n_2] \) is obtained by determining the samples of \( P(\hat{z}_1,\hat{z}_2) \) and the samples of \( Q_1(\hat{z}_1) \) and \( Q_2(\hat{z}_2) \) at equally-spaced points on the unit bi-disk in the \((\hat{z}_1,\hat{z}_2)\)-space. These samples are simply given by the conventional 2D-DFT of the 2D sequence formed by the coefficients of \( P(\hat{z}_1,\hat{z}_2) \), and by the conventional 1D-DFT of the 1D sequences formed by the coefficients of \( Q_1(\hat{z}_1) \) and \( Q_2(\hat{z}_2) \), respectively. We denote a 2D WDFST pair as

\[
x[n_1,n_2] \leftrightarrow \hat{x}[k_1,k_2]
\]

Most properties associated with 2D DFT, such as, linearity and periodicity, also hold for 2D WDFST. Reflection is ob-served straight-forward. Let \( \hat{z}_\ell, 0 \leq k_1 \leq N_1 - 1, \) and \( \hat{z}_\ell, 0 \leq \ell \leq N_2 - 1, \) denote a set of exact points on the z-plane and let \( \hat{z}_k \) and \( \hat{z}_\ell \) denote the corresponding points on the \( \hat{z} \)-plane. We have omitted the indices 1 and 2 from the subscripts of the frequency points for convenience. Then \( \hat{z}_{N_1+k} = \hat{z}_k = \hat{z}_k^\dagger \). It then follows,

\[
\hat{z}_{N_1+k}^{-1} = A(\hat{z}_{N_1-k}) = \frac{\hat{D}(\hat{z}_{N_1-k})}{\hat{D}(\hat{z}_{N_1-k})} = \left(\frac{\hat{M}}{\hat{D}(\hat{z}_k)}\right) = \frac{\hat{M}}{\hat{D}(\hat{z}_k)} = \frac{\hat{M}}{\hat{D}(\hat{z}_k)}, \quad (9)
\]

where \( \hat{D}_k \) is the order of the all-pass transformation function. Some of the other properties that follow Eq. (9) are:

1. Conjugate Symmetry: For real \( x[n_1,n_2] \),

\[
\hat{X}[N_1-k,N_2-\ell] = \hat{X}[k,\ell].
\]

To prove Eq. (10), we note that the left hand side expression are simply the samples of \( \hat{X}(\hat{z}_{N_1-k},\hat{z}_{N_2-\ell}) \) at \( \hat{z}_{N_1-k} = e^{-j2\pi(N_1-k)/N_1} \) and \( \hat{z}_{N_2-\ell} = e^{-j2\pi(N_2-\ell)/N_2} \). Using \( \hat{z}_{N_1-k} = \hat{z}_k = \hat{z}_k^\dagger \), we can write

\[
\hat{X}(\hat{z}_{N_1-k},\hat{z}_{N_2-\ell}) = \hat{X}(\hat{z}_k,\hat{z}_k) = \hat{X}[k,\ell],
\]

with last term obtained by sampling \( \hat{X}(\hat{z}_k^\dagger,\hat{z}_k) \) at \( \hat{z}_k = e^{-j2\pi k/N_1} \) and \( \hat{z}_\ell = e^{-j2\pi \ell/N_2} \).

2. Conjugation:

\[
x^*[n_1,n_2] \leftrightarrow \hat{x}^*[N_1-k,N_2-\ell]
\]

To prove (11), let \( y[n_1,n_2] = x^*[n_1,n_2] \). Then using Eq. (6), we obtain

\[
y(\hat{z}_k,\hat{z}_\ell) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1,n_2) \left[A_1(\hat{z}_k)\right]^* \left[A_2(\hat{z}_\ell)\right]^* = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x^*[n_1,n_2] \left[A_1(\hat{z}_k)\right]^* \left[A_2(\hat{z}_\ell)\right]^*\]

\[
= \left[\sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left[A_1(\hat{z}_k)\right]^* \left[A_2(\hat{z}_\ell)\right]^*\right]^* = \hat{x}^*[N_1-k,N_2-\ell],
\]

using the relation \( \hat{z}_{N_1-k} = \hat{z}_k \) and \( \hat{z}_{N_2-\ell} = \hat{z}_\ell \).
3. APPLICATIONS

We now illustrate through examples the application of the 2D-WDFT to the tuning of 2D FIR filters. Without any loss of generality we consider here only the 2D WDFT obtained by the spectral transformation of one variable.

The prototype filter is a very narrow-band 2D lowpass FIR filter (see figure 1). Three types of basic transformations are considered, i.e., lowpass-to-lowpass transformation, lowpass-to-highpass transformation and lowpass-to-bandpass transformation.

We restrict our attention here only to the case of spectral transformations based on first-order allpass functions. However, higher order allpass transformations have been dealt in the theory in the past [6, 7] and they can be applied to the WDFT. We demonstrate the application of the 2D-WDFT in the tuning of 2D digital filters. One-dimensional tunable FIR filter design using warped DFT has been dealt in [2].

It follows from Eq. (8) that the transfer function of a 2D FIR digital filter obtained after a spectral transformation is a rational function of $\hat{z}_1$ and $\hat{z}_2$ implying that the transformed filter has an infinite impulse response. If the tuning range is very small, we assume that the impulse response decays to zero values very rapidly. Hence, an exact $N_1N_2$-point 2D inverse DFT of the 2D WDFT samples yields the 2D impulse response of the transformed filter.

3.1 Lowpass-to-Lowpass transformation

A first-order lowpass-to-lowpass transformation is given by

$$\hat{z}_i^{-1} \rightarrow A_i(\hat{z}_i) = \frac{d_i\hat{z}_i^{-1}}{1 + d_i\hat{z}_i^{-1}}, \quad i = 1,2. \quad (12)$$

where $|d_i| < 1$ for stability [8]. Substituting for $z_j = e^{j\omega_j}$, $z = e^{j\omega}$ and $\hat{z} = e^{j\omega}$ in Eq. (12), we get after some algebra the relation between the warped frequency and original frequency as

$$\tan\left(\frac{\omega_i}{2}\right) = \frac{1 + d_i}{1 - d_i} \tan\left(\frac{\omega}{2}\right). \quad (13)$$

where $d_i$ acts as a parameter that controls the amount of shift of the cut-off frequency. For negative $d_i$, the frequency points are stretched along the axis. In our particular case, we shift the cut-off frequency to 0.4 $\pi$ from 0.2 $\pi$. The required value of $d_i$ for this shift is given by -0.382. We have tuned the filter in one direction only. We observe that in the other direction the properties of the original filter are not altered where the passband width remains the same as the original one (as we have not applied warping on this axis). The result is illustrated in Figure 2.

![Figure 1](image1.png)  
**Figure 1.** The prototype narrow-band 2-d low-pass filter with cut-off at 0.2 $\pi$

![Figure 2](image2.png)  
**Figure 2.** Tuned low-pass filter with pass band modified to 0.4 $\pi$. Tuning is only along the horizontal frequency direction for better viewing.

3.2 Lowpass-to-Highpass Transformation

The first-order lowpass-to-highpass transformation is given by

$$\hat{z}^{-1} \rightarrow \frac{1 - d_i\hat{z}^{-1}}{1 + d_i\hat{z}^{-1}}, \quad (14)$$

which leads to the relation between the two frequencies as,

$$\tan\left(\frac{\omega}{2}\right) = -\left[\frac{1 - d_i}{1 + d_i} \tan\left(\frac{\omega}{2}\right)^{-1}\right]. \quad (15)$$

Figure 3 shows the response of the obtained high-pass filter from the prototype filter. The cut-off frequency was intended to be at 0.5 $\pi$. For that, the value of the parameter $d_i$ was obtained to be -0.5095. Here again we have applied the tuning in one direction only and it is observed that the filter retains its original characteristics in the other direction including its passband.
3.3 Lowpass-to-Bandpass Transformation

The lowest-order non-trivial transformation is the second-order function given tentatively by,

\[ z^{-1} \rightarrow \frac{z^{-2} - \frac{2\alpha\beta}{\beta+1} z^{-1} + \frac{\beta-1}{\beta+1}}{\beta-1 z^{-2} - \frac{2\alpha\beta}{\beta+1} z^{-1} + 1}, \]  

where \( \alpha \) and \( \beta \) are two parameters that determine the center frequency and band of the resulting filter. However, if the passband of the original 2D digital filter is to be conserved, then Eq. (16) reduces to

\[ z^{-1} \rightarrow \frac{z^{-1} - \alpha + z^{-1}}{1 - \alpha z^{-1}}, \]  

where \( \alpha = \cos(\hat{\omega}_c) \) parameterizes the center frequency. For our case, we chose center frequency, \( \hat{\omega}_c \), as \( \pi/2 \) and the response of the transformed filter is shown in Figure 4. For all our considerations the prototype filter length is 1024 and we tested for the tuning in both the horizontal and vertical frequency directions and results obtained show perfect agreement with the predicted theoretical behavior.

4. CONCLUDING REMARKS

A novel 2D transform (2D WDFT) based on non-uniform discrete Fourier transform is presented in this paper. We have presented a scheme to compute the transform using an efficient and easy algorithm. The properties are discussed and the use of the WDFT in the tuning of 2D digital filters has been demonstrated. We believe that 2D WDFT will find applications in many more areas.

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