

Readings in Fourier Analysis on Finite Non-Abelian Groups

Radomir S. Stanković Claudio Moraga Jaakko Astola

TICSP Series #5
September 1999

TTKK, Monistamo
1999

TICSP Series #5

READINGS IN FOURIER ANALYSIS ON
FINITE NON-ABELIAN GROUPS

Radomir S. Stanković Claudio Moraga Jaakko Astola

TICSP Series

Editor **Jaakko Astola**
Tampere University of Technology

Editorial Board **Moncef Gabbouj**
Tampere University of Technology

Murat Kunt
Ecole Polytechnique Fédérale de Lausanne

Truong Nguyen
University of Wisconsin, Madison

- 1 EGIAZARIAN/SARAMÄKI/ASTOLA. Proceedings of Workshop on Transforms and Filter Banks.
- 2 YAROSLAVSKY. Target Location: Accuracy, Reliability and Optimal Adaptive Filters.
- 3 ASTOLA. Contributions to Workshop on Trends and Important Challenges in Signal Processing.
- 4 CREUTZBURG/ASTOLA. Proceedings of The Second International Workshop on Transforms and Filter Banks.

Tampere International Center for Signal Processing
Tampere University of Technology
P.O. Box 553
FIN-33101 Tampere
Finland

ISBN 952-15-0284-3
ISSN 1456-2774

TTKK, Monistamo
1999

Contents

1	Signals and Their Mathematical Modells	1
1.1	Summary	6
2	Fourier Analysis on non-Abelian groups	11
2.1	Fourier transform on finite non-Abelian groups	15
2.2	Properties of Fourier transform	20
2.3	Matrix interpretation of the Fourier transform on finite non-Abelian groups	22
2.4	Fast Fourier transform on finite non-Abelian groups	24
3	Matrix Interpretation of Fast Fourier Transform on Finite Non-Abelian Groups	31
3.1	Matrix interpretation of FFT on finite non-Abelian groups	33
3.2	Illustrative examples	36
3.3	FFT through decision diagrams	54
3.3.1	Decision diagrams	56
3.3.2	FFT on finite non-Abelian groups through DDs	58
3.3.3	MTDDs for the Fourier spectrum	62
3.3.4	Complexity of DDs calculation methods	62
4	Gibbs Derivatives on Finite Groups	69
4.1	Definition and properties of Gibbs derivatives on finite non-Abelian groups	70
4.2	Gibbs anti-derivative	73
4.3	Partial Gibbs derivatives	74
4.4	Gibbs differential equations	75
4.5	Matrix interpretation of Gibbs derivatives	78

4.6	Fast algorithms for calculation of Gibbs derivatives on finite groups	80
4.6.1	Complexity of Calculation of Gibbs Derivatives	88
4.7	Calculation of Gibbs derivatives through DDs	88
4.7.1	Calculation of partial Gibbs derivatives	91
5	Linear Systems and Gibbs Derivatives on Finite Non-Abelian Groups	99
5.1	Linear shift-invariant systems on groups	100
5.2	Linear shift-invariant systems on finite non-Abelian groups	102
5.3	Gibbs derivatives and linear systems	103
5.3.1	Discussion	104
6	Hilbert Transform on Finite Groups	109
6.1	Some results of Fourier analysis on finite non-Abelian groups	111
6.2	Hilbert transform on finite non-Abelian groups	115
6.3	Hilbert transform in finite fields	120

Preface

We are convinced that the group-theoretic approach to spectral techniques and in particular Fourier analysis offers some important advantages, among which the possibility for an unique consideration of various classes of signals is probably the most important. In particular, that approach is a mean to transfer some important very useful results from classical Fourier analysis on the real line to other algebraic structures and different classes of signals, discrete and digital signals on these structures. Among different possible groups, finite non-Abelian groups have found some interesting and useful applications in different areas of science and engineering practice. The possibly most important are those in electrical engineering and physics.

This monograph reviews some authors' research in the area of abstract harmonic analysis on finite non-Abelian groups. Most of the results discussed are already published in this or the restricted form or presented at conferences and published in conference proceedings.

We have attempted to present them here in a consistent, but self-contained way and uniform notation, but aware of repeating well-known results from abstract harmonic analysis, except those needed for derivation, discussion and appreciation of the results presented. However, the results are accompanied, where that was necessary or appropriate, with a short discussion including the comments concerning their relationship to the existing results in the area.

The aim of this monograph is, therefore, to provide a base for a further eventual study in abstract harmonic analysis on finite not necessarily Abelian groups, which should hopefully result into a further extending of the signal processing methods and techniques to signals modelled by functions on finite non-Abelian groups.

The authors would be grateful for comments on these results, especially those suggesting their improvement or concrete applications in science and engineering practice.

Outline

Pretensions with this book were to offer a condensed and short, but rather self-contained monograph considering some basic and new concepts in Fourier analysis on finite non-Abelian groups providing in that way a mean for a further study and development of signal processing methods, system theory and related topics on these structures.

In the first chapter some general comments about signals and their mathematical models are given offering also the reason and, therefore, explanation for the restriction of the consideration to discrete case and finite non-Abelian structures.

In attempting to determine the place of the concepts considered and to trace their relationship to related notions in a more general theory, the next chapter first briefly review some basic concepts of group representations and Fourier analysis on groups and, then, present the bases of Fourier analysis on finite non-Abelian groups.

As is usually done in the corresponding literature concerning Abelian groups, the discussion of fast algorithms for the calculation of Fourier transform on non-Abelian groups should convict in its efficiency in performing and, therefore, application. The matrix interpretation of fast Fourier transform on non-Abelian groups was intended to provide a mean for an unique consideration of fast algorithms on Abelian and non-Abelian groups and to offer, at the same time, a formalism for an entire use of characteristics inherent in such algorithms in different ways and various possible environments to perform them.

Differential calculus is the second of the two, in our opinion the most important tools in signal processing and related areas. Therefore, in chapter 4 the attention is focused to a special kind of differentiation, the Gibbs differentiation, closely related with Fourier analysis and efficiently characterized through Fourier coefficients.

Chapter 5 briefly introduces the concept of group theoretic models of linear shift invariant systems and discuss the relationship of Gibbs differentiators with such systems.

The Hilbert transform could be interpreted in classical analysis as a transform relating the real and imaginary parts of Fourier spectra of some classes of functions. In chapter 6 an attempt was made to discuss corresponding counterparts of such functions on finite non-Abelian groups and extend the Hilbert transform theory to these functions.

By referring to possible applications to digital signals, the presentation

in this monograph uniformly concerns the complex functions and functions taking their values in some finite fields admitting the existence of Fourier transform on the considered groups.

The chapters in this monograph are written to represent more or less independent units. In this way, the book permits fast reading. The reader can concentrate on the chapters discussing the topics of his particular interest and may skip the others still keeping the continuity in reading. Fig. 0.1 shows relationships among the chapters.

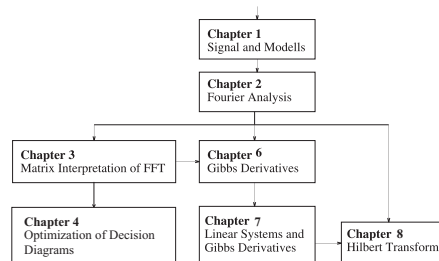


Figure 0.1: Relations among the Chapters.

Acknowledgment

Prof. Mark G. Karpovsky and Prof. Lazar A. Trachtenberg have traced in a series of publications chief directions in research in Fourier analysis on finite non-Abelian groups that we are following in our research in the area, in particular in extending the theory of Gibbs differentiation to non-Abelian structures. For that, we are very indebted to them both.

The first author is very grateful to Prof. Paul L. Butzer, Dr. J. Edmund Gibbs, and Prof. Tsutomu Sasao for continuous support in studying and research work.

A part of the work towards this monograph was done during the stay of R. S. Stanković at the Tampere International Center for Signal Processing (TICSP). The support and facilities provided by TICSP are gratefully acknowledged.

Definitions, Theorems, Examples, Tables and Figures

The following table gives the page numbers where some definitions (D), theorems (T), examples (E), tables (X) and figures (F) appear.

D.2.1	11	T.2.1	12	E.2.1	18	X.2.1	18	F.2.1	27
D.2.2	12	T.2.2	13	E.2.2	19	X.2.2	19	F.2.2	28
D.2.3	13	T.2.3	13	E.2.3	25	X.2.3	19	F.2.3	29
D.2.4	13	T.2.4	14	E.3.1	38	X.2.4	19	F.3.1	43
D.2.5	22	T.2.5	21	E.3.2	42	X.2.5	20	F.3.2	45
D.2.6	23	T.4.1	71	E.3.3	44	X.2.6	20	F.3.3	46
D.2.7	23	T.4.2.	74	E.3.4	48	X.2.7	20	F.3.4	47
D.3.1	64	T.4.3	75	E.3.5	51	X.2.8	25	F.3.5	50
D.3.2	65	T.4.4	76	E.3.6	64	X.2.9	26	F.3.6	53
D.4.1	79	T.6.1	161	E.3.7	65	X.2.10	29	F.3.7	55
D.4.2	79	T.6.2	164	E.3.8	67	X.3.1	39	F.3.8	56
D.4.3	90	T.6.3	165	E.4.1	91	X.3.2	41	F.3.9	57
D.4.4	96	T.6.4	166	E.4.2	92	X.3.3	41	F.3.10	60
D.4.5	109	T.8.1	206	E.4.3	94	X.3.4	49	F.3.11	62
D.4.6	110	T.8.2	210	E.4.4	98	X.3.5	51	F.3.12	63
D.4.7	111			E.4.5	100	X.3.6	52	F.3.13	65
D.4.8	112			E.4.6	111	X.3.7	60	F.3.14	66
D.4.9	114			E.4.7	111	X.3.8	61	F.3.15	70
D.6.1	160			E.4.8	115	X.3.9	61	F.3.16	71
D.6.2	163			E.4.9	120	X.3.10	61	F.4.1	80
D.6.3	165			E.4.10	122	X.3.11	72	F.4.2	80
D.6.4	166			E.4.11	127	X.4.1	98	F.4.3	81
D.6.5	168			E.5.1	132	X.4.2	101	F.4.4	82
D.6.6	168			E.5.2	133	X.4.3	102	F.4.5	82
D.7.1	190			E.5.3	134	X.4.4	104	F.4.6	83
D.7.2	192			E.5.4	138	X.4.5	105	F.4.7	83
D.8.1	199			E.5.5	139	X.4.6	107	F.4.8	84
D.8.2	202			E.5.6	141	X.4.7	108	F.4.9	84
D.8.3	202			E.5.7	142	X.4.8	108	F.4.10	87
D.8.4	206			E.5.8	144	X.4.9	115	F.4.11	87
D.8.5	210			E.5.9	146	X.10	118	F.4.12	91
D.8.6	210			E.5.10	148	X.4.11	122	F.4.13	92

CONTENTS

vii

E.5.111	152	X.4.12	125	F.4.14	93
E.5.12	154	X.4.13	128	F.4.15	93
E.6.1	172	X.4.14	129	F.4.16	95
E.6.2	173	X.4.15	129	F.4.17	95
E.6.3	182	X.5.1	147	F.4.18	96
E.8.1	209	X.5.2	147	F.4.19	99
E.8.2	212	X.5.3	149	F.4.20	103
		X.6.1	173	F.4.21	116
		X.6.2	176	F.4.22	116
		X.6.3	177	F.4.23	117
		X.8.1	2029	F.4.24	117
		X.8.2	210	F.4.25	119
		X.8.3	212	F.4.26	119
		X.8.4	213	F.4.27	119
		X.8.5	213	F.4.28	119
				F.4.29	120
				F.4.30	123
				F.4.31	123
				F.4.32	123
				F.4.33	123
				F.5.1	151
				F.5.2	153
				F.5.3	153
				F.5.4	155
				F.5.5	155
				F.5.6	156
				F.6.1	173
				F.6.2	174
				F.6.3	175
				F.6.4	178
				F.6.5	179
				F.6.6	179
				F.6.7	180
				F.6.8	185
				F.7.2	191

Chapter 1

Signals and Their Mathematical Modells

Our communication with our environment is carried out through signals we perceive and we emit. In that way, our comprehension of the world around is based upon the signals which are conveyors of information. After a signal is registered, it should be processed to extract the information coded in it.

Generally, the signals are physical processes which spread in space-time. A signal can be observed directly in the form and at the moment in which it appears. However, that could be not only inconvenient, but, in many cases, even impossible. In any way, we are always ultimately condemned to finitness and discretness in the observation and study of signals: each signal can be observed only in a finite time interval irrespective to its real duration, and its parameters have been measured, evaluated or estimated only discretely, since any of these activities requires some time, possibly very short, but still finite, to be performed.

The observation of a signal directly by our senses only is globally a rather very subjective procedure and certainly strongly depended on the sensitivity of particular senses which are limited by personal abilities of the observers. The subjectivity can be eliminated or reduced if the observation is carried out by some more or less sophisticated technical aids, but, even in that case, it can be given only under a limited precision determined by the technical performances of the tools used and, at the other hand, by the physical characteristics of the signal observed. It may be said that we are always faced with finitness, discretness and an ultimate finite precision as the constrains imposed by the physical, that means materialistic, nature of the World.

An approach, alternative to direct observation and processing of signals, is to consider mathematical models instead of the signals themselves. It should be noted that we are subjected also in that case to the all limitations mentioned above. First, the mathematical modelling of a signal is, in essence, a procedure of approximation. Second, the mathematical modelling is based, hence highly dependent, on some a priori knowledge about the considered signals since it is performed thanks to the information gained by measuring or estimating of some characteristics of the signal or the characteristics and performances of the system emitting the signals, or the environment which the signal spread through. Therefore, in getting that knowledge, we are already subjected to the constraints and limitations discussed above. Moreover, the very most of mathematical models often appreciate only one of a few relevant parameters of a signal while all other its characteristics irrelevant or not very important in a particular application are disregarded. This is another approximation which should be taken in the mathematical modelling of signals existing in reality, which, from an optimistic point of view, can be regarded as an appropriate, necessary simplification of the signal processing tasks.

We can conclude in that way that, despite of the constraints and approximations mentioned above, the approach works very good as the engineering practice confirms well.

Since the signals are physical processes which spread in space-time they should be modelled by elements of some function spaces. A conviction prevalent for a long time, was that the signals existing in reality can be represented adequately exclusively by the functions of real variables. However, that conviction was considerably changed by the recognition that the information can be transmitted in discrete forms, and that the information change in a system can be measured discretely [26], [33]. It could be claimed that this recognition was provoked by a wish for an extensive use of that time offered technology in communications and signal processing, nevertheless the essence of what is now called the sampling theorem was known to the earlier times mathematicians. The reader is referred to [4], [15], [19], [40] for some discussions about the history, different formulations, extensions and generalizations of the sampling theorem. For sampling theorem on the dyadic group see [31] and later [20]. The extension of the theory to arbitrary locally compact Abelian group is given in [25]. An interpretation of sampling theorem in Fourier analysis on finite dyadic group is given in [29] and extended to arbitrary finite Abelian and non-Abelian groups in [39] and [40], respectively.

In engineering practice the signals modelled by real variable functions are usually called continuous signals. Those represented by discrete function, i.e., by functions whose variables are taken from discrete sets, are called discrete signals. If some restrictions are imposed also on the amplitude of functions, then we can distinguish two subclasses of continuous and discrete signals.

The continuous signals of a real amplitude are analog signals, while the discrete signals whose amplitudes belong to some finite sets are digital signals.

In engineering practice the signals are often identified with their mathematical models. In that setting, the continuous signals are considered as mappings $f : R \rightarrow R$, or $f : R \rightarrow C$, where R and C are the real and complex field, respectively. Therefore, the mathematical tools for the processing of these signals are taken from classical mathematical analysis and in particular from their parts called Fourier analysis and differential calculus.

To take advantage of having similar powerful mathematical operators in dealing with discrete signals, it is very convenient to impose some algebraic structure on their domain as well as range or on the set of all possible functions of given characteristics. In this setting the signals are conveniently regarded as functions on groups into fields. Moreover, as is shown in [42], the structure of a group is the weakest structure which can be imposed on the domain of a signal still providing a practically tractable model for the most of signal processing and system theory tasks.

Discrete signals are considered as signals on some discrete groups, usually identified with the group of integers Z , or with some of its subgroups Z_p of integers modulo less than some given p . In the other words, the discrete signals are considered as mappings $f : Z \rightarrow X$, or $f : Z_p \rightarrow Z_p$, where X could be the complex field C , the real field R , or Z or some finite field. In this way the discrete signals are often considered as functions on some locally compact Abelian groups, uniformly with the continuous signals regarded as signals on the real group R . For example, among different Abelian groups, the dyadic group and finite dyadic groups G_{2^n} $n \in N$ have attained a lot of attention, see for example [1], [3], since the Walsh functions [44], the groups characters of these groups [11], and their discrete counterparts, the discrete Walsh functions, take two values $+1$ and -1 and, therefore, the calculation of the Walsh-Fourier spectra can be carried out without multiplication.

However, there are real-life signals and systems which are naturally modelled by functions and, respectively, relations between functions on non-Abelian groups. We will mention those related with electrical engineering

practice.

As is noted in [22], some relevant examples are a problem of pattern recognition for two colored pictures, which may be considered as a problem of realization of binary matrices, a problem of synthesis of rearrangeable switching networks whose outputs depend on the permutation of input terminals [14], [30], a problem of interconnecting telephone lines, etc. An application of non-Abelian groups in linear systems theory can be found in the approximation of a linear time-invariant system by a system whose input and output are functions defined on non-Abelian groups [23]. See, also [35], [36], [37], [38], [42].

The application of non-Abelian groups in signal filtering is discussed in [24], where a general model of a suboptimal Wiener filter over a group is defined. It is shown that, with respect to some criteria, the use of non-Abelian groups may be more advantageous than the use of an Abelian group. For example, in some cases the use of Fourier transform on various non-Abelian groups results in improving statistical performance of the filter as compared to DFT. See, also [41].

The fast Fourier transform on finite non-Abelian groups [22], [32], have been widely used in different applications [23], [24]. It may be said that in the world of finite non-Abelian groups the quaternion group has a role equal to that played by the finite dyadic group among Abelian groups [40]. Similarly as with the Walsh transform, i.e., the Fourier transform on finite dyadic groups, the calculation of the Fourier transform on the quaternion does not require the multiplication. Regarding efficiency of fast Fourier transform on groups, it is shown [32] for sample evaluations with different groups that in a multiprocessor environment the use of non-Abelian groups, for example quaternions, may result in many cases in optimal, fastest, performance of FFT. Moreover, as is shown in [32], the quaternion groups as components of the direct product for G in many cases show the optimal performance as to the accuracy of calculation.

These performances are estimated taking into consideration the number of arithmetic operations, the number of interprocessor data transfers, and the number of communication lines operating in parallel. In this setting, looking for a suitable finite group structure G which should be imposed on the domain of a discrete signal, it is shown that the combination of small cyclic groups C_2 and non-Abelian quaternions in the direct product for G results in groups exhibiting in the most cases the fastest algorithms for the computation of Fourier transform.

Therefore, there is more than an academic interest in further study of

harmonic analysis on finite non-Abelian groups and in extending the signal processing methods to functions defined on finite non-Abelian groups.

In practical applications, we are often referring to topologic properties of algebraic structures we are dealing with to get mathematical models of signals and systems. The space-time topology of the produced solutions is deduced from the topology (in the mathematical sense) of the related algebraic structures. It was qualified [13] as an historic paradox that some important mathematical notions were introduced first on more complicated structures, and then extended or transferred to the simpler cases. Differential operators were mentioned as an example confirming this statement. Through the Newton-Leibniz derivative, the notion was introduced first for real functions, although the continuum of the real line R is one of the most sophisticated algebraic structures. Extension of differentiation to the probably simplest case of finite dyadic groups, was done at about two centuries after the appearance of first vague ideas of differentiators and their applications in estimating the rate of change and the direction of change of a signal [12]. Moreover, it was motivated by the requirements of technology related to the interest in various applications of two-valued discrete Walsh functions in transmitting and processing of binary coded signals and their realizations within prevalent two-stable state circuits environment. The support set of finite dyadic groups, the n -th order direct product of the basic cyclic group of order 2, produces a binary coding of the sequence of first non-negative integers less than 2^n , representing a base for the Boolean topologies often used in system design, including the logic design as a particular example of systems devoted the processing of a special class of signals, the logic signals [17], [18]. The restriction of the order to 2^n , and some other inconveniences of the Boolean topology, motivated the the recent interest in topologies derived from the binary coding of Fibonacci sequences and their applications [1], [7], [8], [9], [16],[34]. Use of these structures permits introduction of new Fourier-like transforms [1], [6], [10], enriching the class of transforms appearing in Nature and computers [43], [45], [46]. Various extensions and generalizations of the representations of discrete signals and spectral methods in terms of different systems of not necessarily orthogonal basic functions on Abelian groups [2], and the use of non-Abelian groups in signal processing and related areas, suggested probably first in the introduction of [21], offer new interesting research topics as is shown, for example, in [5], [27], [28]. For these reasons, we have found interesting to study Fourier transforms on finite non-Abelian groups, and Fourier-like or generalized discrete Fourier transforms [23] on direct product of finite not necessarily Abelian

groups. These transform are defined in terms of basic functions generated as the Kronecker product of unitary irreducible representations of subgroups in the domain groups. This way of generalization of Fourier transform ensures the existence of fast algorithms for efficient, in terms of space and time, calculation of spectral coefficients. We uniformly denote all these transform as the Fourier transforms, with the excuse that efficient computation is, in many applications, stronger requirement, than possessing counterparts of all the nice properties of Fourier transform on R . We also consider the Gibbs derivatives on finite non-Abelian groups, since they extend the notion of differentiation to functions on finite groups through a generalization of the relationship between the Newton-Leibniz derivative and Fourier coefficients in Fourier analysis on R .

1.1 Summary

It may be said that a customary practice is to identify the signals with their mathematical models and, thus, distinguish the following classes of signals in engineering practice:

1. Signals described by real-variable functions are called continuous signals. The continuous signals of continuous amplitude are called analog signals.
2. Discrete signals are described by functions on discrete sets, usually identified with the set of integers Z , or some of its subsets Z_q of integers less than some given q in the case of periodic signals or signals defined on finite sets.
3. Discrete signals taking their values in finite sets, i.e., of quantized finite amplitude are called digital signals.

As is noted above, the group structures is the weakest algebraic structure which can be imposed on the set upon which a signal is defined.

The signals are often considered as conveyors of information about the state of a system. In this setting, the input and output signals of a system are related through the input-output relation describing the functioning and behavior of the system.

The corresponding classes of systems can be distinguished depending on the input and output signals: continuous, discrete and digital systems.

In what follows the attention will be focused to signals and systems defined on finite non-Abelian groups.

References

- [1] Aгаian, S., Astola, J., Egiazarian, K., *Binary Polynomial Transforms and Non-linear Digital Filters*, Marcel Dekker, 1995.
- [2] Aizenberg, N.N., Aizenberg, I.N., Egiazarian, K., Astola, J., "An introduction to algebraic theory of discrete signals", *Proc. First Int. Workshop on Transforms and filter Banks*, TICSP Series, No. 1, June 1998, 70-94.
- [3] Beauchamp, K.G., *Applications of Walsh and Related Functions: With an Introduction to Sequency Theory*, Academic Press, New York, 1984.
- [4] Butzer, P.L., Splettstösser, W., Stens, R.L., "The sampling theorem and linear prediction in signal analysis", *Jber. d. Dt. Math. -Verein.*, 90, 1988, 1-70.
- [5] Creutzburg, R., Labunets, V.G., Labunets, E.V., "Algebraic foundations of an abstract harmonic analysis of signals and systems", *Proc. First Int. Workshop on Transforms and Filter Banks*, TICSP Series, No. 1, June 1998, 30-69.
- [6] Egiazarian, K., Astola, J., "Discrete orthogonal transforms based on Fibonacci-type recursion", *Proc. IEEE Digital Signal Processing Workshop (DSPWS-96)*, Norway, 1996.
- [7] Egiazarian, K., Astola, J., "Generalized Fibonacci cubes and trees for DSP applications", *Proc. ISCAS-96*, Atlanta, Georgia, May 1996.
- [8] Egiazarian, K., Astola, J., "On generalized Fibonacci cubes and unitary transforms", *Applicable Algebra in Engineering, Communication and Computing*, Vol. AAEECC 8, 1997, 371-377.
- [9] Egiazarian, K., Gevorkian, D., Astola, J., "Time-varying filter banks and multiresolution transforms based on generalized Fibonacci topology", *Proc. 5th IEEE Int. Workshop on Intelligent Signal Proc. and Communication Systems*, Kuala Lumpur, Malaysia, 11-13 Nov. 1997, S16.5.1-S16.5.4.
- [10] Egiazarian, K., Astola, J., Aгаian, S., "Orthogonal transforms based on generalized Fibonacci recursions", *Proc. 2nd Int. Workshop on Spectral Techniques and Filter Banks*, Brandenburg, Germany, March 5-7, 1999.
- [11] Fine, N.J., "On the Walsh functions", *Trans. Amer. Math. Soc.*, Vol. 65, No. 3, 1949, 372-414.

- [12] Gibbs, J.E., "Walsh spectrometry, a form of spectral analysis well suited to binary digital computation", *NPL DES Rept.*, Teddington, Middlesex, United Kingdom, July 1967.
- [13] Gibbs, J.E., Simpson, J., "Differentiation on finite Abelian groups", *National Physical Lab.*, Teddington, England, DES Rept, No.14, 1974.
- [14] Harada, K., "Sequential permutation networks", *IEEE Trans.*, Vol.C-21, 472-479, 1972.
- [15] Higgins, J.R., "Five short stories about the cardinal series", *Bull Amer. Math. Soc.*, 12, 1985, 45-89.
- [16] Hsu, W.-J., "Fibonacci cubes - a new interconnection topology", *IEEE Trans. Parallel Distrib. Syst.*, Vol. 4, 1993, 3-12.
- [17] Hurst, S.L., *The Logical Processing of Digital Signals*, Crane Russak and Edward Arnold, Basel and Bristol, 1978.
- [18] Hurst, S.L., Miller, D.M., Muzio, J.C., *Spectral Techniques in Digital Logic*, Academic Press, 1985.
- [19] Jerri, A.J., "The Shannon sampling theorem - its various extensions and applications: a tutorial review", *proc. IEEE*, Vol. 65, No. 11, 1977, 1565-1596.
- [20] Kak, S.C., "Sampling theorem in Walsh-Fourier analysis", *Electron. Lett.*, Vol. 6, July 1970, 447-448.
- [21] Karpovsky, M.G., *Finite Orthogonal Series in the Design of Digital Devices*, John Wiley and Sons and JUP, New York and Jerusalem, 1976.
- [22] Karpovsky, M.G., "Fast Fourier transforms on finite non-Abelian groups", *IEEE Trans.*, Vol. C-26, 1028-1030, 1977.
- [23] Karpovsky, M.G., Trachtenberg, E.A., "Some optimization problems for convolution systems over finite groups", *Inform. and Control*, 34, 3, 227-247, 1977.
- [24] Karpovsky, M.G., Trachtenberg, E.A., "Statistical and computational performance of a class of generalized Wiener filters", *IEEE Trans. on Inform. Theory*, Vol. IT-32, 1986.
- [25] Ključevnik, I., "Sampling theorem in abstract harmonic analysis", *Mat.-Fiz. Časopis*, Sloven. Akad. Vied., 15, 1965, 43-48.
- [26] Kotelnikov, V.A., "On the carrying capacity of the 'ether' and wire in telecommunications", *Material for the First All-Union Conference on Questions of Communication*, Izd. Red. Upr. Svazy RKKA, Moscow, 1933, (in Russian).
- [27] Labunets, V.G., *Algebraic Theory of Signals and Systems - Computer Signal Processing*, Ural State University Press, Sverdlovsk, 1984, (in Russian).

- [28] Labunets, V.G., "Fast Fourier transform on generalized dihedral groups", in *Design Automation Theory and Methods*, Institute of Technical Cybernetics of the Belorussian Academy of Sciences Press, Minsk, 1985, 46-58, (in Russian).
- [29] Le Dinh, C.T., Le, P., Goulet, R., "Sampling expansions in discrete and finite Walsh-Fourier analysis", *Proc. 1972 Symp. Applic. Walsh Functions*, Washington, D.C., 1972, 265-271.
- [30] Opferman, D.C., Tsao, Wu, N.T., "On a class of rearrangeable switching networks", *Bell System Tech. J.*, 5C, 1579-1618, 1971.
- [31] Pichler, F.R., "Sampling theorem with respect to Walsh-Fourier analysis", Appendix B in *Reports Walsh Functions and Linear System Theory*, Elect. Eng. Dept., Univ. of Maryland, College Park, U.S.A., May 1970.
- [32] Roziner, T.D., Karpovsky, M.G., Trachtenberg, L.A., "Fast Fourier transform over finite groups by multiprocessor systems", *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-38, No. 2, 226-240, 1990.
- [33] Shannon, C.E., 'A mathematical theory of communication", *Bell Systems Tech. J.*, Vol. 27, 1948, 379-423.
- [34] Stakhov, A.P., *Algorithmic Measurement Theory*, Znanie, Moscow, No. 6, 1979, 64 p., (in Russian).
- [35] Stanković, R.S., "Linear harmonic translation invariant systems on finite non-Abelian groups", R. Trappl, Ed., *Cybernetics and Systems Research*, North-Holland, 1986.
- [36] Stanković, R.S., "A note on differential operators on finite non-Abelian groups," *Applicable Anal.*, 21, 1986, 31-41.
- [37] Stanković, R.S., "A note on spectral theory on finite non-Abelian groups," *3rd Int. Workshop on Spectral Techniques*, Univ. Dortmund, 1988, Oct. 4-6, 1988.
- [38] Stanković, R.S., "Fast algorithms for calculation of Gibbs derivatives on finite groups," *Approx. Theory and Its Applications*, 7, 2, June 1991, 1-19.
- [39] Stanković, R.S., Stanković, M.S., "Sampling expansions for complex-valued functions on finite Abelian groups", *Automatika*, 25, 3-4, 147-150, 1984.
- [40] Stanković, R.S., Stojić, M.R., Bogdanović, M.R., *Fourier Representation of Signals*, Naučna knjiga, Beograd, 1988, (in Serbian).
- [41] Trachtenberg, E.A., "Systems over finite groups as suboptimal filters: a comparative study", P.A. Fuhrmann, Ed., *Proc. 5th Int. Symp. Math. Theory of Systems and Networks*, Springer-Verlag, Beer-Sheva, Israel, 1983, 856-863.

- [42] Trachtenberg, E.A., “SVD of Frobenius matrices for approximate and multiobjective signal processing tasks”, in: E.F. Deprettere, Ed., *SVD and Signal Processing*, Elsevier North-Holland, Amsterdam/New York, 1988, 331-345.
- [43] Trakhtman, A.M., Trakhtman, V.A., *Fundamentals of the Theory of Discrete Signals Defined on Finite Intervals*, Sov. Radio, Moscow, 1975, (in Russian).
- [44] Walsh, J.L., “A closed set of orthogonal functions”, *Amer. J. Math.*, 45, 1923, 5-24.
- [45] Yaroslavsky, L., *Digital Picture Processing, an Introduction*, Springer Verlag, Heidelberg, 1985.
- [46] Yaroslavsky, L., “Transforms in Nature and computers: origin, discrete representation, synthesis and fast algorithms”, *Proc. first Int. Workshop on Transforms and Filter Banks*, TICSP Series, No. 1, June 1998, 3-29.

Chapter 2

Fourier Analysis on non-Abelian groups

In this chapter a brief introduction to the representation theory and harmonic analysis on not necessarily Abelian groups is given. For more details and proof of the theorems and statements presented, the reader is referred to [2], [6], [8], [16].

Let G be a locally compact topological group. That is, G is a Hausdorff, locally compact topological space with a group structure which is compatible with the topology in the sense that the group operations are continuous. Let V be a Hilbert space, that is a finite or infinite dimensional complex vector space with inner product (\cdot, \cdot) , which is complete with respect to the norm $\|v\| = (v, v)^{\frac{1}{2}}$, $v \in V$, derived from the inner product.

Definition 2.1. A mapping $x \rightarrow R(x)$ of G into the set $GL(V)$ of all linear bounded operators on V is a representation of G over V if

$$R(xy) = R(x)R(y), \quad (2.1)$$

$$R(e) = I, \quad (2.2)$$

where e is the identity of G , and I is the identity mapping.

The condition (2.1) implies that $x \rightarrow R(x)$ is a homomorphism of G into the set of linear operators on V , while the condition (2.2) provides that this representation is given by invertible operators, i.e.,

$$R(x)R(x^{-1}) = R(x^{-1})R(x) = R(e) = I,$$

from where

$$(R(x))^{-1} = R(x^{-1}).$$

A representation is unitary if each $R(x)$, $x \in G$, is an unitary operator over V , and is trivial if $R(x) = I$ for each $x \in G$.

The following theorem [2] states that in the case of compact groups it is sufficient to restrict the consideration just to the unitary representations without loss of generality.

Theorem 2.1. Let R be an arbitrary representation of a compact group G over V . Then, there is a new inner product in V defining a norm in V equivalent to the starting norm of V , relative to which $x \rightarrow R(x)$ is an unitary representation of G .

For practical application it is usually very convenient to deal with the matrix form of group representations. In that order, let $\{e_i\}$ $i \in \{1, \dots, N; N \leq \infty\}$ be an orthonormal base in V . The operator $R(x)$, $x \in G$ maps an element e_j of this base into $R(x)e_j \in V$, which can be represented by

$$R(x)e_j = D_{ij}(x)e_i, \quad j = 1, \dots, N.$$

Hence,

$$D_{ij}(x) = (R(x)e_j, e_i), \quad i, j = 1, \dots, N, \quad (2.3)$$

where (\cdot, \cdot) is the inner product in V .

It follows that $R(x)$ can be represented in the basis e_i , $i = 1, \dots, N$ by a finite or infinite matrix $\mathbf{D}_{ij}(x)$, with the matrix $\mathbf{D}_{ij}(xy)$ of $R(xy)$ equal to the matrix product of the matrices $\mathbf{D}_{ik}(x)$ and $\mathbf{D}_{kj}(y)$ corresponding to $R(x)$ and $R(y)$, respectively, i.e.,

$$\mathbf{D}_{ij}(x) = \mathbf{D}_{ik}(x)\mathbf{D}_{kj}(y).$$

The order of the matrix $\mathbf{D}_{i,j}(x)$ assigned to the w -th representation $R(x)$ is the dimension of $R(x)$, and is denoted by r_w .

It is shown that each matrix element (2.3) is a continuous function on G , see, for example [2].

Definition 2.2. Let V and V' be the Hilbert spaces.

A representation $x \rightarrow R_w(x)$ of the topological group G over V is equivalent to the representation $x \rightarrow R_q(x)$ over V' , which we denote by $R_w(x) \equiv R_q(x)$, if there is an bounded isomorphism S of V into V' such that

$$S \circ R_w(x) = R_q(x) \circ S, \quad \text{for each } x \in G.$$

Relative to the equivalence relation, the set of all representations of a given group G can be split into the equivalence classes.

A subspace or subset V_1 of V is called invariant relative to R if $R(x)V_1 \subset V_1, \forall R(x) \in GL(V)$.

Each representation exhibits at least two invariant subspaces: the null space O and the whole space V . These subspaces are called trivial subspaces. A non-trivial subspace or subset is called proper subspace or subset.

Definition 2.3. A representation $x \rightarrow R(x)$ of G over V is irreducible if there are no nontrivial invariant subsets of V .

The set of all nonequivalent irreducible representations of G will be denoted by Γ and will be called the dual object of G . Denote by K the cardinal number of Γ .

The following theorems hold for compact groups, see, for example, [8].

Theorem 2.2. Each unitary irreducible representation R for a compact group G is finite dimensional.

Theorem 2.3. Let R_w and R_q be two unitary irreducible representations of a compact group G . Then, for the matrix elements $R_w^{(i,j)}$ and $R_q^{(m,n)}$ holds

$$\int_G R_w^{(i,j)}(x)(R_q^{(m,n)})^* dx = \begin{cases} 0, & R_w \neq R_q, \\ r_w^{-1} \delta_{im} \delta_{jn}, & R_w \cong R_q' \end{cases} \quad (2.4)$$

where r_w is the dimension of R_w , and \mathbf{R}^* denotes the adjoint matrix, i.e., the transposed and complex conjugate of \mathbf{R} .

Recall that for an Abelian group G the Schur lemma states that each unitary representation is one-dimensional. In this setting, the character $\chi(w, x)$ of an Abelian group G is an one-dimensional continuous unitary representation of G over the complex field modulo 1. It follows that the dual object Γ is actually the set of all nonequivalent irreducible continuous representations of G . In the other words, Γ is the set of all characters of G and exhibits the structure of a multiplicative group isomorphic to G .

Since any non-Abelian group contains at least one representation of order $r_w > 1$, the concept of the characters of non-Abelian groups is introduced.

Definition 2.4. The character $\chi(w, x)$ of a finite dimensional representation R_w of a compact group G is defined as the trace of the operator $R_w(x)$, i.e.,

$$\chi(w, x) = Tr R_w(x) = (R_w(x)e_i, e_i) = D_{ii}(x).$$

The main properties of characters of unitary irreducible representations are:

1. $\chi(w, s^{-1} + y) = \chi(w, s)\chi(w, y)$,

2. $\chi(w, x^{-1}) = \chi^*(w, x)$,
3. If $R_w \cong R_q$, then $\chi(w, x) = \chi(q, x)$,
- 4.

$$\int_G \chi(w, x)\chi(q, x)dx = \begin{cases} 0, & R_w \neq R_q, \\ 1, & R_w \cong R_q. \end{cases}$$

The following theorem, in the literature known as the Peter-Weyl theorem [12], provides a base for an extension of the classical Fourier analysis on R to compact groups.

Theorem 2.4. Let Γ be the dual object of a compact group G , i.e., the set (in general infinite) of all irreducible nonequivalent representations. The functions

$$\sqrt{r_w}R_w^{(j,k)}(x), \quad w \in \Gamma, \quad 1 \leq j, k \leq r_w,$$

where r_w is the dimension of R_w and $R_w^{(j,k)}$ are the matrix elements of R_w , form a complete orthogonal system in $L^2(G)$.

This theorem provides a base for definition of the Fourier transform for a function $f \in L^2(G)$ where G is a compact not necessarily Abelian group. In this case the Fourier coefficients $S_f(\cdot)$ are $r_w \times r_w$ matrices whose elements are given by [2]

$$S_f^{(i,j)}(w) = \int_G f(u)(R_w^{(i,j)}(u))^* du. \quad (2.5)$$

The function f can be reconstructed from its Fourier coefficients as follows

$$f(x) = \sum_{w \in \Gamma} r_w \sum_{i,j=1}^{r_w} S_f^{(i,j)}(w)(R_w^{(i,j)}(x)), \quad (2.6)$$

where the convergence of this series is understood in the norm sense in $L^2(G)$.

Note that the uniform convergence can be achieved for the continuous functions instead of norm convergence in $L^2(G)$. In the matrix notation the relations (2.5) and (2.6) read as

$$S_f(w) = \int_G f(u)\mathbf{R}^*(u)du, \quad (2.7)$$

2.1. FOURIER TRANSFORM ON FINITE NON-ABELIAN GROUPS 15

$$f(x) = \sum_{w \in \Gamma} r_w \text{Tr}(\mathbf{S}_f(w) \mathbf{R}_w^T(x)). \quad (2.8)$$

where $\text{Tr} \mathbf{A}$ denotes the trace of \mathbf{A} .

The Fourier transform on compact not necessarily Abelian groups possesses the most of the properties of classical Fourier transform on the real line, for example, the shift theorem, Parseval relation, etc. However, the convolution theorem holds only in one direction for non-Abelian groups, since the dual object Γ in that case does not exhibit any group structure and the multiplication in Γ can hardly be defined. In that case the convolution theorem reads as follows.

Convolution theorem. For a function $g \in C(G)$ and a function $f \in L^2(G)$ the convolution product is defined as

$$(g \star f)(x) = \int_G g(y) f(y^{-1}x) dy,$$

and the Fourier transform of the convolution product is given by

$$S_{g \star f}(w) = S_g(w) S_f(w).$$

2.1 Fourier transform on finite non-Abelian groups

Further consideration in this monograph will be restricted to finite non-Abelian groups and, therefore, in this section we will consider in more details the definition and chief properties of the Fourier transform on these groups.

Recall that any finite group is a compact group, so that all previously mentioned results hold also for finite groups. What should be done is to replace the integral by a finite sum. In the case of finite groups the following holds:

1. Every irreducible representation of a finite group G is equivalent to some unitary representation.
2. Every irreducible representation is finite dimensional.
3. The number of non-equivalent irreducible representations R_w of a finite non-Abelian group G of order g is equal to the number of equivalence classes of the dual object Γ of G . Denoting this number by K it can be written

$$\sum_{w=0}^{K-1} r_w^2 = g.$$

We will use the following notation to discuss the definition and properties of the Fourier transform on finite non-Abelian groups.

Let G be a finite, not necessarily Abelian, group of order g . We associate permanently and bijectively with each group element a non-negative integer from the set $\{0, 1, \dots, g-1\}$, providing that 0 is associated with the group identity. In what follows, each group element will be identified with the fixed non-negative integer associated with it and with no other element. We assume that G can be represented as a direct product of subgroups G_1, \dots, G_n of orders g_1, \dots, g_n , respectively, i.e.,

$$G = \times_{i=1}^n G_i, \quad g = \prod_{i=1}^n g_i \quad g_1 \leq g_2 \leq \dots \leq g_n. \quad (2.9)$$

The convention adopted above for the denotation of group elements applies to the subgroups G_i as well. Provided that the notational bijections of the subgroups and of G are consistently chosen, each $x \in G$ can be uniquely represented as

$$x = \sum_{i=1}^n a_i x_i, \quad x_i \in G_i, \quad x \in G. \quad (2.10)$$

with

$$a_i = \begin{cases} \prod_{j=i+1}^n g_j, & i = 1, \dots, n-1 \\ 1, & i = n, \end{cases}$$

where g_j is the order of G_j .

The group operation \circ of G can be expressed in terms of the group operations $\overset{\circ}{i}$ of the subgroups G_i , $i = 1, \dots, n$ by:

$$x \circ y = (x_1 \overset{\circ}{1} y_1, x_2 \overset{\circ}{2} y_2, \dots, x_n \overset{\circ}{n} y_n), \quad x, y \in G, \quad x_i, y_i \in G_i. \quad (2.11)$$

Denote by P the complex field or a finite field. Henceforth it will be assumed that:

1. $\text{char } P = 0$, or $\text{char } P$ does not divide g .

2.1. FOURIER TRANSFORM ON FINITE NON-ABELIAN GROUPS 17

2. P is a so-called splitting field for G .

Recall that the complex field is the splitting field for any finite group. We denote by $P(G)$ the space of functions f mapping G into P , i.e., $f : G \rightarrow P$. Due to the assumption (2.9) and the relation (2.10), each function $f \in P(G)$ can be considered as an n -variable function $f(x_1, \dots, x_n)$, $x \in G$.

Let K be the number of equivalence classes of irreducible representations of G over P . Each such equivalence class contains just one unitary representation. We shall denote the K unitary irreducible representations of G in some fixed order by $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_{K-1}$. We denote by $R_w(x)$ the value of R at $x \in G$.

Note that $\mathbf{R}_w(x)$ stands for a non-singular r_w by r_w matrix, with elements $R_w^{(i,j)}(x)$, $i, j = 1, 2, \dots, r_w$.

If the group G is representable in the form (2.9), then its unitary irreducible representations can be obtained as the Kronecker product of the unitary irreducible representations of subgroups G_i , $i = 1, \dots, n$. Therefore, the number K of unitary irreducible representations of G can be expressed as,

$$K = \prod_{i=1}^n K_i, \quad (2.12)$$

where K_i is the number of unitary irreducible representations of i the subgroup G_i .

Now, for a given group G of the form (2.9), the index w of each unitary irreducible representation \mathbf{R}_w can be written as:

$$w = \sum_{i=1}^n b_i w_i, \quad w_i \in \{0, 1, \dots, K_i - 1\}, \quad w \in \{0, 1, \dots, K - 1\},$$

with

$$b_i = \begin{cases} \prod_{j=i+1}^n K_j, & i = 1, \dots, n-1, \\ 1, & i = n, \end{cases} \quad (2.13)$$

where K_j is the number of unitary irreducible representations of the subgroup G_j .

We denote by \mathbf{W}_i the vector of order K of the values of w_i considered as an n -variable discrete function.

As is noted above, the functions $R_w^{(i,j)}(x)$, $w = 0, 1, \dots, K-1$, $i, j = 1, \dots, r_w$ form an orthogonal system in the space $P(G)$. Therefore, the

direct and the inverse Fourier transform of a function $f \in P(G)$ are defined respectively by,

$$\mathbf{S}_f(w) = r_w g^{-1} \sum_{s=0}^{g-1} f(u) \mathbf{R}_w(u^{-1}), \quad (2.14)$$

$$f(x) = \sum_{w=0}^{K-1} Tr(\mathbf{S}_f(w) \mathbf{R}_w(x)). \quad (2.15)$$

Here and in the sequel we shall assume, without explicitly saying so, that all arithmetical operations are carried out in the field P .

Note that if in (2.9), there are two equal non-Abelian subgroups $G_i = G_j$, $i \neq j$, then the Kronecker product of unitary irreducible representations of the subgroups does not produce the unitary irreducible representations of G . However, the transform defined in terms of such a generated set of linearly independent functions is still denoted as the Fourier (generalized) transform, with the generalization achieved through the formal application of the decomposition used in FFT-like algorithms, see, for example [13].

Example 2.1. Let $S_3 = (0, (132), (123), (12), (13), (23), \circ)$ be the symmetric group of permutations of order 3. According to the convention adopted in this book, the group elements of S_3 will be denoted by 0,1,2,3,4,5, respectively. Using this notation the group operation of S_3 is shown in Table 2.1. The unitary irreducible representations of S_3 over C are given in Table 2.2. The group characters and the set $R_w^{(i,j)}$ of functions representing a basis in the space of complex on S_3 are given in Table 2.3 and Table 2.4.

Table 2.1. Group operation of S_3 .

\circ	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	0	5	3	4
2	2	0	1	4	5	3
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

2.1. FOURIER TRANSFORM ON FINITE NON-ABELIAN GROUPS 19

Table 2.2. The unitary irreducible representations of S_3 over C .

x	\mathbf{R}_0	\mathbf{R}_1	\mathbf{R}_2
0	1	1	\mathbf{I}
1	1	1	\mathbf{A}
2	1	1	\mathbf{B}
3	1	-1	\mathbf{C}
4	1	-1	\mathbf{D}
5	1	-1	\mathbf{E}

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A} = -2^{-1} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}, \quad \mathbf{B} = -2^{-1} \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{D} = -2^{-1} \begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}, \quad \mathbf{E} = -2^{-1} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix},$$

Table 2.3. The group characters of S_3 over C .

x	χ_0	χ_1	χ_2
0	1	1	2
1	1	1	2
2	1	1	2
3	1	-1	0
4	1	-1	0
5	1	-1	0

Table 2.4. The set $R_w^{(i,j)}(x)$ of S_3 over C .

x	R_0	R_1	$R_2^{(0,0)}$	$R_2^{(0,1)}$	$R_2^{(1,0)}$	$R_2^{(1,1)}$
0	1	1	1	0	0	1
1	1	$-\frac{1}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
2	1	1	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
3	1	-1	1	0	0	-1
4	1	-1	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
5	1	-1	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

Example 2.2. The unitary irreducible representations of S_3 over the Galois field $GF(11)$ are shown in Table 2.5. The group characters and the basis $R_w^{(i,j)}$ of S_3 over $GF(11)$ are given in Table 2.6 and Table 2.7.

Table 2.5. Unitary irreducible representations of S_3 over $GF(11)$.

x	\mathbf{R}_0	\mathbf{R}_1	\mathbf{R}_2
0	1	1	\mathbf{I}
1	1	1	\mathbf{A}
2	1	1	\mathbf{B}
3	1	10	\mathbf{C}
4	1	10	\mathbf{D}
5	1	10	\mathbf{E}

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 5 & 8 \\ 8 & 6 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 5 & 3 \\ 3 & 6 \end{bmatrix}$$

Table 2.6. The group characters of S_3 over $GF(11)$.

x	χ_0	χ_1	χ_2
0	1	1	2
1	1	1	10
2	1	1	10
3	1	10	0
4	1	10	0
5	1	10	0

Table 2.7. The set $R_w^{(i,j)}(x)$ of S_3 over $GF(11)$.

x	R_0	R_1	$R_2^{(0,0)}$	$R_2^{(0,1)}$	$R_2^{(1,0)}$	$R_2^{(1,1)}$
0	1	1	1	0	0	1
1	1	1	5	8	3	5
2	1	1	5	3	8	5
3	1	10	1	0	0	10
4	1	10	5	8	8	6
5	1	10	5	3	3	6

2.2 Properties of Fourier transform

The chief properties of Fourier transform on finite non-Abelian groups are analog to that of the Fourier transform on Abelian groups, as for example, of the Walsh transform [1], the Vilenkin-Chrestenson transform [4], [17], see

also [9], [11], and in particular to that of the DFT Fourier transform on the real line R [3].

Theorem 2.5. The chief properties of Fourier transform on finite non-Abelian groups are the following:

1. Linearity: For all $\alpha_1, \alpha_2 \in P$, $f_1, f_2 \in C(G)$,

$$S_{\alpha_1 f_1 + \alpha_2 f_2}(w) = \alpha_1 S_{f_1}(w) + \alpha_2 S_{f_2}(w).$$

2. Right group translation: For all $\tau \in G$,

$$S_{f(x\tau)}(w) = R_w(\tau)S_f(w).$$

3. Group convolution: For two functions $f_1, f_2 \in C(G)$ the convolution is defined by

$$(f_1 * f_2)(\tau) = \sum_{x \in G} f_1(x)f_2(\tau^{-1}).$$

4. Relative to such defined convolution, the Fourier transform exhibits the following property

$$r_w g^{-1} S_{(f_1 * f_2)(\tau)}(w) = S_{f_1}(w)S_{f_2}(w).$$

It should be noted that unlike the Fourier transform on Abelian groups, a reverse statement cannot be formulated since the dual object Γ does not exhibit a group structure suitable for definition of a convolution of functions on Γ .

5. Parseval theorem: For all $f_1, f_2 \in P(G)$,

$$\sum_{x \in G} f_1(x)\bar{f}_2(x) = g \sum_{R_w \in K(G)} r_w^{-1} Tr(\mathbf{S}_{f_1}(w)\mathbf{S}_{f_2}^*(w)),$$

where \bar{f} denotes the complex-conjugate of f , $\mathbf{S}_{f_2}^*(\cdot)$ is the conjugate transpose of $\mathbf{S}_{f_2}(\cdot)$, i.e., $\mathbf{S}_{f_2}^*(\cdot) = (\overline{\mathbf{S}_{f_2}(\cdot)})^T$.

6. The Wiener-Khintchin theorem: For two functions $f_1, f_2 \in P(G)$, the cross-correlation function is defined by

$$R_{f_1, f_2}(\tau) = \sum_{x \in G} f_1(x)\overline{f_2(x\tau^{-1})}.$$

The autocorrelation function is the cross-correlation function for $f_1 = f_2$.

Denote by F_G and F_G^{-1} the direct and inverse Fourier transform on G , respectively, and by F_G^* the transform such that

$$(F_G^*(f))(w) = S_f^*(w).$$

With this notation the Wiener-Khinchin theorem on G is defined by

$$B_{f_1, f_2} = gF_G^{-1}(r_w^{-1}F_G(f_1F_G^*(f_2))).$$

Proof. Properties 1, 2 and 3 follow immediately from the definition of the Fourier transform and its inverse.

The Parseval theorem can be proved by using the orthogonality relation (2.4) and the unitarity of $R_w(\cdot)$, i.e.,

$$\begin{aligned} & g \sum_{R_w \in K(G)} r_w^{-1} \text{Tr}(\mathbf{S}_{f_1}(w) \mathbf{S}_{f_2}^*(w)) \\ &= g^{-1} \sum_{R_w \in K(G)} r_w \text{Tr} \left(\left(\sum_{x_1 \in G} f_1(x_1) R_w(x_1^{-1}) \right) \left(\sum_{x_2 \in G} \overline{f_2(x_2)} R_w^*(x_2) \right) \right) \\ &= g^{-1} \sum_{x_1, x_2 \in G} f_1(x_1) \overline{f_2(x_2)} \sum_{R_w \in K(G)} r_w \text{Tr}(x_1^{-1} x_2) \\ &= \sum_{x \in G} f_1(x) \overline{f_2(x)}. \end{aligned}$$

The Wiener-Khintchin theorem follows from the unitarity of $R_w(x)$, the definition of the Fourier transform and the convolution theorem.

2.3 Matrix interpretation of the Fourier transform on finite non-Abelian groups

In our further consideration we need the generalized matrix multiplications defined as follows.

Definition 2.5. Let \mathbf{A} be an $(m \times n)$ matrix with elements $a_{ij} \in P$, $i \in \{0, 1, \dots, m-1\}$, $j \in \{0, 1, \dots, n-1\}$. Let $[\mathbf{B}]$ be an $(n \times r)$ matrix whose elements \mathbf{b}_{jk} , $j \in \{0, \dots, n-1\}$, $k \in \{0, 1, \dots, r-1\}$ are $(p \times p)$ matrices of not necessarily mutually equal orders with elements in P . We define the product $\mathbf{A} \odot [\mathbf{B}]$ as an $(m \times r)$ matrix $[\mathbf{Y}]$ whose elements y_{ik} , $i \in$

2.4 MATRIX INTERPRETATION OF THE FOURIER TRANSFORM 23

$\{0, 1, \dots, m-1\}$, $k \in \{0, 1, \dots, r-1\}$ are $(p \times q)$ matrices with elements in P given by

$$y_{ik} = \sum_{i=0}^{n-1} a_{ij} \mathbf{b}_{jk}.$$

The product $[\mathbf{B}] \odot \mathbf{A}$ is defined similarly.

Definition 2.6. Let $[\mathbf{Z}]$ be an $(m \times n)$ matrix whose elements \mathbf{z}_{ij} $i \in \{0, 1, \dots, m-1\}$, $j \in \{0, 1, \dots, n-1\}$ are the square matrices of not necessarily mutually equal orders with elements in P . Let $[\mathbf{B}]$ be an $(n \times r)$ matrix whose elements \mathbf{b}_{jk} , $j \in \{0, 1, \dots, n-1\}$, $k \in \{0, 1, \dots, r-1\}$ are square matrices of not necessarily mutually equal orders with elements in P . Under the condition that the matrices \mathbf{z}_{ij} and \mathbf{b}_{jk} are of the same order or, if not, that one of them is of the order 1, the product of matrices $[\mathbf{Z}]$ and $[\mathbf{B}]$ is defined as an $(m \times r)$ matrix $\mathbf{Y} = [\mathbf{Z}] \circ [\mathbf{B}]$ whose elements $y_{ik} \in P$ are given by

$$y_{ik} = \sum_{i=0}^{n-1} Tr(\mathbf{z}_{ik} \mathbf{b}_{jk}).$$

Definition 2.7. Let $[\mathbf{Z}]$ be an $(m \times n)$ matrix whose elements \mathbf{z}_{ij} , $i \in \{0, 1, \dots, m-1\}$, $j \in \{0, 1, \dots, n-1\}$ are $(p \times q)$ matrices of not necessarily mutually equal orders with elements in P . Let $[\mathbf{B}]$ be an $(n \times r)$ matrix whose elements \mathbf{b}_{jk} , $j \in \{0, 1, \dots, n-1\}$, $k \in \{0, 1, \dots, r-1\}$ are $(s \times t)$ matrices of not necessarily mutually equal orders with elements in P . The elementwise Kronecker product of matrices $[\mathbf{Z}]$ and $[\mathbf{B}]$ is defined as an $(m \times r)$ matrix $[\mathbf{V}] = [\mathbf{Z}] \overset{\circ}{\otimes} [\mathbf{B}]$ whose elements \mathbf{v}_{ik} are given by

$$\mathbf{v}_{ik} = \sum_{i=0}^{n-1} \mathbf{z}_{ij} \otimes \mathbf{b}_{jk},$$

where \otimes denotes the ordinary Kronecker product.

The first author introduced the generalized matrix multiplication concepts, needed to describe the Fourier transform and its inverse on finite non-Abelian groups, in [14], [15]. Although we have never met these definitions, we cannot be sure that they are not to be found in the voluminous literature on matrix calculus.

By using the matrix operations thus introduced, the Fourier transform pair defined by (2.14) and (2.15) can be expressed as follows.

Let $f \in P(G)$ be given as a vector $\mathbf{f} = [f(0), \dots, f(g-1)]^T$. Then its Fourier transform is given by

$$[\mathbf{S}_f] = g^{-1}[\mathbf{R}]^{-1} \odot \mathbf{f},$$

where $[\mathbf{S}_f] = [\mathbf{S}_f(0), \dots, \mathbf{S}_f(K-1)]^T$, and $[\mathbf{R}]^{-1} = [\mathbf{b}_{sq}]$ with $\mathbf{b}_{sq} = r_w \mathbf{R}_s^{-1}(q)$, $s \in \{0, 1, \dots, K-1\}$, $q \in \{0, 1, \dots, g-1\}$.

The inverse Fourier transform is given by

$$\mathbf{f} = [\mathbf{R}] \circ [\mathbf{S}_f],$$

where $[\mathbf{R}] = [\mathbf{a}_{ij}]$ with $\mathbf{a}_{ij} = \mathbf{R}_j(i)$, $i \in \{0, 1, \dots, g-1\}$, $j \in \{0, 1, \dots, K-1\}$.

2.4 Fast Fourier transform on finite non-Abelian groups

The formulation of the fast Fourier transform on a finite decomposable group G of the form (2.9) is based on the consideration of the Fourier transform on G as n -dimensional Fourier transform each of them relative to one of n subgroups G_j of G .

In this setting the Fourier transform can be written as:

$$\begin{aligned} S_f(w) &= S_f(w_1, \dots, w_n) \\ &= r_w g^{-1} \sum_{x_1} \sum_{x_2} \dots \sum_{x_n} f(x_1, x_2, \dots, x_n) \bigotimes_{j=1}^n R_{w_j}(x_j^{-1}) \\ &= r_w g^{-1} \sum_{x_n} (\dots (\sum_{x_2} (\sum_{x_1} (f(x_1, x_2, \dots, x_n) R_{w_1}(x_1^{-1})) \\ &\quad \otimes R_{w_2}(x_2^{-1})) \otimes \dots) \otimes R_{w_n}(x_n^{-1})). \end{aligned}$$

It follows that the Fourier transform can be performed in n steps defined as follows [10].

Step 1

$$\begin{aligned} f_1(x_1, \dots, x_n) &= f(x_1, \dots, x_n), \\ f_2(w_1, x_2, \dots, x_n) &= \sum_{x_1} f_1(x_1, \dots, x_n) R_{w_1}(x_1^{-1}). \end{aligned}$$

Step 2

$$f_3(w_1, w_2, x_3, \dots, x_n) = \sum_{x_2} f_2(w_1, x_2, \dots, x_n) R_{w_2}(x_2^{-1}).$$

Step j

$$f_j(w_1, \dots, w_j, x_{j+1}, \dots, x_n) = \sum_{x_j} f_{j-1}(w_1, \dots, w_{j-1}, x_j, \dots, x_n) R_{w_j}(x_j^{-1}).$$

Step n

$$f_{n+1}(w_1, \dots, w_n) = \sum_{x_n} f_n(w_1, \dots, x_n) R_{w_n}(x_n^{-1}),$$

$$S_f(w_1, \dots, w_n) = r_w g^{-1} f_{n+1}(w_1, \dots, w_n).$$

During the step j , the variables $w_1, \dots, w_{j-1}, x_{j+1}, \dots, x_n$ are fixed and the summation is performed through the variable $x_j \in G_j$.

The algorithm is probably the best explained by an example.

Example 2.3. Let G be the Quaternion (non-Abelian) group Q_2 of order 8. This group has two generators a and b and the group identity is denoted by e . If the group operation is written as abstract multiplication, the following relations hold for the group generators: $b^2 = a^2, bab^{-1} = a^{-1}, a^4 = e$. If the following bijection V is chosen

x	e	a	a^2	a^3	b	ab	a^2b	a^3b
$V(x)$	0	1	2	3	4	5	6	7

then the full group operation is described in Table 2.8.

Table 2.8. Group operation for the quaternion group Q_2 .

\circ	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	5	6	7	2	3	0	1
5	5	6	7	4	3	0	1	2
6	6	7	4	5	0	1	2	3
7	7	4	5	6	1	2	3	0

All the irreducible unitary representations are given in Table 2.9.

The dual object Γ of Q_2 is of order 5, since there are five irreducible unitary representations of this group.

Four of representations are 1-dimensional and one is 2-dimensional. Therefore, the Fourier spectra of a function f on Q_2 consists of five coefficients, four 1-dimensional and one 2-dimensional and can be represented as a vector

Table 2.9. Irreducible unitary representations of Q_2 over C .

x	\mathbf{R}_0	\mathbf{R}_1	\mathbf{R}_2	\mathbf{R}_3	\mathbf{R}_4
0	1	1	1	1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
1	1	1	-1	-1	$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$
2	1	1	1	1	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
3	1	1	-1	-1	$\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$
4	1	-1	1	-1	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
5	1	-1	-1	1	$\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$
6	1	-1	1	-1	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
7	1	-1	-1	1	$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$
	$r_0 = 1$	$r_1 = 1$	$r_2 = 1$	$r_3 = 1$	$r_4 = 2$

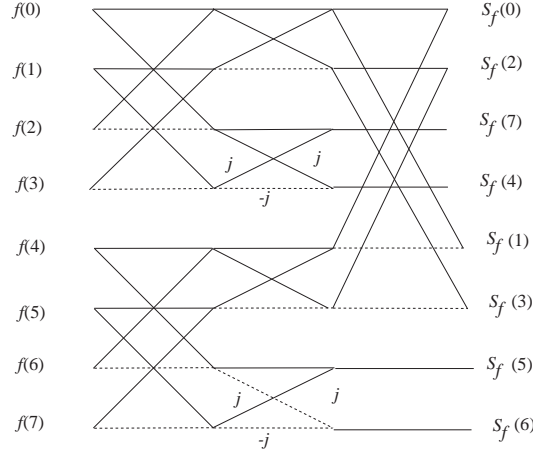


Figure 2.1: FFT on the quaternion group Q_2 .

$[\mathbf{S}_f] = [S_f(0) \ S_f(1) \ S_f(2) \ S_f(3) \ \mathbf{S}_f(4)]^T$. For example, the Fourier spectra of the function f on Q_2 given by the truth-vector $\mathbf{f} = [0\alpha 00\beta\lambda 00]^T$ is given by

$$[\mathbf{S}_f] = \begin{bmatrix} \alpha + \beta + \lambda \\ -\alpha + \beta - \lambda \\ \alpha - \beta - \lambda \\ -\alpha - \beta + \lambda \\ 2 \begin{bmatrix} -i\alpha & \beta + i\lambda \\ -\beta + i\lambda & i\alpha \end{bmatrix} \end{bmatrix}.$$

Direct computation of the Fourier transform requires 64 computation for the quaternion group $G = Q_2$. Using the fast Fourier transform it can be computed by using 20 additions. The multiplications by the complex unity i are not considered. The corresponding flow-graph is shown in Fig. 2.1.

The quaternion group Q_2 is a group structure which can be considered as the domain of signals defined on a set X_8 of eight elements. Thanks to the mapping

$$x = \sum_{i=0}^2 x_i 2^{3-i}, \quad x_i \in \{0, 1\},$$

Table 2.10. The discrete Walsh functions $wal(i, x)$.

i, x	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	-1	1	-1	1	-1	1	-1
2	1	1	-1	-1	1	1	-1	-1
3	1	-1	-1	1	1	-1	-1	1
4	1	1	1	1	-1	-1	-1	-1
5	1	-1	1	-1	-1	1	-1	1
6	1	1	-1	-1	-1	-1	1	1
7	1	-1	-1	1	-1	1	1	-1

the structure of an Abelian group which could be imposed on the same set can be the structure of the dyadic group of order 8.

Recall that the dyadic group of order 2^n consists of the set of binary n -tuples $x = (x_1, \dots, x_n)$, $x_i \in \{0, 1\}$, under the componentwise addition modulo 2. Discrete Walsh functions, the discrete version of Walsh functions are the characters of the dyadic groups [7] and, therefore, form a basis in the space of complex functions on X_8 . For $n = 3$ they are given in Table 2.10.

The algorithm for the computation of Fourier transform on Q_2 can be compared to the algorithm for computation of Fourier transform on the dyadic group of order 8 shown in Fig. 2.2. The number of additions and subtractions to compute Fourier transform on this group is 24 compared to 20 operations on Q_2 and 4 multiplications by the complex unity.

A method for an optimal implementation of the Fourier transform on finite not necessarily Abelian groups in a multiprocessor environment is presented in [13].

We already assumed that the ordering of the subgroups G_j , indicated in (2.9), is not essential from the theoretical point of view. However, it seems logical in some sense, and in our experience it proved as not only notationally convenient, but also useful in some practical considerations. Moreover, as is documented in [13], for a fixed set of constituent subgroups G_j their ordering as is required in (2.9) minimizes the number of data transfers in the implementation of the fast Fourier transform on G in a multiprocessor environment.

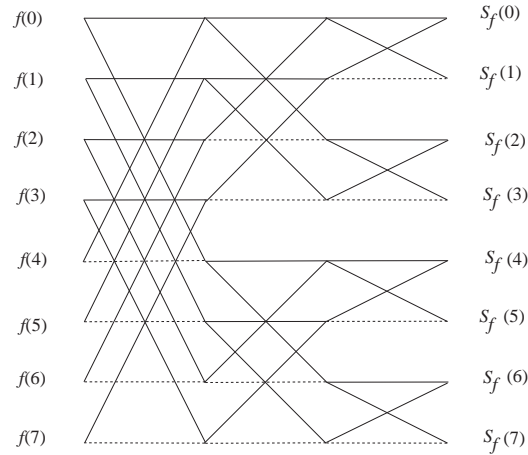


Figure 2.2: FFT on the dyadic group of order 8.

References

- [1] Beauchamp, K.G., *Applications of Walsh and Related Functions: With an Introduction to Sequency Theory*, Academic Press, New York, 1984.
- [2] Borut, A.O., Raczka, R.T., *Theory of Group Representations and Applications*, Warszawa, 1977.
- [3] Butzer, P.L., Nessel, R.J., *Fourier-Analysis and Approximation*, Vol. I, Birkhauser Verlag, Basel and Stuttgart, 1971.
- [4] Chrestenson, N.E., "A class of generalized functions", *Pacific J. Math.*, 5, 1955, 17-31.
- [5] Cooley, J.W., Tukey, C.W., "An algorithm for machine calculation of complex Fourier series", *Math. Computation*, Vol.19, 1965, 297-301.
- [6] Curtis, C.N., Rainer, I., *Representation Theory of Finite Groups and Associative Algebras*, Halsted, New York, London, 1962.
- [7] Fine, N.J., "On the Walsh functions", *Trans. Amer. Math. Soc.*, Vol.65, No.3, 1949, 374-414.
- [8] Hewitt, E., Ross, K.A., *Abstract Harmonic Analysis*, I,II, Springer-Verlag, Berlin, 1963, 1970.

- [9] Hurst, S.L., Miller, D.M., Muzio, J.C., *Spectral Techniques in Digital Logic*, Academic Press, Toronto, 1985.
- [10] Karpovsky, M.G., "Fast Fourier transforms on finite non-Abelian groups", *IEEE Trans.*, Vol.C-26, 1028-1030, 1977.
- [11] Moraga, C., "On a property of the Chrestenson spectrum", *IEE Proc.*, Vol.129, Pt. E, 1982, 217-222.
- [12] Peter, F., Weyl, H., "Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe", *Math. Ann.*, 97, 1927, 737-755.
- [13] Roziner, T.D., Karpovsky, M.G., Trachtenberg, L.A., "Fast Fourier transform over finite groups by multiprocessor systems", *IEEE Trans. Acoust., Speech, Signal Processing*, Vol.ASSP-38, No.2, 226-240, 1990.
- [14] Stanković, R.S., "Matrix interpretation of fast Fourier transform on finite non-Abelian groups", *Res. Rept. in Appl. Math.*, YU ISSN 0353-6491, Ser. Fourier Analysis, Rept. No.3, April 1990, 1-31, ISBN 86-81611-03-8.
- [15] Stanković, R.S., "Matrix interpretation of the fast Fourier transforms on finite non-Abelian groups", *Proc. Int. Conf. on Signal Processing, Beijing/90*, 22.-26.10.1990, Beijing, China, 1187-1190.
- [16] Stanković, R.S., Stojić, M.R., Bogdanović, M.R., *Fourier Representation of Signals*, Naučna knjiga, Beograd, 1988, (in Serbian).
- [17] Vilenkin, N.Ya., "A class of orthonormal series", *Izv. Akad. Nauk SSSR, Ser. Mat.*, Vol.11, 1947, 363-400.

Chapter 3

Matrix Interpretation of Fast Fourier Transform on Finite Non-Abelian Groups

There exist in each area of science some important concepts representing the corner stones of a whole theory and a corresponding practice the further study of which from different aspects never becomes out of interest. The algorithm for efficient computation of the discrete Fourier transform (DFT), generally known as the fast Fourier transform (FFT), is certainly such a concept in digital signal processing. Recall that the great practical application of DFT and its importance in signal processing steams from the publication of the world famous work by Coley and Tukey in 1965 [8]. Although the research community relatively recently was imprisoned by a very interesting discovery about the history of this algorithm [14].

Presently there is variety of algorithms in the quite voluminous literature on FFT, each of them suitable with respect to some a priori assumed criteria of optimality. These criteria are very different and range from the reduction of the time and memory resources needed for the computation to the use of some particular properties of the functions the DFT of which will be determined, or the use of the properties of spectral coefficients which should be calculated. Note for example, the real or pure imaginary functions, the symmetric functions, the functions with a lot of zero values, and similarly for spectral coefficients. For more detail see, for example, [1], [5],[22].

FFT algorithms are extended to be applicable to the calculation of the values of generalized Fourier transform on finite Abelian groups [2], [6] in-

cluding DFT as a particular example.

Some other particular examples of this theory, as the Walsh or Chrestenson transform, found also some important applications in different areas (see, for example, [3], [13], [15], [21]). Together with that, FFT algorithms were a base for the formulation of fast algorithms for the implementation of other discrete transforms on finite sets. Note as examples the discrete Haar transform, the slant transform, the discrete cosine transform (DCT), etc. More information about these algorithms can be found, for example, in [3] and the references mentioned there. Moreover, the practical applicability of the discrete transforms mentioned above and many others is greatly supported by the existence of the fast algorithms for their implementation. In some applications such algorithms are an ultimate request assumed in definition of a transform.

The matrix calculus appears to be the most convenient way for representation of discrete transforms and for dealing with them from the theoretical, practical and educational point of view as well.

Among different discrete transforms, the Fourier transform on finite non-Abelian groups is recommended recently as the best choice in some particular applications [17], [18], [19], [38], [39]. As we already noted in Section 2.4, a fast algorithm for the implementation of this transform based on the classical Coley-Tukey FFT [22], is formulated in an analytical form in [16]. It seems that an earlier related result on this subject can be found in [9]. Matrix interpretation of FFT on finite non-Abelian groups we will discuss was given in [28].

The Fourier transform on finite non-Abelian groups can be studied in a unique setting with the Fourier transform on Abelian groups as well as the classical Fourier transform in the frame of abstract harmonic analysis on groups. However, in the case of Fourier transform on non-Abelian groups there are some important differences which must be greatly appreciated at least in practical applications. According to this fact, our aim in this chapter is twofold. First, we consider a matrix representation of the fast Fourier transform on finite non-Abelian groups introduced in attempting to keep the entire analogy with the Abelian case as much as that is possible in the shape of the derived corresponding fast flow-graphs, and, then, we point out and discuss the chief differences of this transform with respect to the fast Fourier transform on finite Abelian groups.

3.1 Matrix interpretation of FFT on finite non-Abelian groups

To obtain a fast algorithm for the computation of the Fourier transform on finite non-Abelian groups we use the Good-Thomas method as in the case of the FFT on finite Abelian groups [11], [12], [36].

It is well known that the definition of the fast Fourier transform (FFT) on an Abelian group G (an algorithm for the efficient computation of the Fourier transform on G) is based upon the factorization of G into the equivalence classes relative to the subgroups of G . This group theoretical approach to the derivation of fast Fourier transform in the matrix notation can be interpreted as follows.

The disclosure of FFT on a finite Abelian group G of the form (2.9) is based upon the factorization of the Fourier transform matrix into a product of sparse factors. Such a factorization is possible since the Fourier transform matrix on a given Abelian group G of the form (2.9) is representable as the Kronecker product of the Fourier transform matrices on its subgroups G_i . As is noted in Section 2.1, the transforms whose basic functions are generated as the Kronecker products of unitary irreducible representations of equal non-Abelian subgroups are also considered as the generalized Fourier, or short, Fourier transforms on groups [23].

Each of the factor matrices describes uniquely one step of the fast algorithm implementing the Fourier transform with respect to one particular coordinate x_i , $i = 1, \dots, n$, determined by (2.10). In the other words, the i -th step of the FFT can be considered as the restriction of the Fourier transform on the whole group G to the Fourier transform on its i -th subgroup G_i . It follows that the i -th factor matrix can be represented as the Kronecker product of the Fourier transformation matrix on G_i of order g_i at i -th position and the identity matrices of orders g_j , $j \in \{1, \dots, n\} \setminus \{i\}$, at all other positions into that Kronecker product. We will extend the same approach to the non-Abelian groups by using the concepts of the generalized matrix multiplications.

The matrix $[\mathbf{R}]$ in the definition of the Fourier transform on finite non-Abelian groups is the matrix of unitary irreducible representations of G over P . Since G is representable in the form (2.9), the matrix $[\mathbf{R}]$ can be generated as the Kronecker product of $(K_i \times g_i)$ matrices $[\mathbf{R}_i]$ of unitary irreducible

representations of subgroups G , $i \in \{1, \dots, n\}$, i.e.,

$$[\mathbf{R}] = \bigotimes_{i=1}^n [\mathbf{R}_i].$$

Thanks to the well-known properties of the Kronecker product, the same applies to the matrix $[\mathbf{R}]^{-1}$, i.e., for this matrix holds

$$[\mathbf{R}]^{-1} = \bigotimes_{i=1}^n [\mathbf{R}_i]^{-1}.$$

This matrix further can be factorized into the elementwise Kronecker product of n sparse factors $[\mathbf{C}^i]$, $i \in \{1, \dots, n\}$ as

$$[\mathbf{C}^i] = \bigotimes_{j=1}^n [\mathbf{S}_j^i], \quad i = 1, \dots, n,$$

where

$$[\mathbf{S}_j^i] = \begin{cases} \mathbf{I}_{(g_j \times g_j)}, & j < i \\ [\mathbf{R}_j]^{-1}, & j = i, \\ \mathbf{I}_{(K_j \times K_j)}, & j > i. \end{cases} \quad (3.1)$$

where $\mathbf{I}_{a \times a}$ is an $(a \times a)$ identity matrix.

Each matrix $[\mathbf{C}^i]$ describes uniquely one step of the fast Fourier transform performed in n steps. The algorithm is best represented by a flow-graph consisting of nodes connected with branches to which some weights are associated.

The matrix representation and the corresponding fast algorithm obtained in such a way is similar to the FFT on finite Abelian groups, but some important differences appear here.

As it is known, see, for example [15], the flow-graph of the i -th step of the FFT on a finite Abelian group G of order g has g input and g output nodes. The output nodes of the $(i-1)$ -th step are the input nodes for the i -th step. The input nodes of the first step are the input nodes of the algorithm, and respectively, the output nodes of the n -th step are the output nodes of the algorithm except for the normalization with g^{-1} .

In the case of non-Abelian groups, the number of input and output nodes is different for the each step of the algorithm. Only the number of input nodes of the algorithm, i.e., the number of input nodes for the

first step of the algorithm equals g . The number of input nodes of the i -th step is $g_1 g_2 \dots g_{i-1} g_i K_{i+1} \dots K_n$, while the number of output nodes is $g_1 g_2 \dots g_{i-1} K_i K_{i+1} \dots K_n$. Accordingly, the number of output nodes of the algorithm is equal to K .

It is determined from the position of non-zero elements of $[\mathbf{C}^i]$ which nodes will be connected. The weights associated to the branches are equal to the values of these elements. An important difference with respect to FFT on Abelian groups is that in the case of non-Abelian groups the weights may be matrices, and therefore, according to the notation adopted in this paper, will be denoted by bold letters. Denote by $k(i, j)$ the branch connecting the output node i with the input node j in the flow-graph of the k -th step of the FFT on a finite group. The weight $\mathbf{q}^k(i, j)$ associated to this branch is determined by $\mathbf{q}^k(i, j) = \mathbf{C}_{ij}^{n-k}$, where \mathbf{C}_{ij}^{n-k} is the (i, j) -th element of the matrix $[\mathbf{C}^{n-k}]$. The branches for which the weight is equal to zero, i.e., the branches corresponding to zero elements of $[\mathbf{C}^{n-k}]$, do not appear in the flow-graph.

Now let us give a brief analysis of the complexity of the algorithm described here. The number of calculations is usually employed as a first approximation to the complexity of an algorithm.

Taking no into account the g input nodes, the number of nodes in the flow-graph described, and hence, the number of basic operations $L(G)$ in the FFT on a finite non-Abelian group G based on this flow-graph, is equal to

$$L(G) = \sum_{i=1}^n a_i,$$

where $a_i = g_1 g_2 \dots g_{i-1} K_i K_{i+1} \dots K_n$.

Here by a basic operation in the i -th step of algorithm we mean the operation given in a general form by

$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{c} \mathbf{F} \\ \mathbf{G} \end{array} \rightarrow \mathbf{C} = (\mathbf{A} \otimes \mathbf{F}) + (\mathbf{B} \otimes \mathbf{G})$$

where \mathbf{A} , \mathbf{B} are the matrices of order $\prod_{j=1}^{i-1} r_{w_j}$ while the weights \mathbf{F} and \mathbf{G} are the matrices of order r_{w_j} . To obtain the Fourier coefficients \mathbf{S}_f as they are defined by (2.14), it is needed to perform the normalization by g^{-1} after the calculation in the n -th step of algorithm is carried out.

Recall that in the case of Abelian groups, according to the definition of the unitary irreducible representations, all these matrices reduce to the numbers belonging to P , and hence, in that case the Kronecker product and matrix addition in P appearing in the basic operation defined here, reduce to the ordinary multiplication and addition in P .

Note that the flow-graph of the fast direct Fourier transform can be transformed into the flow-graph for the implementation of the inverse Fourier transform by a suitable mutual replacement of the input and output nodes, i.e., by considering the output nodes as the input nodes and vice versa. Clearly, the weights in this flow-graph are determined by the elements of the matrix $[\mathbf{R}]$ factorized as

$$[\mathbf{R}] = [\mathbf{D}^1] \overset{\circ}{\otimes} [\mathbf{D}^2] \overset{\circ}{\otimes} \dots \overset{\circ}{\otimes} [\mathbf{D}^n],$$

where

$$[\mathbf{D}^i] = \bigotimes_{j=1}^n [\mathbf{E}_j^i], \quad i = 1, \dots, n,$$

with

$$[\mathbf{E}_j^i] = \begin{cases} \mathbf{I}_{(K_j \times K_j)}, & j < i \\ [\mathbf{R}_j]^{-1}, & j = i, \\ \mathbf{I}_{(g_j \times g_j)}, & j > i. \end{cases}$$

The main differences of FFT on finite non-Abelian groups relative to FFT on finite Abelian groups are summarized in Table 3.1.

3.2 Illustrative examples

As it is usually case with the problems like that considered here, the algorithm is best explained by some examples.

Example 3.1. Let $G_{2 \times 8}$ be a given group of order 16. The elements of this group will be denoted according to the convention adopted in this monograph by 0,1,...,15. The identity of the group is 0, and the group operation is described in Table 3.2. All the unitary irreducible representations over the complex field C are given in Table 3.3. Note that in our notation, according to the definition of the inverse Fourier transform, the Table 3.3 defines the matrix $[\mathbf{R}]$ in (2.15).

Table 3.1. Summary of differences between FFT on finite Abelian and finite non-Abelian groups.

Group G of order g	
non-Abelian	Abelian
dual object	dual object
Γ -the set of unitary irreducible representations	$\{\chi\}$ -the set of group characters
Γ -does not have a group structure	$\{\chi\}$ -has the structure of a multiplicative group
Direct Fourier transform	
$\mathbf{S}_f(w) = g^{-1} r_w \sum_{x=0}^{g-1} f(x) \mathbf{R}_w(x^{-1})$,	$\hat{f}(w) = g^{-1} \sum_{x=0}^{g-1} f(x) \bar{\chi}(w, x)$
Number of spectral coefficients	
K	g
Inverse Fourier transform	
$f(x) = \sum_{w=0}^{K-1} Tr(\mathbf{S}_f(w) \mathbf{R}_w(x))$	$f(x) = \sum_{w=0}^{g-1} \hat{f}(x) \chi(w, x)$
Fourier transformation matrix	
$[\mathbf{R}]^{-1}$	$[\bar{\chi}]$
Order of the Fourier transformation matrix	
$K \times g$	$g \times g$
Direct Fourier transform in matrix notation	
$[\mathbf{S}_f] = g^{-1} [\mathbf{R}]^{-1} \odot \mathbf{f}$	$\hat{\mathbf{f}} = g^{-1} [\bar{\chi}] \mathbf{f}$
Inverse Fourier transform in matrix notation	
$\mathbf{f} = [\mathbf{R}] \circ [\mathbf{S}_f]$	$\mathbf{f} = [\chi] \hat{\mathbf{f}}$
Number of steps of the fast algorithm	
n	n
Number of input nodes	
1-st step g	g
i -th step $g_1 g_2 \dots g_{i-1} g_i K_{i+1} \dots K_n$	g
n -th step $g_1 K_2 \dots K_n$	g
Number of output nodes	
1-st step g	g
i -th step $g_1 g_2 \dots g_{i-1} K_i K_{i+1} \dots K_n$	g
n -th step K	g
Weights in the flow-graph	
The values of unitary irreducible representations	The values of group characters
$\{r_w \mathbf{R}^{-1}(x)\}$	$\{\bar{\chi}(\cdot)\}$
$w \in \{0, \dots, K-1\}, x \in \{0, \dots, g-1\}$	$w, x \in \{0, 1, \dots, g-1\}$
Factor of normalization after n -th step	
g^{-1}	g^{-1}

Note that the group $G_{2 \times 8}$ defined in this way can be considered as the direct product of the cyclic group C_2 of order 2 with modulo 2 addition as the group operation, and the quaternion group Q_2 of order 8.

The group representations of C_2 are $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, while these of the group Q_2 are in the left upper (5×8) quadrants in Table 3.3 by the dotted lines. The cardinality of the dual object Γ of $G_{2 \times 8}$ is $K = 10$, and, according to (2.12) can be represented as $K = K_1 K_2 = 2 \cdot 5$.

The transformation matrix of the Fourier transform on $G_{2 \times 8}$ can be factorized as follows:

$$[\mathbf{R}_{2 \times 8}]^{-1} = [\mathbf{C}^{-1}] \overset{\circ}{\otimes} [\mathbf{C}^2]$$

where

$$[\mathbf{C}^1] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \mathbf{I}_{5 \times 5}, \quad (3.2)$$

$$[\mathbf{C}^2] = \mathbf{I}_{(2 \times 2)} \otimes [\mathbf{Q}]^{-1}, \quad (3.3)$$

with

$$[\mathbf{Q}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 2\mathbf{I} & 2i\mathbf{B} & -2\mathbf{I} & 2i\mathbf{A} & 2\mathbf{E} & 2i\mathbf{D} & 2\mathbf{C} & -2i\mathbf{D} \end{bmatrix},$$

where the notation is as in Table 3.3.

The flow-graph of the fast Fourier transform on G_{16} corresponding to this factorization is shown in Fig. 3.1. For simplicity the weights corresponding to the branches of this flow-graph are not indicated in this figure. As we noted above, these weights are determined by the values of the elements of the matrices (3.3) and (3.2) for the first and the second step of the flow-graph, respectively. For example, $\mathbf{q}^1(5, 4) = 2i\mathbf{D}$ and $\mathbf{q}^2(3, 2) = 0$.

Table 3.2. Group operation of $G_{2 \times 8}$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	0	5	6	7	4	9	10	11	8	13	14	15	12
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	0	1	2	7	4	5	6	11	8	9	10	15	12	13	14
4	4	7	6	5	2	1	0	3	12	15	14	13	10	9	8	11
5	5	4	7	6	3	2	1	0	13	12	15	14	11	10	9	8
6	6	5	4	7	0	3	2	1	14	13	12	15	8	11	10	9
7	7	6	5	4	1	0	3	2	15	14	13	12	9	8	11	10
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	10	11	8	13	14	15	12	1	2	3	0	5	6	7	4
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	8	9	10	15	12	13	14	3	0	1	2	7	4	5	6
12	12	15	14	13	10	9	8	11	4	7	6	5	2	1	0	3
13	13	12	15	14	11	10	9	8	5	4	7	6	3	2	1	0
14	14	13	12	15	8	11	10	9	6	5	4	7	0	3	2	1
15	15	14	13	12	9	8	11	10	7	6	5	4	1	0	3	2

Table 3.3. Unitary irreducible representations of $G_{2 \times 8}$ over C .

x	\mathbf{R}_0	\mathbf{R}_1	\mathbf{R}_2	\mathbf{R}_3	\mathbf{R}_4	\mathbf{R}_5	\mathbf{R}_6	\mathbf{R}_7	\mathbf{R}_8	\mathbf{R}_9
0	1	1	1	1	\mathbf{I}	1	1	1	1	\mathbf{I}
1	1	-1	1	-1	$i\mathbf{A}$	1	-1	1	-1	$i\mathbf{A}$
2	1	1	1	1	$-\mathbf{I}$	1	1	1	1	$-\mathbf{I}$
3	1	-1	1	-1	$i\mathbf{B}$	1	-1	1	-1	$i\mathbf{B}$
4	1	1	-1	-1	\mathbf{C}	1	1	-1	-1	\mathbf{C}
5	1	-1	-1	1	$-i\mathbf{D}$	1	-1	-1	1	$-i\mathbf{D}$
6	1	1	-1	-1	\mathbf{E}	1	1	-1	-1	\mathbf{E}
7	1	-1	-1	1	$i\mathbf{D}$	1	-1	-1	1	$i\mathbf{D}$
8	1	1	1	1	\mathbf{I}	-1	-1	-1	-1	$-\mathbf{I}$
9	1	-1	1	-1	$i\mathbf{A}$	-1	1	-1	1	$i\mathbf{B}$
10	1	1	1	1	$-\mathbf{I}$	-1	-1	-1	-1	\mathbf{I}
11	1	-1	1	-1	$i\mathbf{B}$	-1	1	-1	1	$i\mathbf{A}$
12	1	1	-1	-1	\mathbf{C}	-1	-1	1	1	\mathbf{E}
13	1	-1	-1	1	$-i\mathbf{D}$	-1	1	1	-1	$i\mathbf{D}$
14	1	1	-1	-1	\mathbf{E}	-1	-1	1	1	\mathbf{C}
15	1	-1	-1	1	$i\mathbf{D}$	-1	1	1	-1	$-i\mathbf{D}$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{E} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

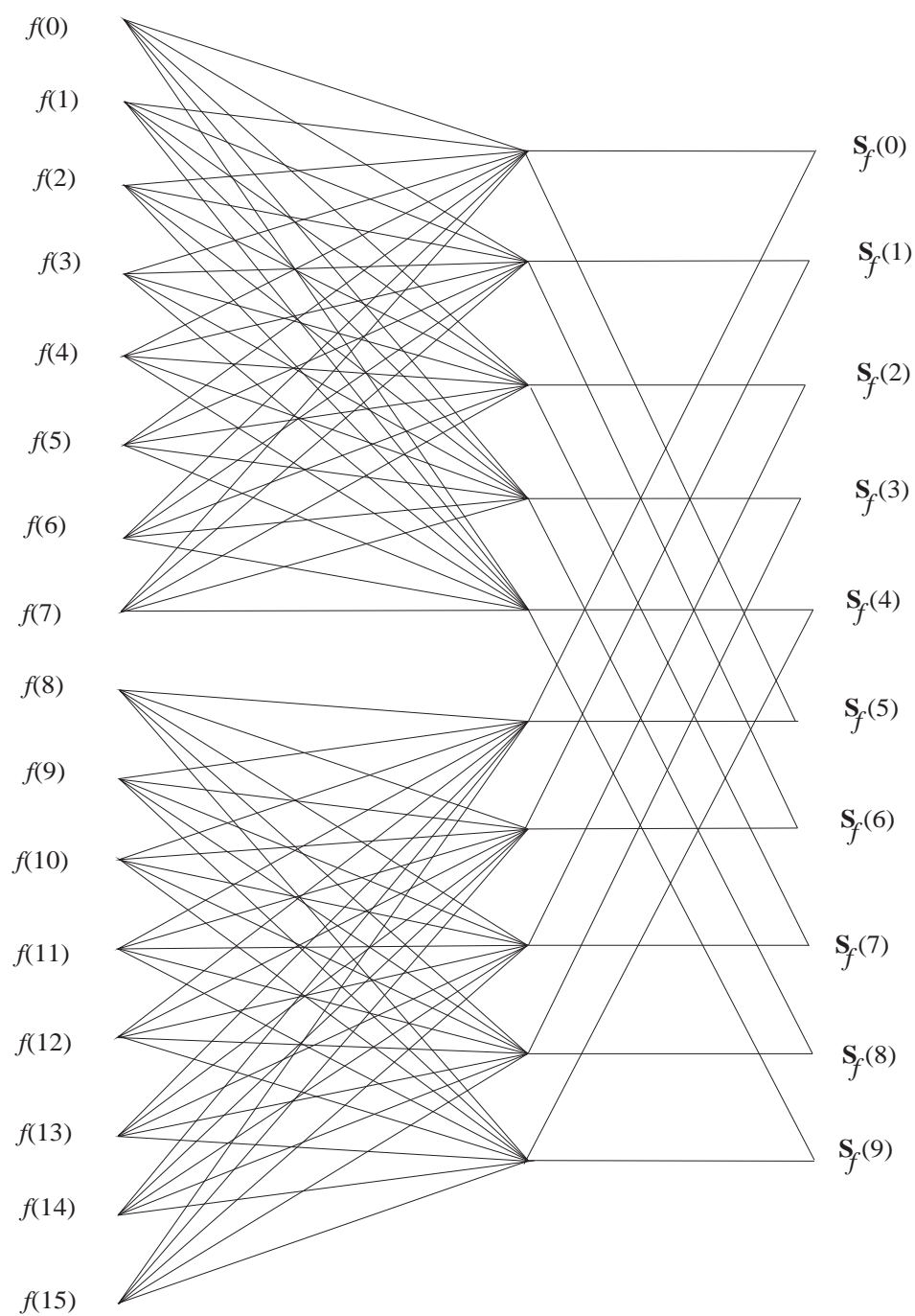
Note that in the example considered above the Fourier transform on the quaternion group Q_2 and the symmetric group of permutations of order 3, S_3 , is used as the basic module, and hence, is performed directly. However, the fast algorithms for the computation of Fourier transform on these groups can be used here at least for the calculation of the coefficients corresponding to the unitary irreducible representations of order 1 and using the matrix operations for the calculation of the remaining coefficients. Of course, by a complete use of these fast algorithms, and therefore, by avoiding the matrix operations, the algorithms shown here can be easily translated into the ordinary-like FFT algorithms like those used in the case of Abelian groups. A discussion of this approach is given in [23].

The presented matrix interpretation of FFT on finite non-Abelian groups requires the implementation of matrix operations in some branches of the algorithm. Therefore, the presented algorithms are efficient providing that the elements of the corresponding matrices are calculated simultaneously on some multiprocessor architectures. In the other words, the matrix interpretation assumes that the constituent subgroups G_i of G are used as the basic modules, as in the case of Abelian groups, the Fourier transform on G is decomposed into n Fourier transforms on G_i , where it is implemented directly.

However, if the calculation of matrix-valued Fourier coefficients is decomposed into the calculation of their matrix elements, extending in that way the size of each step of FFT into g , the fast algorithms for the calculation of FFT on some of the basic constituent subgroups can be applied by taking the advantage of some peculiar properties of group representations of the constituent subgroups. For example, the calculation of Fourier transform on the quaternion group Q_2 can be done without multiplication except the multiplication by the imaginary unit i , see Example 2.3. The statement will be illustrated by the following example.

Example 3.2. Let G_{32} be the direct product of two cyclic groups C_2 of order 2 with modulo 2 addition as the group operation and the quaternion group Q_2 of order 8, i.e., $G_{32} = C_2 \times C_2 \times Q_2$. The Fourier transformation matrix on this group over the complex field can be factorized as

$$[\mathbf{R}]^{-1} = [\mathbf{C}^1] \overset{\circ}{\otimes} [\mathbf{C}^2] \overset{\circ}{\otimes} [\mathbf{C}^3],$$

Figure 3.1: FFT on the group $Q_{2 \times 8}$.

where

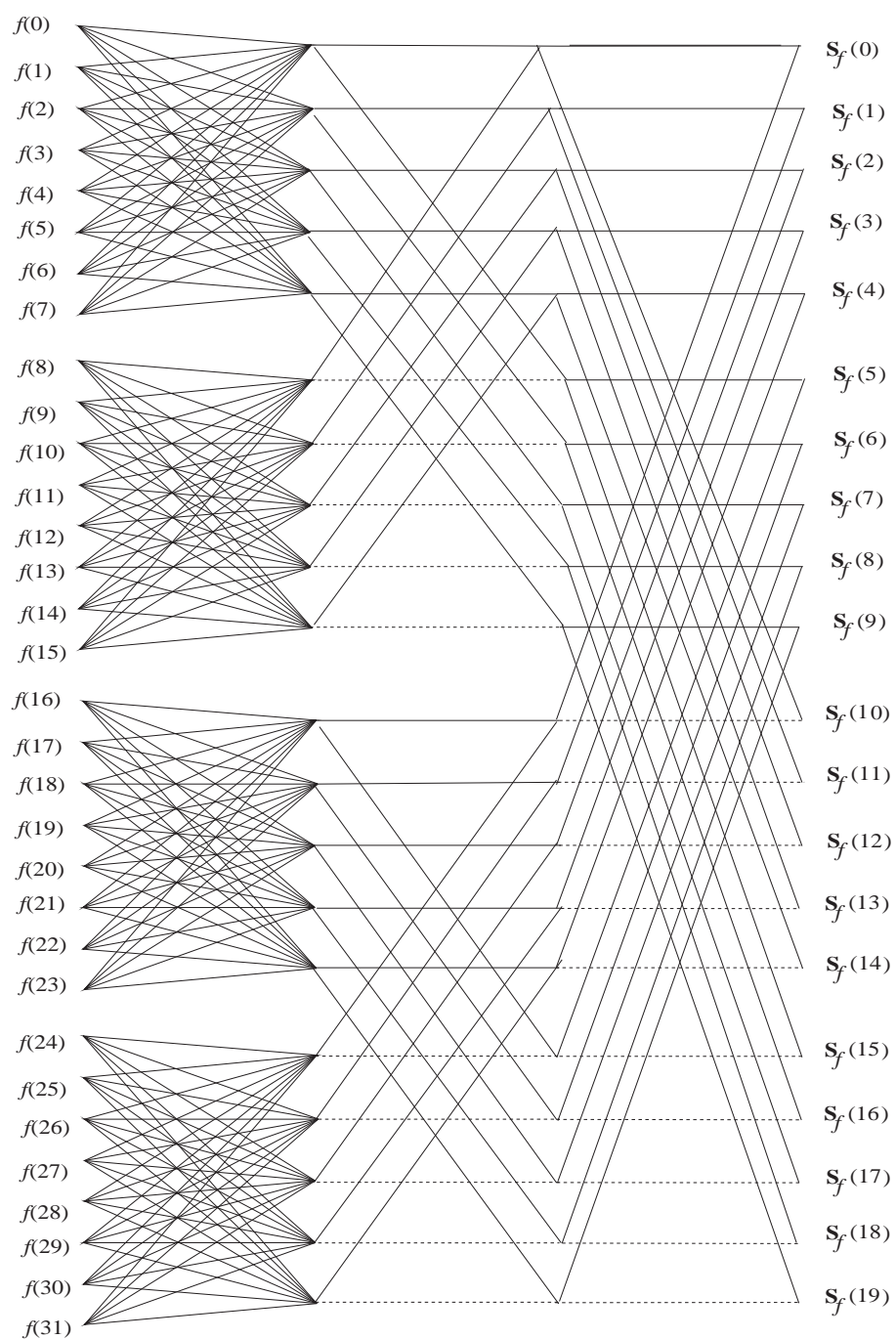
$$\begin{aligned} [\mathbf{C}^1] &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \mathbf{I}_{2 \times 2} \otimes \mathbf{I}_{5 \times 5}, \\ [\mathbf{C}^2] &= \mathbf{I}_{2 \times 2} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \mathbf{I}_{5 \times 5}, \\ [\mathbf{C}^3] &= \mathbf{I}_{2 \times 2} \otimes \mathbf{I}_{2 \times 2} \otimes [\mathbf{Q}_2]^{-1}, \end{aligned}$$

with the matrix $[\mathbf{Q}_2]$ described in Example 2.3.

The flow-graph of FFT on G_{32} derived according to this factorization is given in Fig. 3.2.

In this algorithm the Fourier transforms on the cyclic groups C_2 and the quaternion group Q_2 are used as the basic modules, and hence they are performed directly. The C_2 is the simplest possible case, but the Fourier transform on Q_2 can be carried out by using the corresponding fast algorithm. If we do not like to work with matrix operations in a FFT flow-graph, and if we use the fast algorithm on Q_2 , the algorithm given in Fig. 3.2 can be translated easily into an ordinary FFT like those used in the case of Abelian groups. In that order note that the first four rows of $[Q]^{-1}$ are identical to some particular Walsh functions on the finite group of order 8, and, hence, some of the values representing the output from the first step of our algorithm can be calculated by using a part of the flow-graph of the corresponding fast Walsh transform. The remaining output values from the first step correspond to the group representation of order 2, and therefore they are (2×2) matrices. The elements of these matrices will be calculated independently and for efficiency we will use the fact that each of the matrices \mathbf{I} , \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} , \mathbf{E} contains two zero elements. In this way we derive the algorithm for the calculation of Fourier transform on G_{32} shown in Fig. 3.3. This algorithm can be compared with the corresponding algorithm presented in [23] shown in Fig. 3.4. In this algorithm a different fast algorithm for the calculation of the Fourier transform on the quaternion group described in Example 2.3 is used.

Example 3.3. Let $S_3 = (0, (132), (123), (12), (13), (23), \circ)$ be the symmetric group of permutations of order 3 defined in Example 2.1. Let $G_{6 \times 6} = S_3 \times S_3$ be the direct product of S_3 by itself, i.e., $G_{6 \times 6}$ consists of pairs $(h_1, h_2) = g \in G_{6 \times 6}$, where $h_1, h_2 \in S_3$. The group operation of $G_{6 \times 6}$ is specified as follows: for $(h_1, h_2) = g \in G_{6 \times 6}$, and $(h'_1, h'_2) = g' \in G_{6 \times 6}$, we

Figure 3.2: FFT on the group G_{32} .

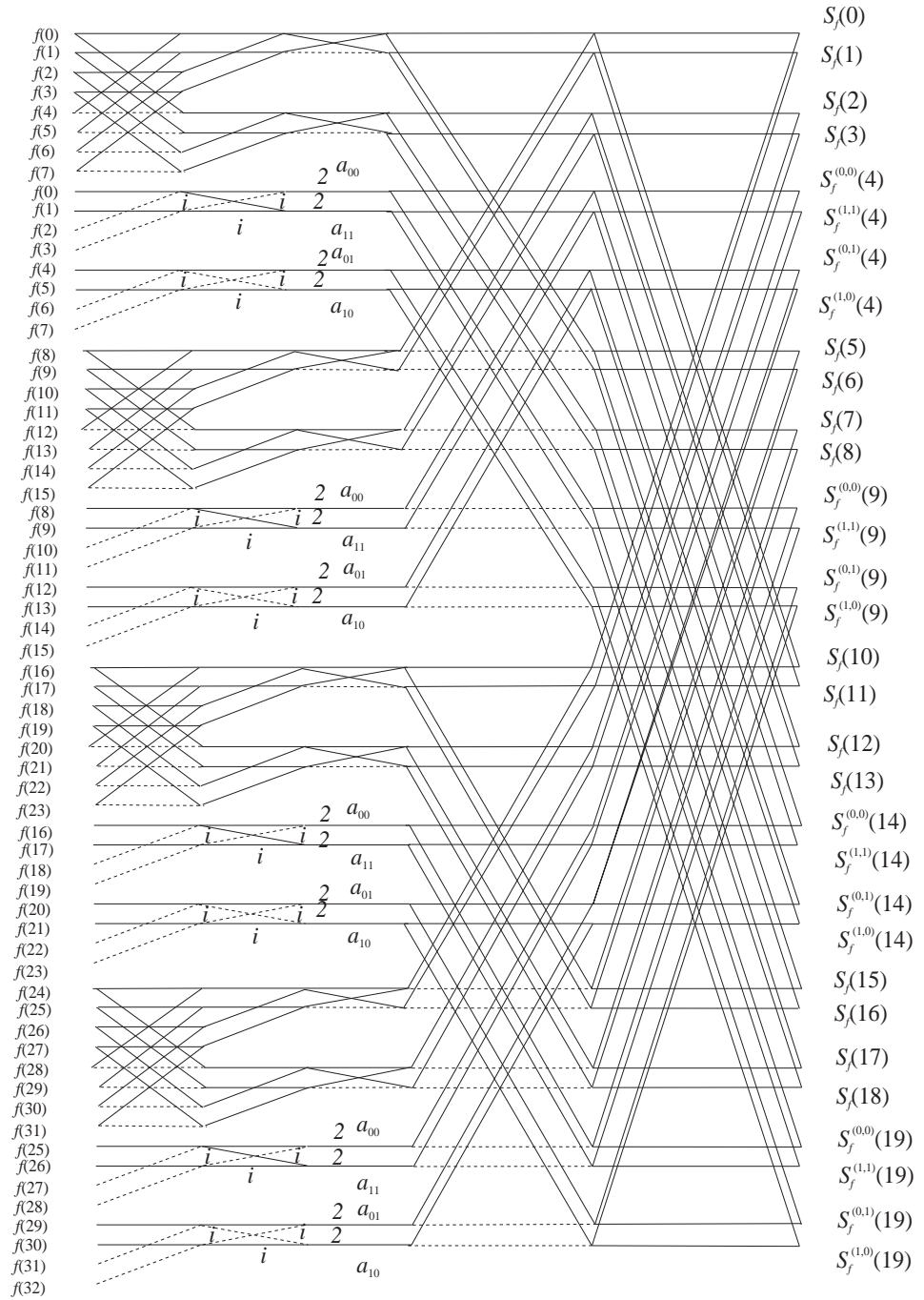


Figure 3.3: FFT on the group G_{32} through a part of fast Walsh transform.

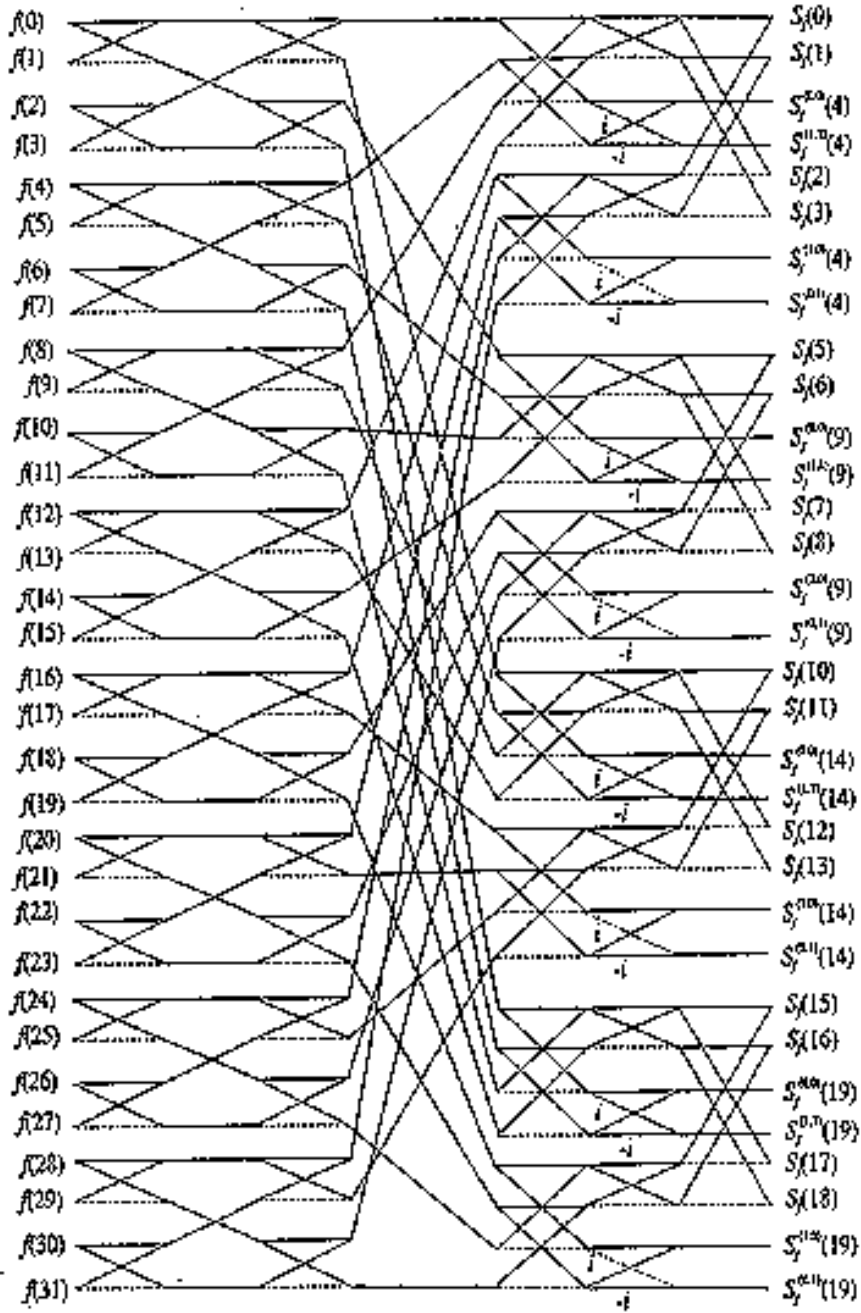


Figure 3.4: FFT on the group G_{32} using FFT on Q_2 .

have $(h_1 \overset{\circ}{s} h'_1, h_2 \overset{\circ}{s} h'_1) = g \circ g' \in G_{6 \times 6}$.

The group operation table is large and we do not write it explicitly. It can be easily derived from the group operation table for S_3 . The unitary irreducible representations of $G_{6 \times 6}$ over the Galois field $\text{GF}(11)$ are given in Table 3.4.

The Fourier transform matrix on $G_{6 \times 6}$ can be factorized as

$$[\mathbf{R}_{6 \times 6}] = [\mathbf{C}^1] \overset{\circ}{\otimes} [\mathbf{C}^2],$$

where

$$\begin{aligned} [\mathbf{C}^1] &= [\mathbf{S}_3] \otimes [\mathbf{I}_{3 \times 3}], \\ [\mathbf{C}^2] &= [\mathbf{I}_{6 \times 6}] \otimes [\mathbf{S}_3], \end{aligned}$$

with

$$[\mathbf{S}_3] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 10 & 10 & 10 \\ 2\mathbf{I} & 2\mathbf{A} & 2\mathbf{B} & 2\mathbf{C} & 2\mathbf{D} & 2\mathbf{E} \end{bmatrix},$$

with the notation as in Table 2.5 since $\mathbf{A} = \mathbf{B}$, and $\mathbf{C}, \mathbf{D}, \mathbf{E}$ are the symmetric matrices.

The flow-graph of the fast Fourier transform on $G_{6 \times 6}$ based on this factorization is shown in Fig. 3.5.

Example 3.4. Let $G_{3 \times 6}$ be the direct product of the group $Z_3 = (0, 1, 2, \overset{\circ}{3})$ of integers less than 3 with modulo 3 addition as the group operation, and the symmetric group of permutations of order 3, S_3 described in Example 2.1. The group Z_3 is an Abelian group, and therefore, their representations are given by the matrix of characters $[\chi]$ as follows:

$$[\chi] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e_1 & e_2 \\ 1 & e_2 & e_1 \end{bmatrix},$$

where $e_1 = -2^{-1}(1 - i\sqrt{3}), e_2 = -2^{-1}(1 + i\sqrt{3})$. The unitary irreducible representations of S_3 over C are given in Table 2.2. Hence, $G_{3 \times 6}$ consists of pairs $(h_1, h_2) = g \in G_{3 \times 6}$ where $h_1 \in Z_3$ and $h_2 \in S_3$. The group operation \circ of $G_{3 \times 6}$ is specified as follows: for $(h_1, h_2) = g \in G_{3 \times 6}$ and $(h'_1, h'_2) = g' \in G_{3 \times 6}$ we have $(h_1 \overset{\circ}{3} h'_1, h_2 \overset{\circ}{3} h'_2) = g \circ g' \in G_{3 \times 6}$. The group table of $G_{3 \times 6}$ is

Table 3.4. The unitary irreducible representations of $G_{6 \times 6}$ over $\text{GF}(11)$.

x	\mathbf{R}_0	\mathbf{R}_1	\mathbf{R}_2	\mathbf{R}_3	\mathbf{R}_4	\mathbf{R}_5	\mathbf{R}_6	\mathbf{R}_7	\mathbf{R}_8
0	1	1	I	1	1	1	I	I	I \otimes I
1	1	1	A	1	1	A	I	I	I \otimes A
2	1	1	B	1	1	B	I	I	I \otimes B
3	1	10	C	1	10	C	I	10 I	I \otimes C
4	1	10	D	1	10	D	I	10 I	I \otimes D
5	1	10	E	1	10	E	I	10 I	I \otimes E
6	1	1	I	1	1	I	A	A	A \otimes I
7	1	1	A	1	1	A	A	A	A \otimes A
8	1	1	B	1	1	B	A	A	A \otimes B
9	1	10	C	1	10	C	A	10 A	A \otimes C
10	1	10	D	1	10	D	A	10 A	A \otimes D
11	1	10	E	1	10	E	A	10 A	A \otimes E
12	1	1	I	1	1	I	B	B	B \otimes I
13	1	1	A	1	1	A	B	B	B \otimes A
14	1	1	B	1	1	B	B	B	B \otimes B
15	1	10	C	1	10	C	B	10 B	B \otimes C
16	1	10	D	1	10	D	B	10 B	B \otimes D
17	1	10	E	1	10	E	B	10 B	B \otimes E
18	1	1	I	10	10	10 I	C	C	C \otimes I
19	1	1	A	10	10	10 A	C	C	C \otimes A
20	1	1	B	10	10	10 B	C	C	C \otimes B
21	1	10	C	10	1	10 C	C	10 C	C \otimes C
22	1	10	D	10	1	10 D	C	10 C	C \otimes D
23	1	10	E	10	1	10 E	C	10 C	C \otimes E
24	1	1	I	10	10	10 I	D	D	D \otimes I
25	1	1	A	10	10	10 A	D	D	D \otimes A
26	1	1	B	10	10	10 B	D	D	D \otimes B
27	1	10	C	10	1	10 C	D	10 D	D \otimes C
28	1	10	D	10	1	10 D	D	10 D	D \otimes D
29	1	10	E	10	1	10 E	D	10 D	D \otimes E
30	1	1	I	10	10	10 I	E	E	E \otimes I
31	1	1	A	10	10	10 A	E	E	E \otimes A
32	1	1	B	10	10	10 B	E	E	E \otimes B
33	1	10	C	10	1	10 C	E	10 E	E \otimes C
34	1	10	D	10	1	10 D	E	10 E	E \otimes D
35	1	10	E	10	1	10 E	E	10 E	E \otimes E

I, A, B, C, D, E as in Table 2.5.

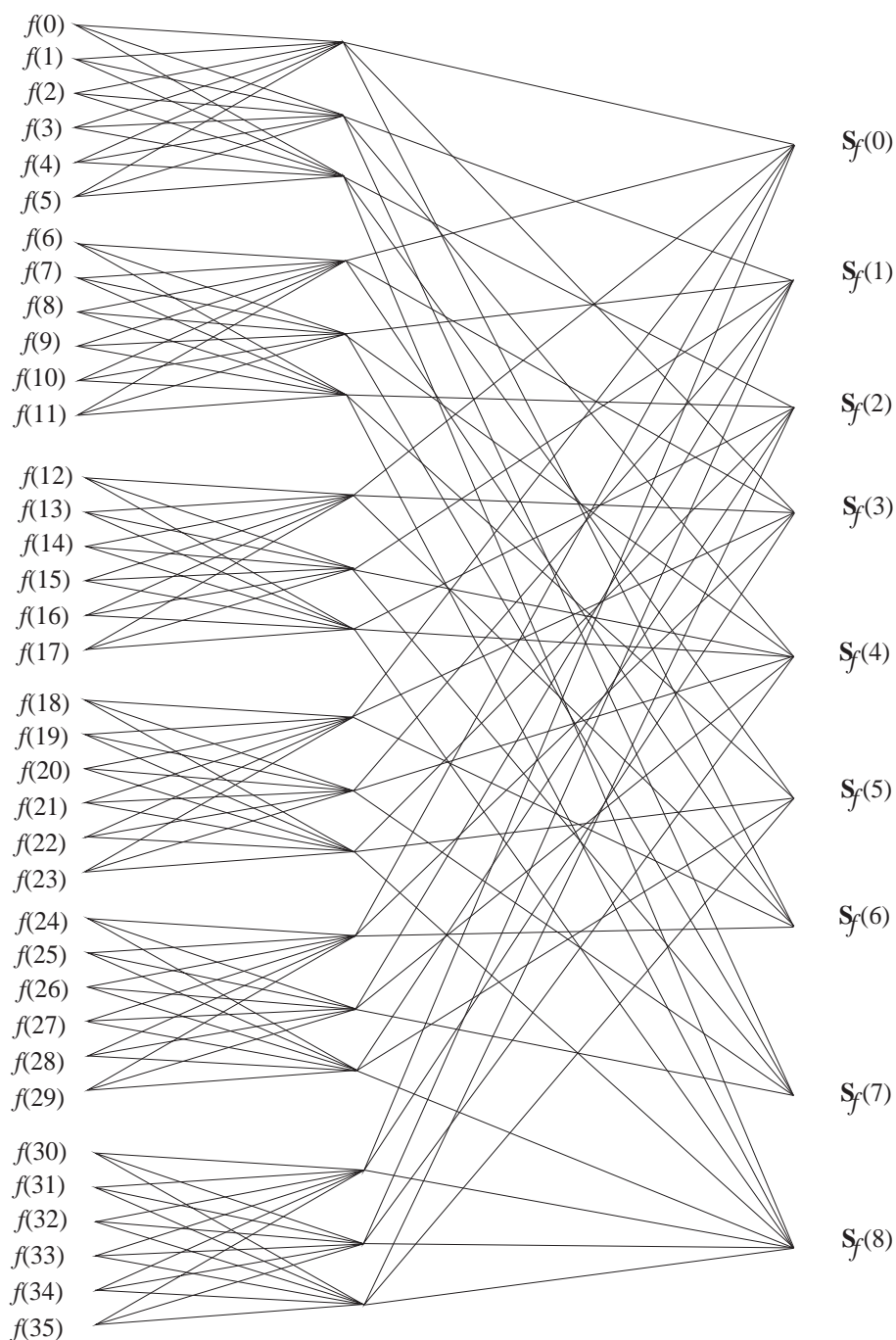
Figure 3.5: FFT on the group $G_{6 \times 6}$.

Table 3.5. The group operation of $G_{3 \times 6}$

o	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	2	0	5	3	4	7	8	6	11	9	10	13	14	12	17	15	16
2	2	0	1	4	5	3	8	6	7	10	11	9	14	12	13	16	17	15
3	3	4	5	0	1	2	9	10	11	6	7	8	15	16	17	12	13	14
4	4	5	3	2	0	1	10	11	9	8	6	7	16	17	15	14	12	13
5	5	3	4	1	2	0	11	9	10	7	8	6	17	15	16	13	14	12
6	6	7	8	9	10	11	12	13	14	15	16	17	0	1	2	3	4	5
7	7	8	6	11	9	10	13	14	12	17	15	16	1	2	0	5	3	4
8	8	6	7	10	11	9	14	12	13	16	17	15	2	0	1	4	5	3
9	9	10	11	6	7	8	15	16	17	12	13	14	3	4	5	0	1	2
10	10	11	9	8	6	7	16	17	15	14	12	13	4	5	3	2	0	1
11	11	9	10	7	8	6	17	15	16	13	14	12	5	3	4	1	2	0
12	12	13	14	15	16	17	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	12	17	15	16	1	2	0	5	3	4	7	8	6	11	9	10
14	14	12	13	16	17	15	2	0	1	4	5	3	8	6	7	10	11	9
15	15	16	17	12	13	14	3	4	5	0	1	2	9	10	11	6	7	8
16	16	17	15	14	12	13	4	5	3	2	0	1	10	11	9	8	6	7
17	17	15	16	13	14	12	5	3	4	1	2	0	11	9	10	7	8	6

given in Table 3.5, while the unitary irreducible representations of $G_{3 \times 6}$ are given in Table 3.6.

The Fourier transformation matrix on $G_{3 \times 6}$ can be factorized as

$$[\mathbf{R}]^{-1} = [\mathbf{C}^1] [\mathbf{C}^2],$$

where

$$[\mathbf{C}^1] = [\chi]^{-1} \otimes \mathbf{I}_{3 \times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e_1 & e_2 \\ 1 & e_2 & e_1 \end{bmatrix} \otimes \mathbf{I}_{3 \times 3},$$

$$[\mathbf{C}^2] = \mathbf{I}_{3 \times 3} \otimes [\mathbf{S}_3]^{-1},$$

with

$$[\mathbf{S}_3]^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 2\mathbf{I} & 2\mathbf{B} & 2\mathbf{A} & 2\mathbf{C} & 2\mathbf{D} & 2\mathbf{E} \end{bmatrix}.$$

The flow-graph of the fast Fourier transform on $G_{3 \times 6}$ derived according to this factorization is given in Fig. 3.6.

Example 3.5. Consider the group $G_{24} = C_2 \times C_2 \times S_3$, where C_2 is the cyclic group of order 2 and S_3 is the symmetric group of permutations of order three.

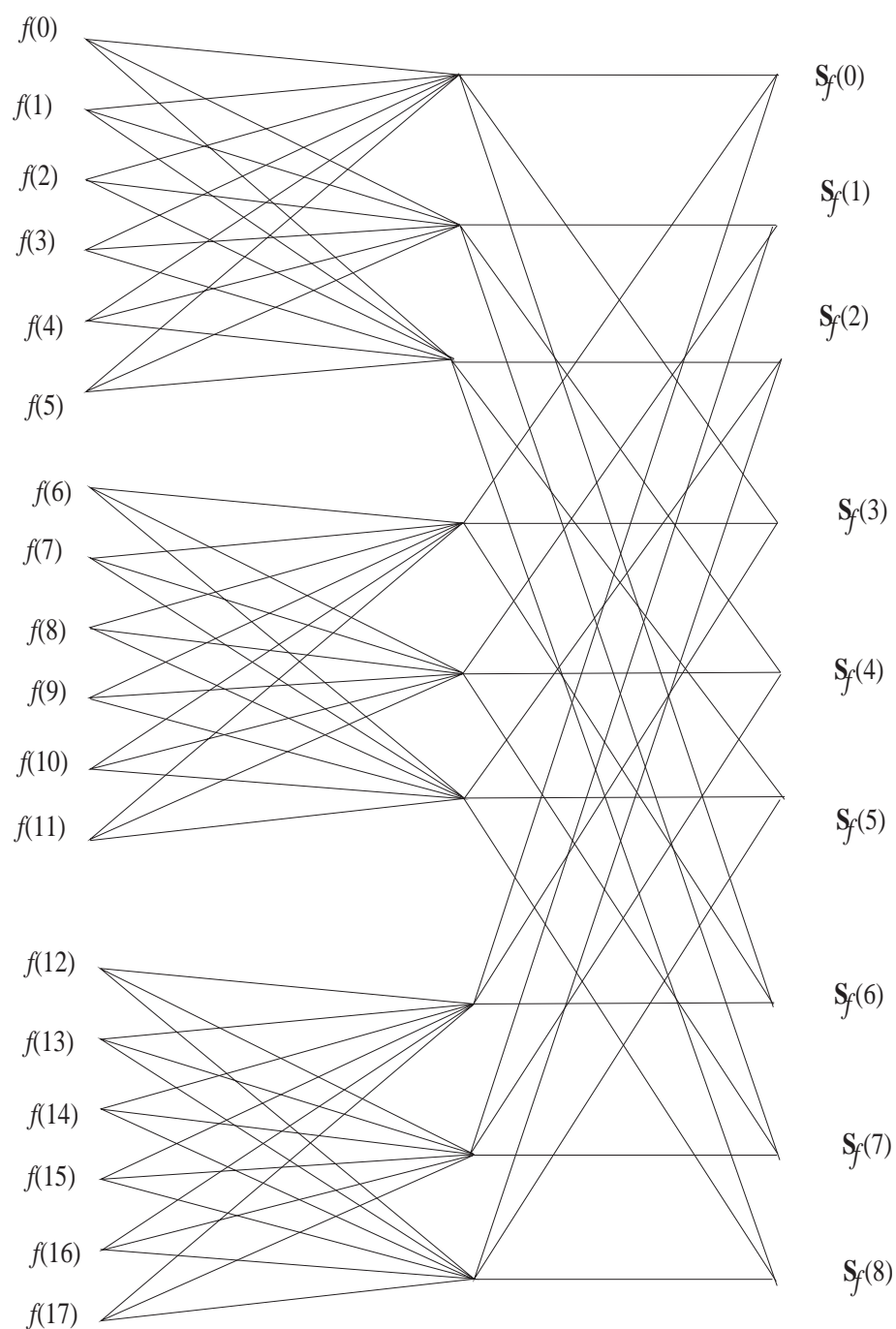
Group representations of C_2 over $GF(11)$ are given by the columns of the matrix

$$\mathbf{W}(1) = \begin{bmatrix} 1 & 1 \\ 1 & 10 \end{bmatrix}.$$

Table 3.6. The unitary irreducible representations of $G_{3 \times 6}$ over C

x	\mathbf{R}_0	\mathbf{R}_1	\mathbf{R}_2	\mathbf{R}_3	\mathbf{R}_4	\mathbf{R}_5	\mathbf{R}_6	\mathbf{R}_7	\mathbf{R}_8
0	1	1	\mathbf{I}	1	1	\mathbf{I}	1	1	\mathbf{I}
1	1	1	\mathbf{A}	1	1	\mathbf{A}	1	1	\mathbf{A}
2	1	1	\mathbf{B}	1	1	\mathbf{B}	1	1	\mathbf{B}
3	1	-1	\mathbf{C}	1	-1	\mathbf{C}	1	-1	\mathbf{C}
4	1	-1	\mathbf{D}	1	-1	\mathbf{D}	1	-1	\mathbf{D}
5	1	-1	\mathbf{E}	1	-1	\mathbf{E}	1	-1	\mathbf{E}
6	1	1	\mathbf{I}	e_1	e_1	$e_1\mathbf{I}$	e_2	e_2	$e_2\mathbf{I}$
7	1	1	\mathbf{A}	e_1	e_1	$e_1\mathbf{A}$	e_2	e_2	$e_2\mathbf{A}$
8	1	1	\mathbf{B}	e_1	e_1	$e_1\mathbf{B}$	e_2	e_2	$e_2\mathbf{B}$
9	1	-1	\mathbf{C}	e_1	$-e_1$	$e_1\mathbf{C}$	e_2	$-e_2$	$e_2\mathbf{C}$
10	1	-1	\mathbf{D}	e_1	$-e_1$	$e_1\mathbf{D}$	e_2	$-e_2$	$e_2\mathbf{D}$
11	1	-1	\mathbf{E}	e_2	$-e_2$	$e_2\mathbf{E}$	e_2	$-e_2$	$e_2\mathbf{E}$
12	1	1	\mathbf{I}	e_2	e_2	$e_2\mathbf{I}$	e_2	e_2	$e_2\mathbf{I}$
13	1	1	\mathbf{A}	e_2	e_2	$e_2\mathbf{A}$	e_1	e_1	$e_1\mathbf{A}$
14	1	1	\mathbf{B}	e_2	e_2	$e_2\mathbf{B}$	e_1	e_1	$e_1\mathbf{B}$
15	1	-1	\mathbf{C}	e_2	$-e_2$	$e_2\mathbf{C}$	e_1	$-e_1$	$e_1\mathbf{C}$
16	1	-1	\mathbf{D}	e_2	$-e_2$	$e_2\mathbf{D}$	e_1	$-e_1$	$e_1\mathbf{D}$
17	1	-1	\mathbf{E}	e_2	$-e_2$	$e_2\mathbf{E}$	e_1	$-e_1$	$e_1\mathbf{E}$

The notation as in Table 2.2.

Figure 3.6: FFT on the group $G_{3 \times 6}$.

This matrix is self-inverse up to a multiplicative constant, and therefore, the Fourier transform on C_2 is defined by the transform matrix

$$\mathbf{W}^{-1}(1) = 6 \begin{bmatrix} 1 & 1 \\ 1 & 10 \end{bmatrix}.$$

Table 2.5 shows the group representations of S_3 over $GF(11)$. The Fourier transform matrix on G_{24} is defined by

$$[\mathbf{R}_{24}]^{-1}(3) = \left(\mathbf{W}^{-1}(1) \otimes \mathbf{W}^{-1}(1) \otimes [\mathbf{S}_3]^{-1}(1) \right) \bmod (11).$$

$$[\mathbf{R}_{24}]^{-1}(3) = [\mathbf{C}^1][\mathbf{C}^2][\mathbf{C}^3],$$

where

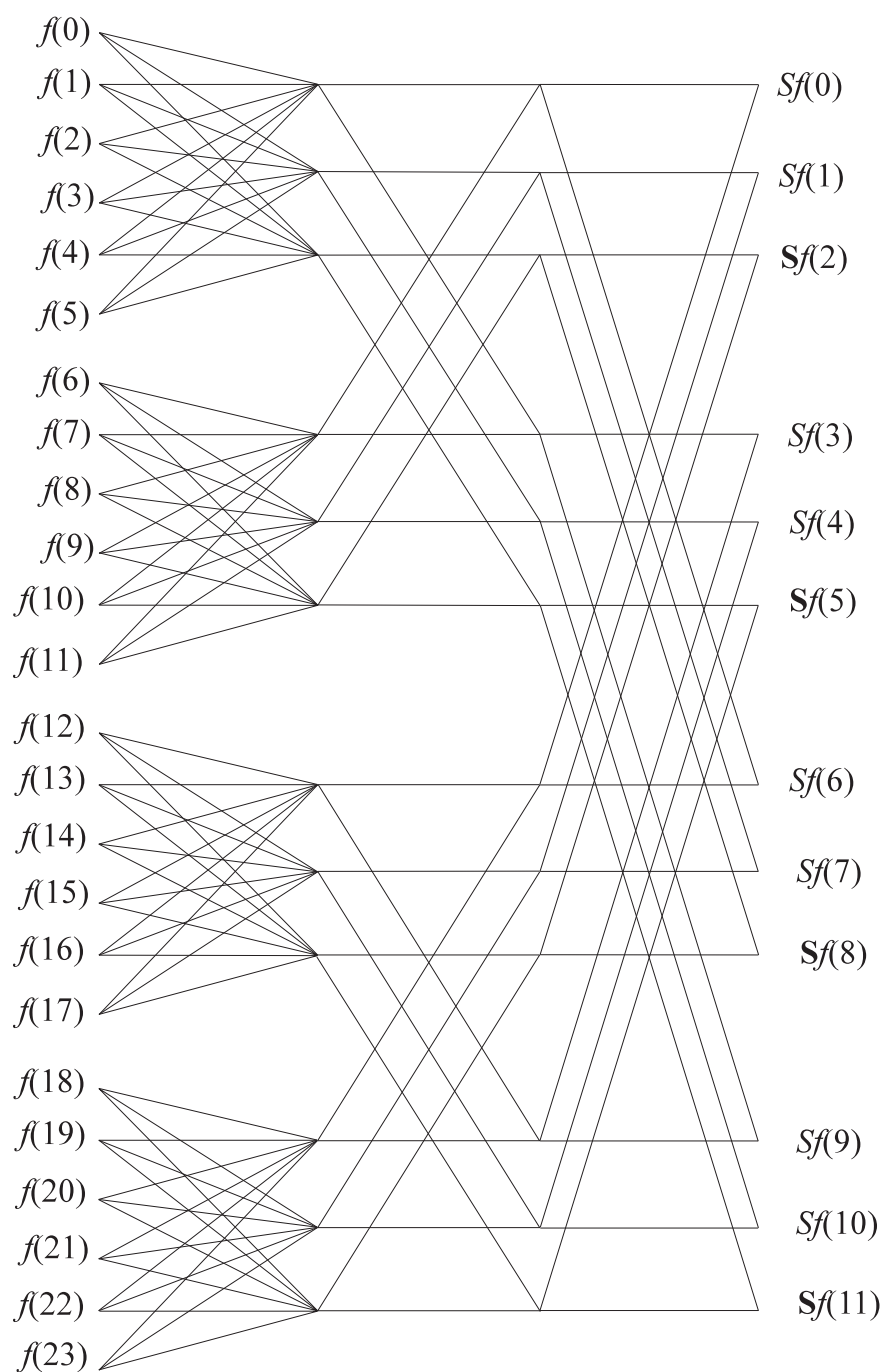
$$\begin{aligned} [\mathbf{C}^1] &= \mathbf{W}^{-1}(1) \otimes \mathbf{I}_{(2 \times 2)} \otimes \mathbf{I}_{(3 \times 3)}, \\ [\mathbf{C}^2] &= \mathbf{I}_{(2 \times 2)} \otimes \mathbf{W}^{-1}(1) \otimes \mathbf{I}_{(3 \times 3)}, \\ [\mathbf{C}^3] &= \mathbf{I}_{(2 \times 2)} \otimes \mathbf{I}_{(2 \times 2)} \otimes [\mathbf{S}_3]^{-1}(1). \end{aligned}$$

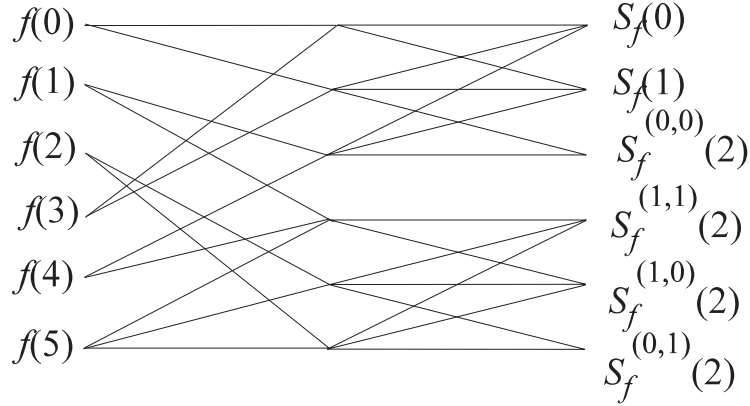
Fig. 3.7 shows the flow-graph of FFT on G_{24} derived from this factorization of $[\mathbf{R}_{24}]^{-1}(3)$. In this flow-graph, the weights at the edges are elements of $[\mathbf{S}_3](1)$, and of $\mathbf{W}(1)$.

As described above, in Example 3.2, with FFT the calculation of one-dimensional Fourier transform on G of the form (2.9) is transferred into successive calculation of n Fourier transforms on the constituent subgroups G_i of G . Each block in the related flow-graph corresponds to a subgroup G_i . In these blocks, calculation of the Fourier transform on G_i is performed directly by definition of the transform. Some further reduction of computations can be achieved if it is possible to develop some FFT on the constituent subgroups [38].

The Fourier transform matrix on S_3 written in terms of functions $\mathbf{R}_w^{(i,j)}(x)$ shown in Table 2.7, is given by

$$\mathbf{S}_3^{-1}(1) = 2 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 10 & 10 & 10 \\ 2 & 10 & 10 & 2 & 10 & 10 \\ 0 & 5 & 6 & 0 & 5 & 6 \\ 0 & 6 & 5 & 0 & 5 & 6 \\ 2 & 10 & 10 & 9 & 1 & 1 \end{bmatrix}.$$

Figure 3.7: FFT on G_{24} .

Figure 3.8: FFT on S_3 .

This matrix can be factorized as follows

$$\mathbf{S}_3(1) = 2 \begin{bmatrix} 1 & 0 & 0 & 10 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 10 & 0 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 10 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 10 & 10 & 0 & 0 & 0 \\ 0 & 5 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 10 & 10 \\ 0 & 0 & 0 & 9 & 1 & 1 \\ 0 & 0 & 0 & 0 & 6 & 5 \end{bmatrix} \pmod{11}.$$

This factorization produces FFT on S_3 over $GF(11)$. Fig. 3.8 shows the flow-graph of this FFT on S_3 . In this figure, the ordering of elements of $\mathbf{S}_f(2)$ as $S_f(2)^{(0,0)}$, $S_f(2)^{(1,1)}$, $S_f(2)^{(1,0)}$, $S_f(2)^{(0,1)}$, makes the graph symmetric.

Fig. 3.9 shows FFT on G_{24} derived by using FFT on S_3 . In this flow-graph all the weights are numbers. The output is the vector of number-valued Fourier coefficients $S_f(0)$, $S_f(1)$, and elements of the matrix-valued Fourier coefficients $S_f^{(i,j)}(2)$, $i, j = 0, 1$, for f .

3.3 FFT through decision diagrams

FFT algorithms on whatever Abelian and non-Abelian groups are based upon the vector representations of discrete functions. It follows from definition of FFT and their matrix description that space complexity of FFT on a

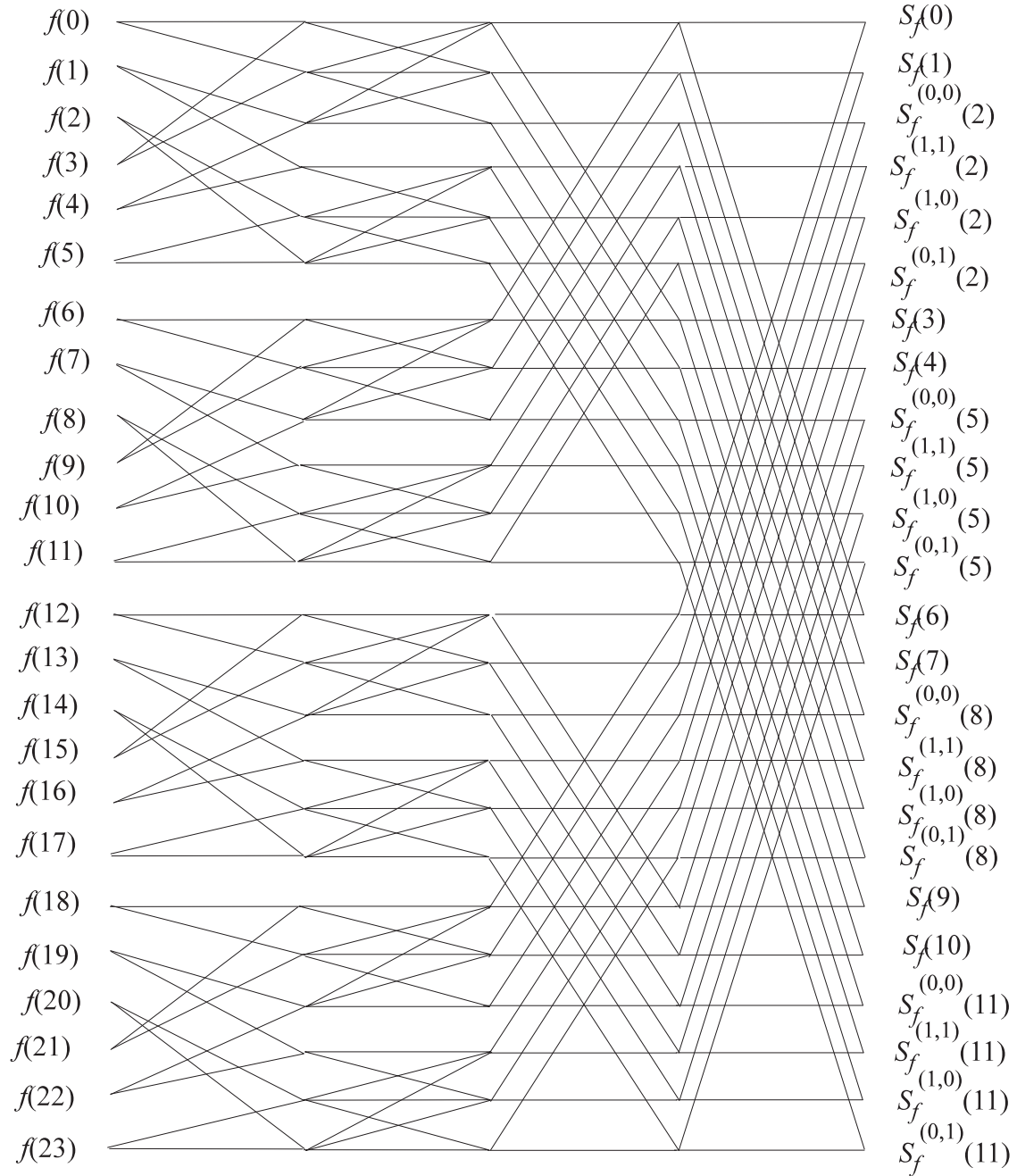


Figure 3.9: FFT on G_{24} with FFT on S_3 .

decomposable group G of order g approximates $O(g)$. The time complexity is O/ng . Thus, the application of FFT is restricted to groups of relatively small orders. This restriction can be overcome by performing FFT on the decision diagram representations of discrete functions in the same way as that were done for various discrete transforms on finite Abelian groups.

3.3.1 Decision diagrams

Decision diagrams (DDs) are a data structure permitting compact representations of functions on finite groups [25]. DDs are derived by the reduction of Decision trees (DTs). In this paper, we use Multi-terminal DDs (MT-DDs) [20] to represent discrete functions on finite not necessarily Abelian groups. MTDDs are a generalization of Multi-terminal binary DDs (MTB-DDs) [7]. MTBDDs were defined as a generalization of Binary DDs (BDDs) [4] by permitting complex numbers as the values of constant nodes. Thus, they can be used to represent complex-valued functions on dyadic groups. Multiple-place decision diagrams (MDDs) are a generalization of MTBDDs from dyadic groups to p -adic groups [20], [26], [33], [34]. They are derived by allowing p outgoing edges for each node in the DD. MTDDs are a further generalization of MDDs derived by allowing that nodes at different levels in the MTDD may have different number of outgoing edges.

MTDDs are derived by the reduction of Multi-terminal decision trees (MTDTs), which can be introduced through the following considerations.

A function f on G is defined by enumerating its values at all the points of G . Since G is of the form (2.9), these points are defined by a recursive assignment of the values for variables in f . This recursive assignment means that for $x_i = s_j$, s_j a particular value in G_i , the variable x_{i-1} takes all the values in G_{i-1} , before x_i takes the value s_{j+1} in G_i . In a MTDD, x_i is assigned to a node with g_i outgoing edges. Each of them points to a node to which x_{i-1} is assigned and thus, with g_{i-1} edges. In that way, a decision tree is derived, where outgoing edges of the tree point to the lexicographically ordered set of n -tuples, representing points in G . The values of f at these points are the constant nodes in the derived MTDT. Thus, a MTDT is the graphical representation of this procedure of enumerating function values for F at the points determined by the recursive assignment of values to variables in f .

A MTDD for a given f is derived from the MTDT by sharing isomorphic subtrees and deleting the redundant information in the DT [25]. Formally, a MTDD can be defined as follows.

Definition 3.1 A MTDD for representation of $f \in P(G)$ is a rooted directed acyclic graph $D(V, E)$ with the node set V consisting of non-terminal nodes and terminal or constant nodes. A non-terminal node is labeled with a variable x_i of f and has g_i successors denoted by $succ_k(v) \in V$ with $k \in G_i$. A constant node v is labeled with an element from P and has no successors.

In a MTDT, the i -th level consists of all the nodes to which the variable x_i is assigned. In a MTDD, edges connecting nodes at non successive levels may appear. Cross points are points where such an edge crosses levels in the MTDD. Through cross points, the impact of the deleted nodes from the MTDT is taken into account. The concept of MTDD is explained and illustrated by the following example.

Example 3.6. Consider a function f on $G_{24} = C_2 \times C_2 \times S_3$, described in the Example 3.5. If f is given by the truth-vector

$$\mathbf{f} = [0, 6, 2, 1, 0, 0, 2, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2]^T,$$

then it can be represented by the MTDD shown in Fig. 3.10. In this figure, x_i^j denotes that the variable x_i takes the value j .

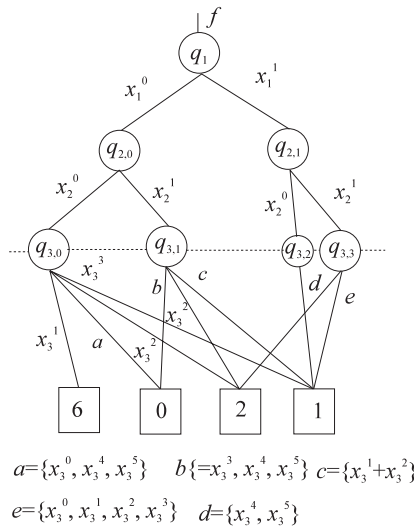


Figure 3.10: MTDD for f in Example 3.5.

3.3.2 FFT on finite non-Abelian groups through DDs

From the theory of Good-Thomas FFT, the calculation of the Fourier transform on a decomposable group G of order g , can be performed through n Fourier transforms on the constituent subgroups G_i of orders g_i .

Calculation of Fourier and Fourier-like transforms through DDs is possible thanks to the recursive structure of DTs, which is compatible with the recursive structure of Kronecker product representable and related transform matrices.

Calculation procedures based on DDs representations of discrete functions are proposed for various discrete transforms on Abelian groups [7], [33], [34], [37]. That approach can be extended to finite non-Abelian groups thanks to the matrix interpretation of the Fourier transform. Calculation through MTDDs consists in the processing of non-terminal nodes and cross points.

A MTDD represents the vector \mathbf{F} for f . Each non-terminal node, or the cross point, in the MTDD can be considered as the root node of a subtree of the MTDD. Each subtree represents a subvector in \mathbf{F} . In that way, the processing of nodes and cross points in the MTBDD means calculations over subvectors represented by the subtrees rooted at the nodes where arrive the outgoing edges of the processed nodes or the cross points.

The calculations are performed through some rules that may be conveniently described by matrices. If this is an identity matrix, the processing reduces to the concatenation of subvectors.

Definition 3.2 The operation of concatenation, denoted by \diamond , over an ordered set of n vectors $\{\mathbf{A}_1, \dots, \mathbf{A}_n\}$ of order m is the operation producing a vector \mathbf{S} of order nm consisting of n successive subvectors $\mathbf{A}_1, \dots, \mathbf{A}_n$.

Example 3.7. Application of the operation of concatenation over the set of three vectors $\mathbf{A} = [a_1 a_2 a_3]^T$, $\mathbf{B} = [b_1 b_2 b_3]^T$, $\mathbf{C} = [c_1 c_2 c_3]^T$ produces the vector $\mathbf{D} = \mathbf{A} \diamond \mathbf{B} \diamond \mathbf{C} = [a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2 c_3]^T$.

It follows that the calculation of the Fourier coefficients of f on a finite decomposable group G of order g given by the DD can be carried out through the following procedure.

Calculation procedure

Given a function f on the decomposable group G of the form (2.9).

1. Represent f by a MTDD. Denote by Q_i the number of non-terminal nodes at the i -th level, i.e., the level corresponding to the variable x_i

```

begin {procedure}
  for i = n to 1
    for k = 0 to Qi do
      Determine qi,k by using the rule (3.4).
      Store [Sf] = q1.
end{procedure}

```

Figure 3.11: Calculation procedure for the Fourier transform.

of f , in the MTDD.

2. Descent the MTDD in a recursive way level by level starting from the constant nodes at $(n + 1)$ -st level up to the root node at the level 1.
3. For $i = n$ to 1, process the nodes and cross points by using the rule

$$\begin{aligned}
 q_{i,k}(w_i) &= r_{w_i} g_i^{-1} \sum_{j=0}^{g_i-1} q_{i+1,j} \mathbf{R}_{w_i}(x_j^{-1}), \\
 k &= 0, \dots, Q_i - 1, w_i = 0, \dots, K_i - 1,
 \end{aligned} \tag{3.4}$$

easily derived from the matrix factorization of $[\mathbf{R}]^{-1}$.

Fig. 3.11 shows this procedure expressed in a programming pseudo code.

It should be pointed out that there is no matrix computation, and only vector operations are used in the computation of Fourier coefficients through this procedure. This ensures efficiency of the procedure. The vectors are represented by DDs, which permits processing of functions on groups of large orders. The matrix-valued vector determined in the root node q_1 is the Fourier spectrum of f .

The procedure is probably the best explained through an example by using the matrix notation.

Example 3.8 Consider the function f in Example 3.6. The Fourier spectrum for f is calculated as

$$[\mathbf{S}_f] = [\mathbf{R}_{24}]^{-1}(3)\mathbf{F} \text{ mod } (11).$$

Thus,

$$[\mathbf{S}_f] = 6[5, 9, \mathbf{S}_f(2), 3, 5, \mathbf{S}_f(5), 10, 2, \mathbf{S}_f(8), 7, 1, \mathbf{S}_f(11)]^T \text{ mod } (11),$$

where

$$\mathbf{S}_f(2) = \begin{bmatrix} 5 & 2 \\ 9 & 5 \end{bmatrix}, \mathbf{S}_f(5) = \begin{bmatrix} 5 & 2 \\ 9 & 8 \end{bmatrix}, \mathbf{S}_f(8) = \begin{bmatrix} 9 & 2 \\ 9 & 1 \end{bmatrix}, \mathbf{S}_f(11) = \begin{bmatrix} 1 & 2 \\ 9 & 1 \end{bmatrix},$$

since the scaling factor $6 \cdot 6 \cdot 2$ modulo 11 reduces to 6. Finally,

$$[\mathbf{S}_f] = \left[8, 10, \begin{bmatrix} 8 & 1 \\ 10 & 8 \end{bmatrix}, 7, 8, \begin{bmatrix} 8 & 1 \\ 10 & 4 \end{bmatrix}, 5, 1, \begin{bmatrix} 10 & 1 \\ 10 & 6 \end{bmatrix}, 9, 6, \begin{bmatrix} 6 & 1 \\ 10 & 6 \end{bmatrix} \right]^T.$$

In (3.1), each matrix $[\mathbf{C}^i]$ describes uniquely one step of the fast Fourier transform performed in n steps. In DDs, the operation in the j -th step of FFT is performed over the nodes and cross points at the j -th level in the DD. Therefore, the Fourier spectrum of f in Example 3.6 is calculated through MTDD in Fig. 3.10 as follows. Note that in this example, all the calculations are in $GF(11)$.

1. The non-terminal nodes $q_{3,0}, q_{3,1}, q_{3,3}$ and the cross point $q_{3,2}$ are processed first by using the matrix $[\mathbf{S}_3]^{-1}$ in $[\mathbf{R}]^{-1}$. The input data for the procedure are the values of constant nodes. In that way, it is determined

$$q_{3,0} = [\mathbf{S}_3]^{-1} \begin{bmatrix} 0 \\ 6 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ \begin{bmatrix} 10 & 4 \\ 7 & 2 \end{bmatrix} \end{bmatrix}, q_{3,1} = [\mathbf{S}_3]^{-1} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \end{bmatrix},$$

$$q_{3,2} = [\mathbf{S}_3]^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}, q_{3,3} = [\mathbf{S}_3]^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix} \end{bmatrix}.$$

2. The non-terminal nodes $q_{2,0}, q_{2,1}$ are processed by using $\mathbf{W}(1)$. It is determined

$$q_{2,0} = \mathbf{W}^{-1}(1) \begin{bmatrix} q_{3,0} \\ q_{3,1} \end{bmatrix} = 6 \left(\begin{bmatrix} 7 \\ 3 \\ \begin{bmatrix} 10 & 4 \\ 7 & 2 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 8 \\ 8 \\ \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \end{bmatrix} \right)$$

$$\diamond 6 \left(\left(\begin{array}{c} 7 \\ 3 \\ \left[\begin{array}{cc} 10 & 4 \\ 7 & 2 \end{array} \right] \end{array} \right) + 10 \left(\begin{array}{c} 8 \\ 8 \\ \left[\begin{array}{cc} 4 & 0 \\ 0 & 4 \end{array} \right] \end{array} \right) \right),$$

$$\begin{aligned} q_{2,1} &= \mathbf{W}^{-1}(1) \begin{bmatrix} q_{3,2} \\ q_{3,3} \end{bmatrix} = 6 \left(\left(\begin{array}{c} 1 \\ 0 \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \end{array} \right) + \left(\begin{array}{c} 5 \\ 7 \\ \left[\begin{array}{cc} 7 & 0 \\ 0 & 4 \end{array} \right] \end{array} \right) \right) \\ &\diamond 6 \left(\left(\begin{array}{c} 1 \\ 0 \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \end{array} \right) + 10 \left(\begin{array}{c} 5 \\ 7 \\ \left[\begin{array}{cc} 7 & 0 \\ 0 & 4 \end{array} \right] \end{array} \right) \right). \end{aligned}$$

3. The root node is processed by using $\mathbf{W}(1)$. It is determined

$$\begin{aligned} q_1 &= \mathbf{W}^{-1}(1) \begin{bmatrix} q_{2,0} \\ q_{2,1} \end{bmatrix} = 6 \left(\left(\begin{array}{c} 2 \\ 0 \\ \left[\begin{array}{cc} 7 & 2 \\ 9 & 3 \end{array} \right] \\ 5 \\ 3 \\ \left[\begin{array}{cc} 3 & 2 \\ 9 & 10 \end{array} \right] \end{array} \right) + \left(\begin{array}{c} 3 \\ 9 \\ \left[\begin{array}{cc} 9 & 0 \\ 0 & 2 \end{array} \right] \\ 9 \\ 2 \\ \left[\begin{array}{cc} 2 & 0 \\ 0 & 9 \end{array} \right] \end{array} \right) \right) \\ &\diamond 6 \left(\left(\begin{array}{c} 2 \\ 0 \\ \left[\begin{array}{cc} 7 & 2 \\ 9 & 3 \end{array} \right] \\ 5 \\ 3 \\ \left[\begin{array}{cc} 3 & 2 \\ 9 & 10 \end{array} \right] \end{array} \right) + 10 \left(\begin{array}{c} 3 \\ 9 \\ \left[\begin{array}{cc} 9 & 0 \\ 0 & 2 \end{array} \right] \\ 9 \\ 2 \\ \left[\begin{array}{cc} 2 & 0 \\ 0 & 9 \end{array} \right] \end{array} \right) \right) \\ &= \left[8, 10, \left[\begin{array}{cc} 8 & 1 \\ 10 & 8 \end{array} \right], 7, 8, \left[\begin{array}{cc} 8 & 1 \\ 10 & 4 \end{array} \right], 5, 1, \left[\begin{array}{cc} 10 & 1 \\ 10 & 6 \end{array} \right], 9, 6, \left[\begin{array}{cc} 6 & 1 \\ 10 & 6 \end{array} \right] \right]^T \\ &= [\mathbf{S}_f]. \end{aligned}$$

Thus, the Fourier spectrum of f is equal to the matrix-valued vector determined in q_1 and it is equal to that calculated by definition of the Fourier transform.

Each step of the calculation can be represented by a MTDD, which results in the MTDD for the Fourier spectrum of f . Fig. 3.12 shows MTDD for the Fourier spectrum generated by using the proposed procedure.

3.3.3 MTDDs for the Fourier spectrum

MTDD for the Fourier spectrum differs from that representing f in the same way as the FFT algorithms on Abelian groups differ from FFT on non-Abelian groups [28]. DDs representing the Fourier spectrum of f on finite non-Abelian groups are matrix-valued, since the values of constant nodes are the Fourier coefficients. Number of outgoing edges of nodes at the i -th level is determined by the cardinality K_i of the dual object Γ_i of G_i . In MTDD for f , the number of outgoing edges of nodes at the i -th level is equal to the order g_i of G_i .

Efficiency of MTDD representation of f depends on the number of different values f can take. In the same way, the efficiency of DDs representation of the Fourier spectrum of f depends on the number of different Fourier coefficients. In this way, it depends indirectly on the number of different values of f .

In a matrix-valued MTDD (mvMTDD) for \mathbf{S}_f , the matrices representing values of constant nodes can be represented by MTDDs [30], in the same way as any matrix can be represented by a MTDD [7]. In that way, the number-valued MTDDs (nvMTDDs) are derived [30].

Fig. 3.13 shows a nvMTBDD for \mathbf{S}_f in Example 3.8. It is derived from the mvMTDD in Fig. 3.12. The columns of matrices representing values of constant nodes are written as subvectors in a vector, which is then represented by a MTDD. In this figure, to make it clear, some constant nodes are repeated.

Thanks to the spectral interpretation of DDs [32], MTDDs for the Fourier spectrum of f can be interpreted as the Fourier DDs for f [29], [30], [31].

3.3.4 Complexity of DDs calculation methods

The structure of a FFT is described by the number of levels and by connections of nodes within levels. For a given G , the structure of FFT is determined by the assumed decomposition into the product of subgroups G_i

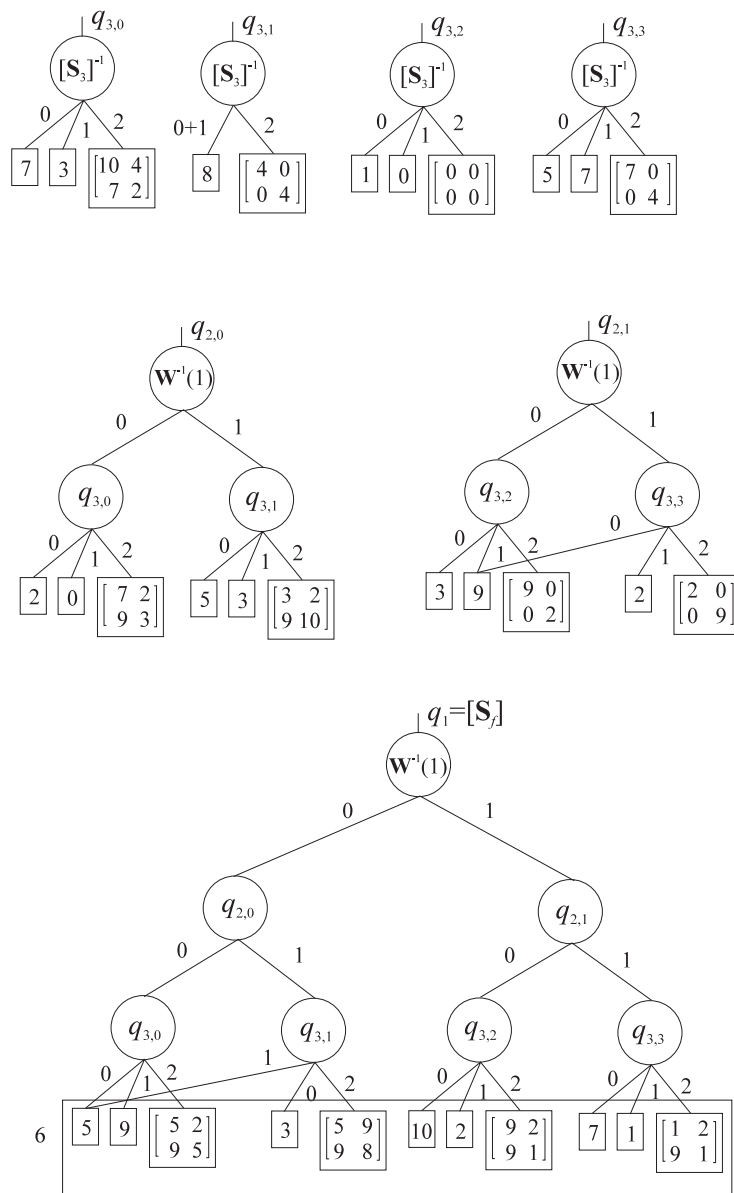


Figure 3.12: Calculation of the Fourier spectrum through MTDD.

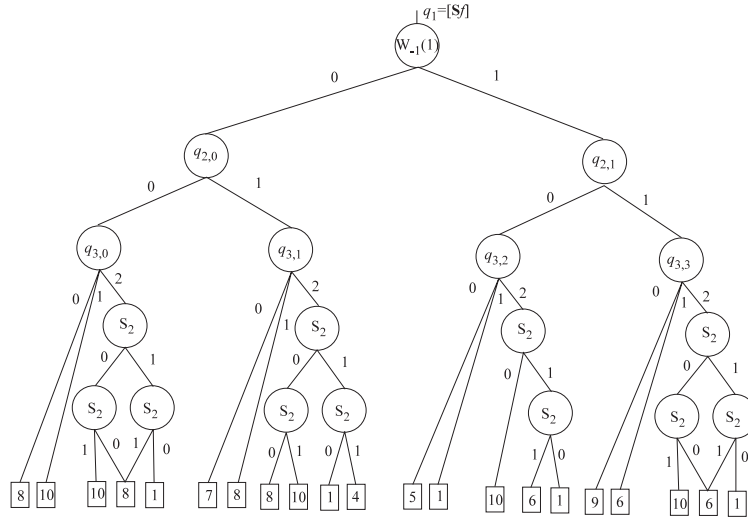


Figure 3.13: nvMTDD for the Fourier spectrum for f in Example 3.5.

of smaller orders (2.9). For the assumed decomposition, FFT on G has the same structure, thus the same complexity, for any f . Therefore, in calculation of the Fourier transform through FFT we do not take into account some peculiar properties a function may have.

Compactness of a DD for f is based upon deleting isomorphic parts in the decision tree for f . Thus, in calculation of the Fourier transform through DD for f , we do not repeat calculations over identical subvectors in the vector representing values of f . In that way, unlike FFT, we take into account properties of the processed functions.

In each node and the cross point at the i -th level, we perform calculations determined by the Fourier transform on G_i . Thus complexity of calculations through DDs is proportional to the number of nodes and cross points in the DD for f , usually denoted as the size of the DD. If a function f has some peculiar properties, as for example symmetry, or decomposability, then MTDD for f has smaller size, and calculation of the Fourier transform is simpler. In reporting experimental results, the size of a DD is usually considered as the number of non-terminal nodes. It is assumed that for an arbitrary function, the number of cross points is smaller than 30 – 40% of the number of non-terminal nodes [25].

The same as in FFT, the complexity of calculations depends on the complexity of calculation rules derived from the Fourier transforms on G_i .

Table 3.7 shows the sizes of MTDDs for some *mcnc* benchmark switching functions and their Fourier transforms. In this table, the multi-output switching functions are represented by Shared binary DDs (SBDDs) [25]. The domain group is of the form C_2^n . For non-Abelian groups, the assumed decompositions are shown. C_2 and C_4 are the cyclic groups of orders 2 and 4, and Q_2 denotes the quaternion group of order 8. The price for the reduced size is the increased number of outgoing edges of nodes. Advantage is the reduced depth of the DDs.

Table 3.7 Sizes of MTDDs for f and nvMTDDs for \mathbf{S}_f .

f	SBDD for f	nvMTDD for \mathbf{S}_f	group
5xp1	90	167	$C_2Q_2^2$
bw	116	34	$C_4Q_2^2$
con1	20	25	$C_2Q_2^2$
rd53	25	31	C_4Q_2
xor5	11	11	$C_4Q_2^2$
add4	103	28	C_2Q_2
add5	226	34	C_4Q_2
add6	477	33	Q_2^2
mul4	159	66	C_2Q_2
mul5	473	87	C_4Q_2
mul6	788	94	Q_2^2

References

- [1] Agaian, S., Astola, J., Egiazarian, K., *Binary Polynomial Transforms and Non-linear Digital Filters*, Marcel Dekker, 1995.
- [2] Apple, G., Wintz, P., "Calculation of Fourier transforms on finite Abelian groups", *IEEE Trans.*, Vol.IT-16, pp.233-234, 1970.
- [3] Beauchamp, K.G., *Applications of Walsh series and Related Functions: With an Introduction to Sequency Theory*, Academic Press, New York, 1984.
- [4] Bryant, R.E. "Graph-based algorithms for Boolean functions manipulation", *IEEE Trans. Comput.*, Vol.C-35, No.8, 1986, 667-691.
- [5] Burrus, C.S., Parks, T.W., *DFT/FFT and Convolution Algorithms: Theory and Implementation*, John Wiley, 1985.

- [6] Cairns, T.W., "On the fast Fourier transform on finite Abelian groups", *IEEE Trans. Comput.*, Vol.C-20, pp.569-571, 1971.
- [7] Clarke, E.M., McMillan, K.L., Zhao, X., Fujita, M., "Spectral transforms for extremely large Boolean functions", in: Keschull, U., Schubert, E., Rosentiel, W., Eds., *Proc. IFIP WG 10.5 Workshop on Applications of the Reed-Muller Expansion in Circuit Design*, 16-17.9.1993, Hamburg, Germany, 86-90.
- [8] Cooley, J.W., Tukey, J.W., "An algorithm for the machine calculation of complex Fourier series", *Math. Computation*, Vol.19, 297-301, 1965.
- [9] Depeyrot, M., "Fondements algebriques de la transformation de Fourier rapide", 1E/11, Centre d'Automatique, Ecole des Mines, Fontainebleau, France, 1969.
- [10] Gibbs, J.E., Stanković, R.S., "Matrix interpretation of Gibbs derivatives on finite groups", private communication, 1988.
- [11] Good, I.J., "The interaction algorithm and practical Fourier analysis", *J. Roy. Statist. Soc.*, ser. B, Vol.20, 361-372, Addendum, Vol.22, 1960, 372-375.
- [12] Good, I.J., "The relationship between two fast Fourier transforms", *IEEE Trans. Comput.*, Vol.C-20, No.3, 310-317, 1971.
- [13] Harmuth, H.F., *Sequency theory: Foundations and Applications*, Academic Press, New York, 1977.
- [14] Heideman, M.T., Johnson, D.H., Burrus, C.S., "Gauss and the history of the fast Fourier transform", *Archiv for History of Exact Science*, Vol.34, No.3, 265-277, 1985. Also in: *IEEE ASSP Magazine*, Vol.1, No.4, 14-21, Oct.1984.
- [15] Karpovsky, M.G., *Finite Orthogonal Series in the Design of Digital Devices*, Wiley, New York, 1976.
- [16] Karpovsky, M.G., "Fast Fourier transforms on finite non-Abelian groups", *IEEE Trans. Comput.*, Vol.C-26, No.10, 1028-1030, Oct. 1977.
- [17] Karpovsky, M.G., Trachtenberg, E.A., "Some optimization problems over finite groups", *Inform. Control.*, 34, pp.1-22, 1977.
- [18] Karpovsky, M.G., Trachtenberg, E.A., "Fourier transform over finite groups for error detection and error correction in computation channels", *Inform. Control*, Vol.40, No.3, 335-358, March, 1979.
- [19] Karpovsky, M.G., Trachtenberg, E.A., "Statistical and computational performance of a class of generalized Wiener filters", *IEEE Trans.*, Vol.IT-32, May 1986.

- [20] Miller, D.M., "Spectral transformation of multiple-valued decision diagrams", *Proc. 24th Int. Symp. on Multiple-valued Logic*, Boston, Massachusetts, USA, 22.-25.5.1994, 89-96.
- [21] Moraga, C., "On some applications of the Chrestenson functions in logic design and data processing", *Mathematics and Computers in Simulation*, 27, 431-439, 1985.
- [22] Nussbaumer, H.J., *Fast Fourier Transform and Convolution Algorithms*, Springer-Verlag, Berlin, Heidelberg, New York, 1981.
- [23] Roziner, T.D., Karpovsky, M.G., Trachtenberg, L.A., "Fast Fourier transforms over finite groups by multiprocessor system", *IEEE Trans.*, Vol.ASSP-38, No.2, 226-240, 1990.
- [24] Sasao, T., "Representations of logic functions by using EXOR operators", *Proc. IFIP WG 10.5 Workshop on Application of the Reed-Muller Expansion in Circuit Design*, 27-29.8.1995, Makuhari, Chiba, Japan, 11-20.
- [25] Sasao, T., Fujita, M., *Representations of Discrete Functions*, Kluwer Academic Publishers, 1996.
- [26] Srinivasan, A., Kam, T., Malik, Sh., Brayant, R.K., "Algorithms for discrete function manipulation", in: *Proc. Inf. Conf. on CAD*, 1990, pp.92-95.
- [27] Stanković, R.S., "Matrix interpretation of fast Fourier transform on finite non-Abelian groups", *Res. Rept. in Appl. Math.*, YU ISSN 0353-6491, Ser. Fourier Analysis, Rept. No.3, April 1990, 1-31, ISBN 86-81611-03-8.
- [28] Stanković, R.S., "Matrix interpretation of the fast Fourier transforms on finite non-Abelian groups", *Proc. Int. Conf. on Signal Processing, Beijing/90*, 22.-26.10.1990, Beijing, China, 1187-1190.
- [29] Stanković, R.S., "Fourier decision diagrams for optimization of decision diagrams representations of discrete functions", *Proc. Workshop on Post Binary-Ultra Large Scale Integration*, Santiago de Campostela, Spain, 1996, 8-12.
- [30] Stanković, R.S., "Fourier decision diagrams on finite non-Abelian groups with preprocessing," *Proc. 27th Int. Symp. on Multiple-Valued Logic*, Antigonish, Nova Scotia, Canada, May 1997, 281-286.
- [31] Stanković, R.S., *Spectral Transform Decision Diagrams in Simple Questions and Simple Answers*, Nauka, Belgrade, 1998.
- [32] Stanković, R.S., Sasao, T., Moraga, C., "Spectral transform decision diagrams", *Proc. IFIP WG 10.5 Workshop on Application of the Reed-Muller Expansion in Circuit Design*, 27-29.8.1995, Makuhari, Chiba, Japan, 46-53.

- [33] Stanković, R.S., Stanković, M., Moraga, C., Sasao, T., “Calculation of Vilenkin-Chrestenson transform coefficients of multiple-valued functions through multiple-place decision diagrams”, *Proc. 5th Int. Workshop on Spectral Techniques*, March 15-17, 1994, Beijing, China, 107-116.
- [34] Stanković, R.S., Stanković, M., Moraga, C., Sasao, T., “Calculation of Reed-Muller-Fourier coefficients of multiple-valued functions through multiple-place decision diagrams”, *Proc. 24th Int. Symp. on Multiple-valued Logic*, Boston, Massachusetts, USA, May 22-25, 1994, 82-88.
- [35] Stanković, R.S., Stojić, M.R., Stanković, M.S., (eds.), *Recent Developments in Abstract Harmonic Analysis with Applications in Signal Processing*, Nauka, Belgrade and Elektronski fakultet, Niš, 1996.
- [36] Thomas, L.H., “Using a computer to solve problems in physics”, In: *Application of Digital Computers*, Boston, Mass., Ginn, 1963.
- [37] Thornton, M.A., “Modified Haar transform calculation using digital circuit output probabilities”, *Proc. IEEE Int. Conf. on Information, Communications and Signal Processing* (1st ICICS), Singapore, Vol. 1, September 1997, 52-58.
- [38] Trachtenberg, E.A., “Applications of Fourier Analysis on Groups in Engineering Practices”, in: [35], 331-403.
- [39] Trachtenberg, E.A., Karpovsky, M.G., “Filtering in a communication channel by Fourier transforms over finite groups”, In: M.G. Karpovsky, Ed. *Spectral Techniques and Fault Detection*, , Academic Press, Orlando, Florida, 1985.

Chapter 4

Gibbs Derivatives on Finite Groups

Differential operators are a very powerful tool for the mathematical modelling of natural phenomena. Usually the differentiation is with respect to time or to a spatial coordinate, modelled by the real line R . In this setting, using the differential operators, the direction, as well as the rate, of change of a quantity can be adequately described. Moreover, by forming linear differential equations with constant coefficients, we get a very convenient way of expressing the principle of superposition inherent in many natural phenomena. Linearity offers an easily tractable model, usually sufficiently good as a first approximation.

Fourier analysis, having linearity and the superposition principle in its essence, is another very efficient tool used for the same purposes.

It is known from classical analysis that there is a strong relationship between the Newton-Leibniz derivative f' of a function f on R and its Fourier transform, which can be expressed by

$$F'(w) = iwF(w), \quad (4.1)$$

where F' and F denote the Fourier transforms of f' and f , respectively.

Replacing the real group R by a locally compact Abelian, or a compact non-Abelian group extends classical Fourier analysis into abstract harmonic analysis. In this setting it has been natural to think about differentiation on groups in a way preserving as many as possible of the useful properties of Newton-Leibniz differentiation.

The Gibbs derivatives on groups [4], [8], [10], [20], [24], [33], [34], [40]

form a class of differential operators extending the relation (4.1) to the functional spaces on other groups. In this more general context the role of the Euler functions $\exp(jtx)$ (the characters of the real group R) is taken over by the characters of locally compact Abelian groups [4], [6], [8], [10], [18], [20], or by the unitary irreducible representations of compact non-Abelian groups [22]. In this book attention is focused on Gibbs derivatives on finite, not necessarily Abelian, groups. We consider in detail one of the matrix representations suggested in [11], which is suitable for the numerical evaluation of Gibbs derivatives of a given function. Using this matrix representation we present some FFT-like algorithms for calculation of the values of Gibbs derivatives on finite groups [29].

For a review of Gibbs differentiation see [30] and for some particular examples the bibliography [12] given in [3]. Some very recent results are reviewed in [30] and [38], [39].

4.1 Definition and properties of Gibbs derivatives on finite non-Abelian groups

As is noted above, Gibbs differential operators on Abelian groups are defined as linear operators having the group characters as their eigenfunctions, see for example [10]. Since the group characters are the kernels of Fourier transforms on locally compact Abelian groups, it is very convenient to characterize the Gibbs derivatives by Fourier coefficients. Moreover, the strong relationship between the Gibbs derivatives and Fourier coefficients is somewhere used as the starting point for introduction of Gibbs derivatives on some particular groups, see for example [18]. By using the same approach the Gibbs derivatives on finite non-Abelian groups are defined in terms of Fourier coefficients as follows [22].

Definition 4.1. Gibbs derivative Df of a function $f \in P(G)$ whose Fourier transform is S_f is defined by

$$(Df)(x) = \sum_{w=0}^{K-1} wTr(S_f(w)R_w(x)). \quad (4.2)$$

As is noted in [22] this definition is unique only by virtue of the fixed order adopted for the elements of Γ . If a different notation was adopted, then (4.2), though unchanged in appearance, would define a distinct differentiator. This phenomenon is already present in the definition of the dyadic Gibbs

derivative [7] which depends upon the order assumed for the Walsh functions. The same applies to all other Gibbs derivatives on various groups.

In what follows the Gibbs derivatives will be denoted by Df or, alternatively, by $f^{(1)}$.

An interpretation of the Gibbs derivative on finite non-Abelian groups can be given by following the approach used in [17] for the finite dyadic Gibbs derivative and later in [31] for the Gibbs derivatives on finite Abelian groups.

Define the partial sum $f_p(x)$, $p \leq K$ by

$$f_p(x) = \sum_{w=0}^{p-1} Tr(\mathbf{S}_f(w)\mathbf{R}_w(x)). \quad (4.3)$$

Define also the Fejèr sum as

$$\sigma_q(x) = q^{-1} \sum_{p=1}^q f_p(x). \quad (4.4)$$

Substituting (4.3) into (4.4) we have, after a simple calculation,

$$f(x) - \sigma_K(x) = K^{-1} \sum_{w=0}^{K-1} w Tr(\mathbf{S}_f(w)\mathbf{R}_w(x)).$$

The left member of this equality is the error in the approximation of f by its Fejèr sum $\sigma_K(x)$. Thus, the Gibbs derivative on a finite non-Abelian group G can be interpreted as that error multiplied by K .

The chief properties of Gibbs derivatives are analogs to the corresponding properties of the classical Newton-Leibniz derivative, and they are given by the following theorem.

Theorem 4.1. If $f \in P(G)$, then

1. $D(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 Df_1 + \alpha_2 Df_2$, $\alpha_1, \alpha_2 \in P$, $f_1, f_2 \in P(G)$.
2. $Df = 0 \in P$ iff f is a constant function.
3. If the Fourier transform of f is S_f , then that of $f^{(1)}$ is given by $S_{f^{(1)}}(w) = wS_f$, $w = 0, \dots, K-1$.

This property can be interpreted as the fact that the set $\{R_w^{(i,j)}(x)\}$ is the set of eigenfunctions of the Gibbs derivative, i.e.,

$$DR_w^{(i,j)}(x) = wR_w^{(i,j)}(x).$$

From that, thanks to the linearity of Gibbs derivatives,

$$DTrR_w(x) = wTrR_w(x).$$

4. From the property 3, it easily follows that

$$D(f_1 * f_2) = (Df_1) * f_2 = f_1 * (Df_2), \quad f_1, f_2 \in P(G),$$

where $*$ denotes the convolution on G .

5. The Gibbs derivative commutes with the translation (shift) operator T on G defined by $(T^\tau f)(x) = f(\tau \circ x^{-1})$, i.e.,

$$D(T^\tau f) = T^\tau(Df), \quad \text{for each } \tau \in G.$$

6. It is known that the Gibbs differential operators do not obey the product rule. The same applies to the Gibbs derivatives on finite non-Abelian groups, i.e., it is false that for each f_1 and f_2

$$D(f_1 f_2) = f_1(Df_2) + (Df_1)f_2.$$

The Gibbs derivatives can be extended to an arbitrary complex order k by way of the definition of the delta function:

$$\delta(x) = g^{-1} \sum_{w=0}^{K-1} r_w TrR_w(x).$$

The δ -function thus defined has the property

$$\delta(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

The Gibbs derivative of order k of the δ -function is obtained by a direct application of Property 3

$$\delta^{(k)}(x) = g^{-1} \sum_{w=0}^{K-1} w^k r_w TYrR_w(x).$$

By using Property 4 of Theorem 2.5:

$$(D^k f)(x) = ((D^k \delta) * f)(x) = \sum_{w=0}^{K-1} w^k Tr(\mathbf{S}_f(w)\mathbf{R}_w(x)). \quad (4.5)$$

4.2 Gibbs anti-derivative

In this section we consider the determination of the values of a function from the values of its Gibbs derivative.

It is obvious from (4.5) that the Gibbs derivative can be considered as a convolution operator on $P(G)$. More precisely, if we introduce a function W_k defined by its Fourier coefficients as

$$S_{W_k}(w) = \begin{cases} 0, & w = 0, \\ r_w g^{-1} w^k \mathbf{I}_{r_w}, & w = 1, \dots, K-1, \end{cases}$$

where \mathbf{I}_{r_w} is the $(r_w \times r_w)$ identity matrix, then from (4.5) the Gibbs derivative of order k of a function $f \in P(G)$ is given by

$$(Df)(w) = (W_k * f)(x).$$

From here we immediately deduce the concept of the Gibbs anti-derivative. Introduce a function W_{-k} defined in the transform domain by

$$S_{W_{-k}}(w) = \begin{cases} 1, & w = 0 \\ r_w w^{-k} \mathbf{I}_{r_w} & w = 1, \dots, K-1. \end{cases}$$

After the inverse Fourier transform

$$W_{-k}(x) = 1 + \sum_{w=1}^{K-1} w^{-k} r_w Tr(\mathbf{R}_w(x)).$$

Functions of this kind for the particular case of the dyadic group were apparently first investigated in [37] in the Walsh-Fourier multiplier theory. Such functions were later used for the dyadic derivatives in [4] for the same purposes as those considered here. To be consistent with these particular definitions we omitted the factor g^{-1} the appearance of which could be expected from the convolution theorem.

By using the function W_{-k} we introduce an inverse operator called the Gibbs anti-derivative on finite non-Abelian groups [23]

Definition 4.2. For a function $f \in P(G)$ the Gibbs anti-derivative of order k , denoted by I^k , is defined by

$$(I^k f)(x) = (W_k * f)(x).$$

The Gibbs anti-derivative can be considered as a Fourier multiplier operator, thus having all properties characteristic for these operators. Therefore, there is no need for any particular consideration of these properties here.

Having the concept of Gibbs anti-derivative, we can deduce a theorem which shows how to determine the values of a function f from the values of its Gibbs derivative of order k .

Theorem 4.2. Let $f \in P(G)$ be such that $S_f(0) = 0$. Then,

$$f(x) = g^{-1}I^k(D^k f)(x),$$

or, equivalently,

$$f(x) = g^{-1}D^k(I^k f)(x).$$

Here the facto g^{-1} appears at the right hand side of the above equations since it was omitted in the definition of $S_{W^{-k}}$.

Note that Theorem 4.2 can be regarded as a kind of counterpart of the so-called fundamental theorem for dyadic analysis due to Butzer and Wagner [4]. Moreover, as is noted in [2], see also [21], theorems of this kind are a kind of counterpart of the fundamental theorem of the Newton-Leibniz calculus in abstract harmonic analysis.

4.3 Partial Gibbs derivatives

As we noted in 2.2 a given function $f \in P(G)$, G -decomposable group, can be considered as a function of several variables $f(x_1, \dots, x_n)$, $x_i \in G_i$. Therefore, partial Gibbs derivative of f with respect to the variable x_i can be defined [26].

From Definition 4.1 and some well-known properties of unitary irreducible representations

$$(Df)(x) = \sum_{w=0}^{K-1} wTr(r_w g^{-1} \sum_{u=0}^{g-1} f(u)\mathbf{R}_w(u^{-1} \circ x)).$$

From the invariance under translation of the Haar integral

$$\sum_{y \in G} g(y) = \sum_{y \in G} g(z \circ y), \forall z \in G, g \in P(G),$$

it follows

$$(Df)(x) = g^{-1} \sum_{u=0}^{g-1} f(u \circ x) \sum_{w=0}^{K-1} wr_w Tr(\mathbf{R}_w(u^{-1})),$$

which suggests the following definition.

Definition 4.3. Partial Gibbs derivative $(\Delta_i f)(x)$ at a point $x = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \in G$ with respect to the i -th variable x_i of a function $f \in P(G)$ is defined as the Gibbs derivative $(Df_i)(x_i)$, at x_i , of the function $f_i(y) = f(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n)$. Thus,

$$(\Delta_i f)(x) = (Df_i)(x_i) = g^{-1} \sum_{u_i}^{g_i-1} f(x_1, \dots, x_{i-1}, u_i, x_{i+1}, \dots, x_n) \sum_{w=0}^{K_i-1} w r_w^i \text{Tr}(\mathbf{R}_w^i(u_i^{-1})),$$

where g_i is the order of G_i , K_i denotes the number of nonequivalent unitary irreducible representations of G_i , and r_w^i is the dimension of the representation \mathbf{R}_w^i of G_i .

Actually, the partial Gibbs differentiator Δ_i thus defined is the restriction on G_i of the Gibbs differentiator on G . It follows that the partial Gibbs derivatives have properties corresponding to those of the Gibbs derivative.

Theorem 4.3. Let $f \in P(G)$, Then,

1. $\Delta_i f = 0$ iff f is a constant on G_i , i.e., iff f has the same value for each $x_I \in G_i$. Moreover, $\Delta_i c = 0$ for any constant $c \in P(G)$.
2. $\Delta_i(c_1 f_1 + c_2 f_2) = c_1 \Delta_i f_1 + c_2 \Delta_i f_2$, $c_1, c_2 \in P$, $f_1, f_2 \in P(G)$.
3. If the Fourier transform of $f \in P(G)$ is S_f , then that of $\Delta_i f$ is given by

$$S_{\Delta_i f}(w) = B_i(w) S_f(w), \quad w = 0, \dots, K = 1,$$

where $B_i(w)$ is given by the vector of order g of the form $B_i(w) = [0, 1, \dots, K_i - 1, 0, 1, \dots, K_i - 1, \dots, 0, 1, \dots, K_i - 1]^T$.

4. $\Delta_i(f * g) = \Delta_i f * g = f * \Delta_i g$.

4.4 Gibbs differential equations

Relation (4.5) introduces the Gibbs derivative of an arbitrary complex order. The Gibbs derivative of a positive integer order n can be defined recursively by $D^{n+1}f = D(D^n f)$, $n = 1, 2, \dots$. This permits linear equations with constant coefficients in terms of Gibbs derivatives to be defined and solved.

These equations can be considered as a particular case of the generalized linear equations studied in [16].

Definition 4.4. A linear Gibbs discrete differential equation with constant coefficients is an equation of the form

$$\sum_{k=0}^n a_k y^{(k)} = \sum_{k=0}^m b_k f^{(k)}, \quad (4.6)$$

where a_k, b_k are real numbers, $f \in P(G)$ and y is the required solution.

As in the case of ordinary differential equations we get the general solution, y , of the equation (4.6) as the sum of the solution y_{zi} of the homogeneous equation and the partial solution y_{zs} of the inhomogeneous equation, i.e.,

$$y = y_{zi} + y_{zs}. \quad (4.7)$$

In order to find y_{zi} one looks for roots of the characteristic equation of (4.6)

$$\sum_{k=0}^n a_k z^k = 0.$$

Now, we have the following theorem.

Theorem 4.4. If the roots $\{z_i\}, i = 0, \dots, n$ of the characteristic equation are distinct and belong to the set $\{0, \dots, K-1\}$, then the homogeneous solution of (4.6) is

$$y_{zi}(x) = \sum_{i=0}^n \sum_{j,k=1}^i c_{jk}^{z_i} R_{z_i}^{(j,k)}(x),$$

where the constants $c_{jk}^{z_i}$ depend on the boundary conditions.

There are some important differences encountered in solving linear Gibbs discrete differential equations with constant coefficients compared to solving of ordinary differential equations. A homogeneous equation of order n does not always have n linearly independent solutions. The following statements are, in a way, often taken for granted, however, we could not find a proof for these statements, anywhere.

If t roots of the characteristic equation are repetitions of the other roots, then the number of linearly independent solutions of a linear Gibbs discrete

differential equation of order k is $\sum_{i=0}^{k-t} r_{z_i}^2$, provided that each root of the characteristic equation is in the set $\{0, \dots, K-1\}$.

If s of the roots are not in this set, then the number of linearly independent solutions of the given equation is $\sum_{i=0}^{k-s-t} r_{z_i}^2$. This is not any peculiarity of the case considered here. A corresponding statement for so-called logical differential equations with Gibbs derivatives on the dyadic group is given in [9]. Moreover, it seems that an analogous statement is valid more generally, as it is noted without proof in [16].

To get the particular solution of (4.6) we apply the Fourier transform on both sides of (4.6), and with property 3 of Theorem 4.1 we obtain:

$$\sum_{k=0}^n a_k w^k S_y(w) = \sum_{k=0}^m b_k w^k S_f(w). \quad (4.8)$$

From there, providing that equation (4.8) is compatible, that is, $S_f(0) = 0$ for all $w \in \{z_i\}, i = 0, \dots, n$, we have

$$S_y(w) = \frac{P}{Q} S_f(w),$$

where

$$P = \sum_{k=0}^m b_k w^k, \quad Q = \sum_{k=0}^m a_k w^k.$$

By introducing the notation

$$H(w) = r_w g^{-1} \frac{P}{Q}, \quad (4.9)$$

we have

$$S_y(w) = r_w^{-1} H(w) S_f(w). \quad (4.10)$$

From (4.10) by using the convolution property, the inverse Fourier transform produces the particular solution

$$y_{zs}(x) = \sum_{u=0}^{g-1} h(u) f(xu^{-1}). \quad (4.11)$$

It follows that (4.6) has a general solution of the form

$$y(x) = \sum_{i=0}^n \sum_{j,k=1}^m c_{jk}^{z_i} R_{z_i}^{(j,k)}(x) + \sum_{u=0}^{g-1} h(u) f(xu^{-1}).$$

4.5 Matrix interpretation of Gibbs derivatives

In this section we will consider a matrix representation of Gibbs differential operators suitable for their numerical evaluation. One of the main properties characterizing Gibbs derivatives is given by,

$$\mathbf{S}_{Df}(w) = w\mathbf{S}_f(w), \quad w \in \{0, 1, \dots, K-1\}, \quad (4.12)$$

where $\mathbf{S}_f(w)$ are the Fourier coefficients of a function $f \in P(G)$, while $\mathbf{S}_{Df}(w)$ denotes the Fourier coefficients of its Gibbs derivative $D_g f$. Moreover, a wish to have a differential operator satisfying this property motivated the introduction of the class of differential operators considered here. In this setting, the relation (4.12) is used by some authors as a starting point for defining certain Gibbs derivatives on groups in terms of formal Fourier series (see, for example, [18],[24], [25], [40], [33], [34]). Using this approach, the Gibbs derivative on a finite group can be defined in matrix notation as follows.

Definition 4.5. The Gibbs derivative \mathbf{D}_g on a finite, not necessarily Abelian, group G of order g is defined [15], [29] as

$$\mathbf{D}_g = g^{-1}[\mathbf{R}] \circ \mathbf{G} \odot [\mathbf{R}]^{-1},$$

where $[\mathbf{R}]$ is the matrix of unitary irreducible representations of G over P , i.e., $[\mathbf{R}] = [\mathbf{a}_{ij}]$ with $\mathbf{a}_{ij} = \mathbf{R}_j(i)$, $j \in \{0, 1, \dots, g-1\}$, $i \in \{0, 1, \dots, K-1\}$, \mathbf{G} is a diagonal $(K \times K)$ matrix given by $\mathbf{G} = \text{diag}(0, 1, \dots, K-1)$, and $[\mathbf{R}]^{-1} = [\mathbf{b}_{sq}]$ with $\mathbf{b}_{sq} = r_s \mathbf{R}_s^{-1}(q)$, $s \in \{0, 1, \dots, K-1\}$, $q \in \{0, 1, \dots, g-1\}$.

For a function $f(x) = f(x_1, \dots, x_n) \in P$ we define the partial Gibbs derivative with respect to the variable $x_i \in G$ as a restriction on G_i of the previously introduced Gibbs derivative on G .

Definition 4.6. Let G be representable in the form (2.9). The partial Gibbs derivative Δ_i with respect to the variable x_i is defined as:

$$\Delta_i = \bigotimes_{j=1}^n \mathbf{A}_j,$$

with

$$\mathbf{A}_j = \begin{cases} g_j \mathbf{D}_{g_j}, & j = i, \\ \mathbf{I}_{(g_j \times g_j)}, & j \neq i, \end{cases}$$

where $\mathbf{I}_{(g_j \times g_j)}$ is a $(g_j \times g_j)$ identity matrix, and \otimes denotes the Kronecker product.

Note that, using the representation (2.13) for the non-negative integers $\{0, 1, \dots, K-1\}$, the matrix \mathbf{G} can be expressed as:

$$\mathbf{G} = \sum_{i=1}^n \mathbf{w}_i \mathbf{Q}_i,$$

where,

$$\mathbf{Q}_i = \bigotimes_{j=1}^n \mathbf{z}_j^i,$$

with

$$\mathbf{z}_j^i = \begin{cases} \mathbf{G}_j, & i = j, \\ \mathbf{I}_{(K_i \times K_i)}, & i \neq j, \end{cases}$$

where \mathbf{G}_j is a diagonal $(K_j \times K_j)$ matrix given by $\mathbf{G}_j = \text{diag}(0, b_j, b_j, \dots, b_j)$ with b_j defined by (2.13).

Recall that the matrix $[\mathbf{R}]$ is the matrix of unitary irreducible representations of G over P . Since G is representable in the form (2.9), the matrix $[\mathbf{R}]$ can be generated as the Kronecker product of $(K_i \times g_i)$ matrices $[\mathbf{R}_i]$ of unitary irreducible representations of subgroups G_i , $i = 1, \dots, n$, i.e.,

$$[\mathbf{R}] = \bigotimes_{i=1}^n [\mathbf{R}_i].$$

Thanks to the well-known properties of the Kronecker product, the same applies to the matrix $[\mathbf{R}]^{-1}$, i.e, for this matrix holds

$$[\mathbf{R}]^{-1} = \bigotimes_{i=1}^n [\mathbf{R}_i]^{-1}.$$

By using the representations introduced above for the matrices $[\mathbf{R}]$, $[\mathbf{R}]^{-1}$ and \mathbf{G} , the matrix \mathbf{D}_g of the Gibbs derivative can be rewritten as

$$\mathbf{D}_g = g^{-1} \left[\bigotimes_{i=1}^n [\mathbf{R}_i] \right] \circ \left[\sum_{i=1}^n \mathbf{w}_i \mathbf{Q}_i \right] \odot \left[\bigotimes_{i=1}^n [\mathbf{R}_i]^{-1} \right].$$

After a short calculation, by using well-known properties of the Kronecker product we prove the following.

Proposition 1. The matrix \mathbf{D}_g representing the Gibbs derivative on a finite group G of order g can be expressed in terms of partial Gibbs derivatives as

$$\mathbf{D}_g = \sum_{i=1}^n b_i \Delta_i, \quad (4.13)$$

where the coefficients b_i are defined by (2.13).

4.6 Fast algorithms for calculation of Gibbs derivatives on finite groups

In this section we will disclose fast algorithms for computation of Gibbs derivatives on finite groups. As is noted in [29], the application of Definition 4.5 leads to an algorithm for the computation of Gibbs derivatives of which the complexity is obviously approximately equal to the complexity of calculation of one direct and one inverse Fourier transform. The advantage is that the application of fast Fourier transform on groups is immediately possible without any considerable modification. However, from the computational point of view a more efficient algorithm for the computation of Gibbs derivatives on finite groups can be disclosed defining the Gibbs derivative in terms of partial Gibbs derivatives, that is, starting from the relation (4.13). Moreover, the algorithm thus obtained is quite suitable for a parallel implementation. The idea comes from the following facts.

As is noted in Section 3.1, the definition of the fast Fourier transform (FFT) on group G is based upon the factorization of G into the equivalence classes relative to some subgroups of G . On the other hand, the i -th partial Gibbs derivative on a group G representable in the form (2.9) is defined as the restriction of Gibbs differentiation on G to the differentiation on G_i . Therefore, it is natural to search for a fast algorithm for the computation of Gibbs derivatives through the partial Gibbs derivatives.

Note that the i -th partial Gibbs derivative is defined (Definition 4.6) by a relation of the form (3.1). Therefore, comparing the matrices Δ_i , $i \in \{1, \dots, n\}$, for a given group G with the matrices $[\mathbf{C}^{n-k}]$ appearing in the factorization of the Fourier transformation matrix, we infer a strong similarity. It follows that the algorithm for the computation of i -th partial Gibbs derivative will be similar to the i -th step of the FFT, and, hence, may be described by a flow-graph similar to that describing the i -th step of the FFT. Naturally, the similarity is less strong in the case of non-Abelian groups than

in the case of Abelian groups, for the dual object Γ of a non-Abelian group G does not have the structure of a group isomorphic with G as is the case for Abelian groups. More precisely, the cardinality of Γ is not equal to the order of G , and the consequence is that in the case of non-Abelian groups the number of input nodes in the flow-graph is different for each step of the FFT, as we noted above, while all partial Gibbs derivatives are by definition applicable to the vector \mathbf{f} whose order is g . Nevertheless, the similarity in the overall structure of the corresponding flow-graphs is retained and can be efficiently used for the disclosure of the fast algorithms for the computation of partial Gibbs derivatives.

As we noted above, the flow-graph for the computation of the k -th partial Gibbs derivative of a function f on a finite group G of order g consists of g input and g output nodes of which some are connected by branches. It is determined by the overall structure of the matrix Δ_k which nodes will be mutually connected, as in the case of the FFT. More precisely, the output node j will be connected with the input node i iff the element d_{ij} of Δ_k describing the k -th partial Gibbs derivative is not equal to zero. As in the case of the FFT, a weighting coefficient is associated to each branch. However, all weights in the fast algorithm for the computation of Gibbs derivatives are numbers belonging to P even for non-Abelian groups, which is another considerable difference relative to FFT for this case. Denoting by $k(i, j)$ the branch connecting the output node i with the input node j , the weight $w^k(i, j)$ associated with this branch is given by $w(i, j) = d_{ij}$, where d_{ij} is the (i, j) -th element of Δ_k .

Having the fast algorithms for the computation of partial Gibbs derivatives, one obtains the fast algorithm for the computation of the Gibbs derivative of a function $f \in P(G)$ according to (4.13) simply by adding the output nodes of the flow-graphs for the calculation of partial Gibbs derivatives multiplied by the weight coefficients b_i defined by (2.13).

Now, we give a brief analysis of the complexity of the algorithm described above.

The number of calculations is usually employed as a first approximation to the complexity of an algorithm.

It is obvious from Definition 4.6 that the number of calculations required to calculate $\Delta_i f$ is equal to gg_i , since there are at most g_i non-zero elements in each row of the matrix Δ_i . According to (4.13), the number of operations needed to calculate the Gibbs derivative $D_g f$ is equal to $g(\sum_{i=1}^n g_i)$ followed by $(n-1)g$ multiplications with the weighting factors b_i and ng additions. Recall that the number of calculations in an FFT to which our algorithm

can be compared is $g(\sum_{i=1}^n g_i)$.

However, to confirm the efficiency of the algorithm proposed it is important to give at least a rough estimate of its overall time complexity.

It is obvious from Definition 4.6 that, unlike the FFT, which is a sequential algorithm in its essence for the input to one stage is the output from the preceding stage, the fast algorithm for the calculation of Gibbs derivatives (FGD) is quite suitable for a parallel implementation, since the calculation of the partial Gibbs derivatives can be carried out simultaneously. This parallelism is over and above the parallelism in each step of the calculation of each Δ_I as in the corresponding step of the FFT. It follows that the time complexity of the FGD does not depend on the number of subgroups of G as is the case with the FFT. The FGD is considerably faster than the corresponding FFT, since it can always be implemented in only two steps, compared with the n steps required in the FFT. Of course, the price is the number of processors operating in parallel and the memory storage requirements, which in this case should certainly be greater for the FGD than for the FFT.

A more accurate analysis of the complexity of the FGD is certainly needed, but to be correct it may only be done after rather precise specification of the facilities used for the implementation.

As is usually the case in the study of problems like that considered here, the procedure for the numerical calculation is best explained by some examples. We shall therefore consider two examples, the first for Abelian groups and the second for non-Abelian groups.

Example 4.1 Let $G = Z_9 = (\{0, 1, 2, 3, 4, 5, 6, 7, 8\}, \circ)$ be the group of non-negative integers less than 9 with componentwise addition modulo 3 of 3-adic expansions of group elements as the group operation. For convenience the group operation is shown in Table 4.1. The group representations of Z_9 over the complex field are the Vilenkin-Chrestenson functions shown in a matrix form in Fig. 4.1. Note that the group Z_9 can be considered as the product $Z_9 = Z_3 \times Z_3$ where $Z_3 = (\{0, 1, 2\}, \dot{3})$ is the group of non-negative integers less than 3 with addition modulo 3 as group operation. Therefore, any complex-valued function f on Z_9 can be considered as a two-variable function $f(x_1, x_2)$, $x_1, x_2 \in Z_3$.

The matrices Δ_1^9 and Δ_2^9 of the partial Gibbs derivatives relative to the variables x_1 and x_2 , respectively, and the matrix \mathbf{D}_9 of the Gibbs derivative on Z_9 calculated according to (4.13) as $\mathbf{D}_9 = 3\Delta_1^9 + \Delta_1^9$ are shown in Fig.4.2 a,b,c, respectively. The flow-graph of the fast algorithm for the computation

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e_1 & e_2 & 1 & e_1 & e_2 & 1 & e_1 & e_2 \\ 1 & e_2 & e_1 & 1 & e_2 & e_1 & 1 & e_2 & e_1 \\ 1 & 1 & 1 & e_1 & e_1 & e_1 & e_2 & e_2 & e_2 \\ 1 & e_1 & e_2 & e_1 & e_2 & 1 & e_2 & 1 & e_1 \\ 1 & e_2 & e_1 & e_1 & 1 & e_2 & e_2 & e_1 & 1 \\ 1 & 1 & 1 & e_2 & e_2 & e_2 & e_1 & e_1 & e_1 \\ 1 & e_1 & e_2 & e_2 & 1 & e_1 & e_1 & e_2 & 1 \\ 1 & e_2 & e_1 & e_2 & e_1 & 1 & e_1 & 1 & e_2 \end{bmatrix},$$

$$e_1 = -\frac{1}{2}(1 - i\sqrt{3}), \quad e_2 = -\frac{1}{2}(1 + i\sqrt{3})$$

Figure 4.1: The group representations of Z_9 over C .

of the Gibbs derivative \mathbf{D}_9 of a complex-valued function f on Z_9 given by its truth vector $\mathbf{f} = [f(0), \dots, f(8)]^T$ is shown in Fig.4.3.

Table 4.1. Group operation of Z_9 .

\circ	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	6	7	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

Now, let us consider as the second example the calculation of the Gibbs derivatives of functions defined on the non-Abelian group of binary matrices described in [15].

Example 4.2 Let G be the multiplication group of the twelve (3×3) matrices $\mathbf{t} = [t_{ij}]$, $i, j = 0, 1, 2$, over the complex field represented in Table 4.3. For convenience the group operation is explicitly shown in Table 4.2. Note that G is isomorphic to the direct product of the cyclic group $C_2 = (\{0, 1\}, \circ)$ of order 2 with generating element 1 and the symmetric group of permutations S_3 (see Table 2.1 and Table 2.5). Table 4.3 lists also all absolutely irreducible representations for the given group $G = C_2 \times S_3$

$$\Delta_1^9 = \begin{bmatrix} 1 & 0 & 0 & a & 0 & 0 & b & 0 & 0 \\ 0 & 1 & 0 & 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & 1 & 0 & 0 & a & 0 & 0 & b \\ b & 0 & 0 & 1 & 0 & 0 & a & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 & 0 & a & 0 \\ 0 & 0 & b & 0 & 0 & 1 & 0 & 0 & a \\ a & 0 & 0 & b & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & b & 0 & 0 & 1 \end{bmatrix},$$

a.

$$\Delta_2^9 = \begin{bmatrix} 1 & a & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ a & b & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & b & 1 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b & 1 \end{bmatrix},$$

b.

$$\mathbf{D}_9 = \begin{bmatrix} 4 & a & b & 3a & 0 & 0 & 3b & 0 & 0 \\ b & 4 & a & 0 & 3a & 0 & 0 & 3b & 0 \\ a & b & 4 & 0 & 0 & 3a & 0 & 0 & 3b \\ 3b & 0 & 0 & 4 & a & b & 3a & 0 & 0 \\ 0 & 3b & 0 & b & 4 & a & 0 & 3a & 0 \\ 0 & 0 & 3b & a & b & 4 & 0 & 0 & 3a \\ 3a & 0 & 0 & 3b & 0 & 0 & 4 & a & b \\ 0 & 3a & 0 & 0 & 3b & 0 & b & 4 & a \\ 0 & 0 & 3a & 0 & 0 & 3b & a & b & 4 \end{bmatrix},$$

c.

$$a = \frac{1}{3}(e_1 - 1), \quad b = \frac{1}{3}(e_2 - 1)$$

Figure 4.2: a. The partial Gibbs derivative Δ_1^9 on Z_9 , b. The partial Gibbs derivative Δ_2^9 on Z_9 , c. The Gibbs derivative \mathbf{D}_9 on Z_9 .

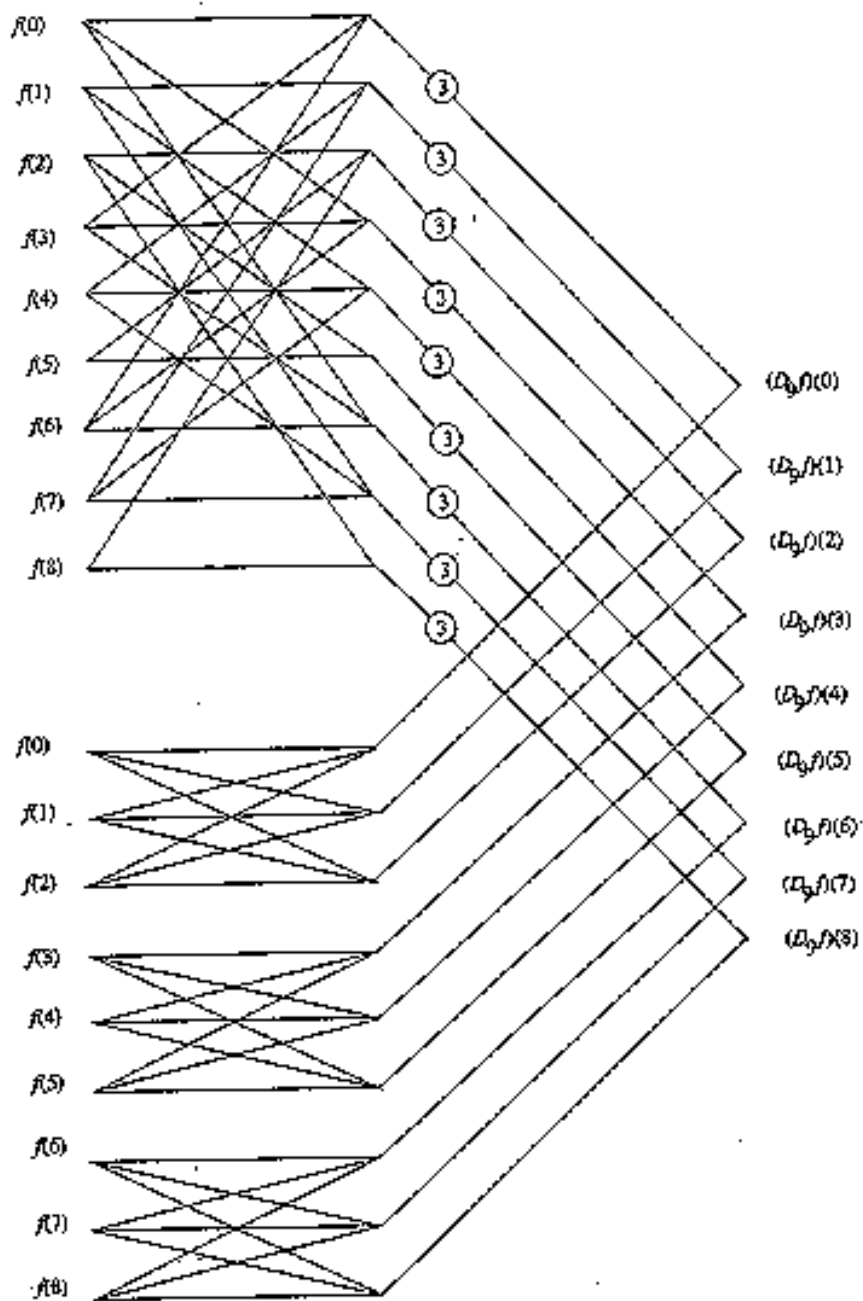


Figure 4.3: The flow-graph of the fast algorithm for the calculation of the Gibbs derivative D_9 on Z_9 .

Table 4.2. The group operation of G_{12} .

\circ	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	5	3	4	7	8	6	11	9	10
2	2	0	1	4	5	3	8	6	7	10	11	9
3	3	4	5	0	1	2	9	10	11	6	7	8
4	4	5	3	2	0	1	10	11	9	8	6	7
5	5	3	4	1	2	0	11	9	10	7	8	6
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	6	11	9	10	1	2	0	5	3	4
8	8	6	7	10	11	9	2	0	1	4	5	3
9	9	10	11	6	7	8	3	4	5	0	1	2
10	10	11	9	8	6	7	4	5	3	2	0	1
11	11	9	10	7	8	6	5	3	4	1	2	0

over the Galois field $\text{GF}(11)$ ($\text{GF}(11)$ is a splitting field for G). A given function $f : G \rightarrow \text{GF}(11)$ can be considered as a two-variable function $f(x_1, x_2)$, $x_1 \in \{0, 1\}$, $x_2 \in \{0, 1, 2, 3, 4, 5\}$.

The matrices Δ_1^{12} and Δ_2^{12} of partial Gibbs derivatives with respect to the variables x_1 and x_2 are shown in Fig. 4.4 and Fig. 4.5, respectively. For the given group G the number of unitary irreducible representations is $K = K_1 K_2$ with $K_1 = 2$, $K_2 = 3$, and, therefore, the matrix \mathbf{D}_{12} of the Gibbs derivative may be calculated according to (4.13) as $\mathbf{D}_{12} = 3\Delta_1^{12} + \Delta_2^{12}$. This matrix is shown in Fig. 4.6.

The flow-graph of the fast algorithm for the calculation of the Gibbs derivative of a function f mapping G into $\text{GF}(11)$ is shown in Fig. 4.7.

Table 4.3. The representations of G_{12} over GF(11).

x	t	\mathbf{R}_0	\mathbf{R}_1	\mathbf{R}_2	\mathbf{R}_3	\mathbf{R}_4	\mathbf{R}_5
0	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$	1	1	$\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$	1	1	$\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$
2	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$	1	1	$\begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$	1	1	$\begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$
3	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	1	10	$\begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$	1	10	$\begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$
4	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	1	10	$\begin{bmatrix} 5 & 8 \\ 8 & 6 \end{bmatrix}$	1	10	$\begin{bmatrix} 5 & 8 \\ 8 & 6 \end{bmatrix}$
5	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$	1	10	$\begin{bmatrix} 5 & 3 \\ 3 & 6 \end{bmatrix}$	1	10	$\begin{bmatrix} 5 & 3 \\ 3 & 6 \end{bmatrix}$
6	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	10	10	$\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$
7	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$	1	1	$\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$	10	10	$\begin{bmatrix} 6 & 3 \\ 8 & 6 \end{bmatrix}$
8	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$	1	1	$\begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$	10	10	$\begin{bmatrix} 6 & 8 \\ 3 & 6 \end{bmatrix}$
9	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	1	10	$\begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$	10	1	$\begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$
10	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	1	10	$\begin{bmatrix} 5 & 8 \\ 8 & 6 \end{bmatrix}$	10	1	$\begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix}$
11	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$	1	10	$\begin{bmatrix} 5 & 3 \\ 3 & 6 \end{bmatrix}$	10	1	$\begin{bmatrix} 6 & 8 \\ 8 & 5 \end{bmatrix}$

$$\Delta_1^{12} = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 5 \\ 5 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Figure 4.4: The partial Gibbs derivative Δ_1^{12} on G_{12} .

4.6.1 Complexity of Calculation of Gibbs Derivatives

There are different approaches to efficiently calculate the Gibbs derivatives on finite groups. From Definition 6, the Gibbs derivatives can be regarded as convolution operators and, therefore, can be calculated through convolution algorithms [19]. These algorithms can be derived in analogy to the convolution algorithms defined in terms of FFT [1], [19]. For Gibbs derivatives in $P(G)$, time and space complexity approximate to $O(2n + 1)$, and $O(g)$, respectively, if in-place computation [35] is assumed.

FFT-like algorithms for calculation of Gibbs derivatives are derived through the application of the Good-Thomas factorization [13], [36], to the matrix representing the Gibbs derivative on a given group G [29]. Unlike the algorithms for calculation of Fourier transform on groups, the steps in FFT-like algorithms for Gibbs derivatives can be performed simultaneously. That approach reduces the time complexity at the price of the space complexity. Gibbs derivative on any finite group can be calculated in two steps. However, the space complexity approximates to $O(ng + g)$.

4.7 Calculation of Gibbs derivatives through DDs

Both convolution and FFT-like algorithms for Gibbs derivatives are based upon the truth-vector representation of a given function f on G . Therefore,

$$\Delta_2^{12} = \begin{bmatrix} 7 & 5 & 5 & 9 & 9 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 7 & 5 & 9 & 9 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 7 & 9 & 9 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 9 & 9 & 7 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 9 & 9 & 5 & 7 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 9 & 9 & 5 & 5 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 5 & 5 & 9 & 9 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 7 & 5 & 9 & 9 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 5 & 7 & 9 & 9 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 & 9 & 7 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 & 9 & 5 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 & 9 & 5 & 5 & 7 \end{bmatrix}$$

Figure 4.5: The partial Gibbs derivative Δ_2^{12} on G_{12} .

$$\mathbf{D}_{12} = \begin{bmatrix} 3 & 5 & 5 & 9 & 9 & 9 & 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 5 & 9 & 9 & 9 & 0 & 4 & 0 & 0 & 0 & 0 \\ 5 & 5 & 3 & 9 & 9 & 9 & 0 & 0 & 4 & 0 & 0 & 0 \\ 9 & 9 & 9 & 3 & 5 & 5 & 0 & 0 & 0 & 4 & 0 & 0 \\ 9 & 9 & 9 & 5 & 3 & 5 & 0 & 0 & 0 & 0 & 4 & 0 \\ 9 & 9 & 9 & 5 & 5 & 3 & 0 & 0 & 0 & 0 & 0 & 4 \\ 4 & 0 & 0 & 0 & 0 & 0 & 3 & 5 & 5 & 9 & 9 & 9 \\ 0 & 4 & 0 & 0 & 0 & 0 & 5 & 3 & 5 & 9 & 9 & 9 \\ 0 & 0 & 4 & 0 & 0 & 0 & 5 & 5 & 3 & 9 & 9 & 9 \\ 0 & 0 & 0 & 4 & 0 & 0 & 9 & 9 & 9 & 3 & 5 & 5 \\ 0 & 0 & 0 & 0 & 4 & 0 & 9 & 9 & 9 & 5 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 4 & 9 & 9 & 9 & 5 & 5 & 3 \end{bmatrix}$$

Figure 4.6: The Gibbs derivative \mathbf{D}_{12} .

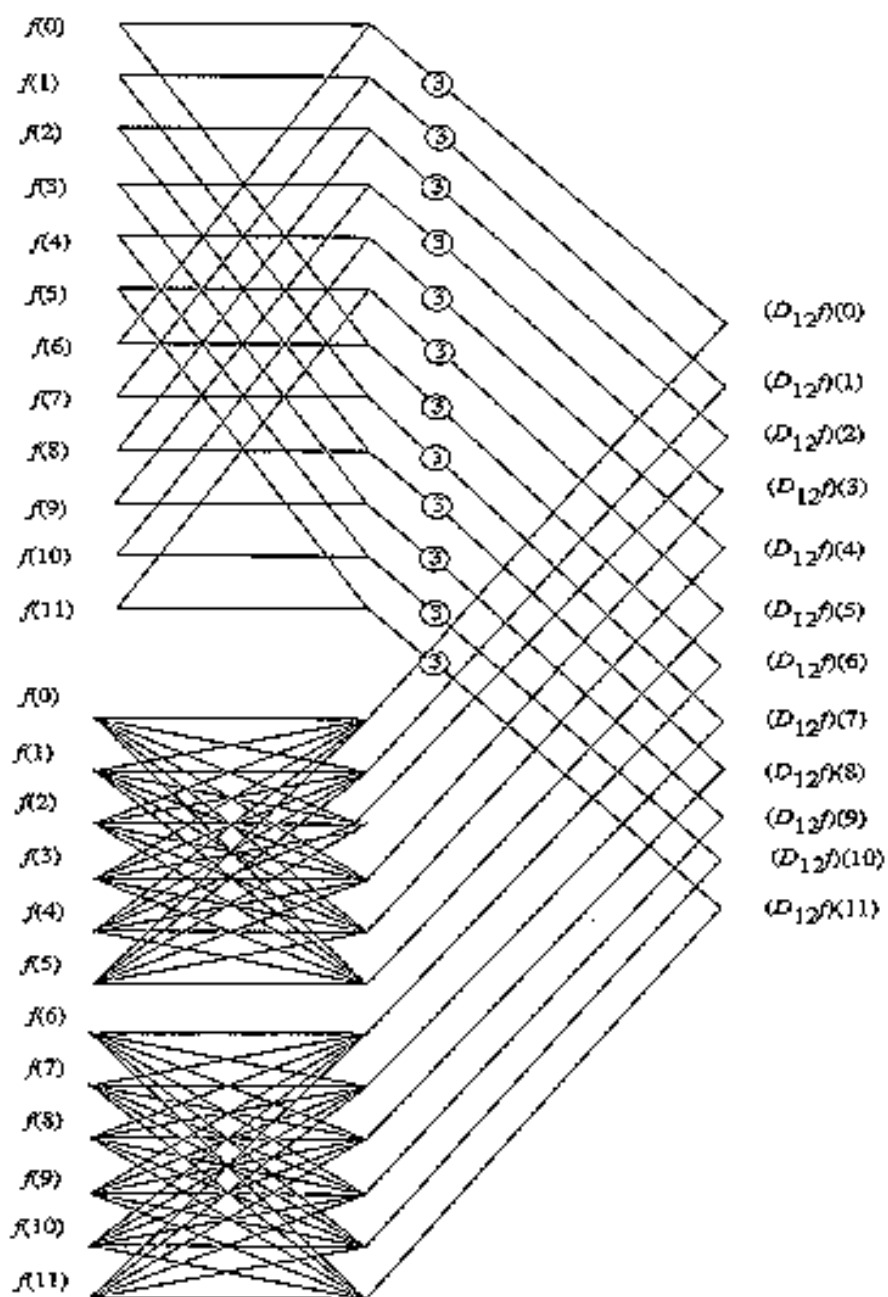


Figure 4.7: The flow graph of the fast algorithm for calculation of the Gibbs derivative D_{12} on G_{12} .

their complexity is determined by the order g of G . In practical applications that limits the use of these algorithms to functions of a relatively small number of variables. Algorithms based on MTDDs permit calculation of Gibbs derivatives of functions of a considerable number of variables.

A procedure to calculate Gibbs derivatives is based on decomposition of Gibbs derivative into the linear combination of partial Gibbs derivatives in (3). It is derived as a generalization of the procedure for calculation of the Fourier transform on non-Abelian groups and as a modification of the procedure for calculation of Gibbs derivatives on finite Abelian groups through DDs [32]. The procedure consists of the following steps.

Procedure for calculation of Gibbs derivatives

1. Represent f by the MTDD.
2. Determine partial Gibbs derivatives.
3. Determine the Gibbs derivative as the linear combination of partial Gibbs derivatives.

The partial Gibbs derivatives are calculated through MTDD for f and represented again by MTDDs. The Gibbs derivative is determined by adding MTDDs representing the partial Gibbs derivatives.

4.7.1 Calculation of partial Gibbs derivatives

The procedure for calculation of partial Gibbs derivatives is similar to that for FFT. The difference is in the processing rules applied at the nodes and cross points in the MTDD for f . For the partial Gibbs derivative with respect to x_i , the nodes and cross points at the i -th level are processed by the rule determined by \mathbf{D}_i . The nodes and cross points at the other levels are processed by the rules determined by the identity matrices of the corresponding orders as determined in Definition 7. We assume that the x_1 is assigned to the root node, and the other variables are assigned to the other levels in the increasing order.

Procedure for calculation of partial Gibbs derivative D_i

Given a function f on the decomposable group G of the form (1).

1. Represent f by the MTDD.

2. Process the nodes and cross points in the MTDD in a recursive way level by level starting from the nodes at the level to which x_n is assigned up to the root node.
3. For $j = n$ to 1, process the nodes and the cross points at the j -th level by using the rule determined by \mathbf{D}_{G_j} if $j = i$, $\mathbf{I}_{(g_j \times g_j)}$ if $j < i$, and $\mathbf{I}_{(K_j \times K_j)}$ if $j > i$. The output from the processing of the root node is the partial Gibbs derivative of f with respect to the variable x_i .

The procedure for calculation of the Gibbs derivatives through MTDDs is explained and illustrated by the following example.

Example 4.3. The Gibbs derivative for functions on the group G_{24} in Example 3.5 is defined by $\mathbf{D}_f = 6\mathbf{D}_1 + 3\mathbf{D}_2 + \mathbf{D}_3$, where the partial Gibbs derivatives are given by $\mathbf{D}_1 = \mathbf{D}_{C_2} \otimes \mathbf{I}_{(2 \times 2)} \otimes \mathbf{I}_{(6 \times 6)}$, $\mathbf{D}_2 = \mathbf{I}_{(2 \times 2)} \otimes \mathbf{D}_{C_2} \otimes \mathbf{I}_{(6 \times 6)}$, $\mathbf{D}_3 = \mathbf{I}_{(2 \times 2)} \otimes \mathbf{I}_{(2 \times 2)} \otimes \mathbf{D}_{S_3}$, with

$$\mathbf{D}_{C_2} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}, \quad \mathbf{D}_{S_3} = \begin{bmatrix} 7 & 5 & 5 & 9 & 9 & 9 \\ 5 & 7 & 5 & 9 & 9 & 9 \\ 5 & 5 & 7 & 9 & 9 & 9 \\ 9 & 9 & 9 & 7 & 5 & 5 \\ 9 & 9 & 9 & 5 & 7 & 5 \\ 9 & 9 & 9 & 5 & 5 & 7 \end{bmatrix}.$$

For f in Example 3.5 calculation of the Gibbs derivative goes as follows.

To calculate the partial Gibbs derivative with respect to x_3 we perform calculations determined by definition of \mathbf{D}_{S_3} at the nodes $q_{3,0}$, $q_{3,1}$, $q_{3,3}$ and the cross point $q_{3,2}$. In the nodes at the levels corresponding to x_2 and x_1 , we perform the identical mapping defined by $\mathbf{I}_{(2 \times 2)}$.

To calculate the partial Gibbs derivative with respect to x_2 , we perform the identical mapping determined by $\mathbf{I}_{(6 \times 6)}$ at the nodes and cross point at the level for x_3 , the calculations determined by \mathbf{D}_{C_2} at the nodes for x_2 and the identical mapping determined by $\mathbf{I}_{(2 \times 2)}$ at the root node.

Similar, to calculate partial Gibbs derivative with respect to x_1 , we perform \mathbf{D}_{C_2} at the root node, while at the other nodes and the cross points the identical mappings $\mathbf{I}_{(2 \times 2)}$ are performed.

1. Partial Gibbs derivative with respect to x_3 :

$$\begin{aligned} q_{3,0} &= \mathbf{D}_{S_3} [0, 6, 2, 1, 0, 0]^T = [5, 6, 9, 2, 0, 0]^T, \\ q_{3,1} &= \mathbf{D}_{S_3} [2, 1, 1, 0, 0, 0]^T = [2, 0, 0, 3, 3, 3]^T \end{aligned}$$

$$\begin{aligned} q_{3,2} &= \mathbf{D}_{S_3} [1, 1, 1, 1, 1, 1]^T = [0, 0, 0, 0, 0, 0]^T, \\ q_{3,3} &= \mathbf{D}_{S_3} [1, 1, 1, 1, 2, 2]^T = [7, 7, 7, 10, 1, 1]^T. \end{aligned}$$

$$\begin{aligned} q_{2,0} &= \mathbf{I}_{(2 \times 2)} \begin{bmatrix} q_{3,0} \\ q_{3,1} \end{bmatrix} = [5, 6, 9, 2, 0, 0, 2, 0, 0, 3, 3, 3]^T, \\ q_{2,1} &= \mathbf{I}_{(2 \times 2)} \begin{bmatrix} q_{3,2} \\ q_{3,3} \end{bmatrix} = [0, 0, 0, 0, 0, 0, 7, 7, 7, 10, 3, 3]^T. \end{aligned}$$

$$\begin{aligned} \mathbf{D}_3 &= \mathbf{I}_{(2 \times 2)} \begin{bmatrix} q_{2,0} \\ q_{2,1} \end{bmatrix} \\ &= [5, 6, 9, 2, 0, 0, 2, 0, 0, 3, 3, 3, 0, 0, 0, 0, 0, 0, 7, 7, 7, 10, 1, 1]^T. \end{aligned}$$

2. Partial Gibbs derivative with respect to x_2 :

$$\begin{aligned} q_{3,0} &= \mathbf{I}_{(6 \times 6)} [0, 6, 2, 1, 0, 0]^T = [0, 6, 2, 1, 0, 0]^T, \\ q_{3,1} &= \mathbf{I}_{(6 \times 6)} [2, 1, 1, 0, 0, 0]^T = [2, 1, 1, 0, 0, 0]^T, \\ q_{3,2} &= \mathbf{I}_{(6 \times 6)} [1, 1, 1, 1, 1, 1]^T = [1, 1, 1, 1, 1, 1]^T, \\ q_{3,3} &= \mathbf{I}_{(6 \times 6)} [1, 1, 1, 1, 2, 2]^T = [1, 1, 1, 1, 2, 2]^T. \end{aligned}$$

$$\begin{aligned} q_{2,0} &= \mathbf{W}(1) \begin{bmatrix} q_{3,0} \\ q_{3,1} \end{bmatrix} = \begin{bmatrix} 6q_{3,0} + 5q_{3,1} \\ 5q_{3,0} + 6q_{3,1} \end{bmatrix} = [10, 8, 6, 6, 0, 0, 1, 3, 5, 5, 0, 0]^T, \\ q_{2,1} &= \mathbf{W}(1) \begin{bmatrix} q_{3,2} \\ q_{3,3} \end{bmatrix} = \begin{bmatrix} 6q_{3,2} + 5q_{3,3} \\ 5q_{3,2} + 6q_{3,3} \end{bmatrix} = [0, 0, 0, 0, 5, 5, 0, 0, 0, 0, 6, 6]^T. \end{aligned}$$

$$\mathbf{D}_2 = \mathbf{I}_{(2 \times 2)} \begin{bmatrix} q_{2,0} \\ q_{2,1} \end{bmatrix} = [10, 8, 6, 6, 0, 0, 1, 3, 5, 5, 0, 0, 0, 0, 0, 0, 5, 5, 0, 0, 0, 0, 6, 6]^T.$$

3. Partial Gibbs derivative with respect to x_1 :

$q_{3,0}$, $q_{3,1}$, $q_{3,2}$, and $q_{3,3}$ are as in calculation of D_2 .

$$\begin{aligned}
q_{2,0} &= \mathbf{I}_{(2 \times 2)} \begin{bmatrix} q_{3,0} \\ q_{3,1} \end{bmatrix} = [0, 6, 2, 1, 0, 0, 2, 1, 1, 0, 0, 0]^T \\
q_{2,1} &= \mathbf{I}_{(2 \times 2)} \begin{bmatrix} q_{3,2} \\ q_{3,3} \end{bmatrix} = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2]^T.
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}_1 &= \mathbf{W}(1) \begin{bmatrix} q_{2,0} \\ q_{2,1} \end{bmatrix} = \begin{bmatrix} 6q_{2,0} + 5q_{2,1} \\ 5q_{2,0} + 6q_{2,1} \end{bmatrix} \\
&= [5, 8, 6, 0, 5, 5, 6, 0, 0, 5, 10, 10, 6, 3, 5, 0, 6, 6, 5, 0, 0, 6, 1, 1]^T.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{D}_{24} &= 6\mathbf{D}_1 + 3\mathbf{D}_2 + \mathbf{D}_3 \\
&= [10, 1, 8, 9, 8, 8, 8, 9, 4, 4, 8, 8, 3, 7, 8, 0, 7, 7, 4, 7, 7, 2, 3, 3]^T.
\end{aligned}$$

Each step of calculation can be represented through MTDDs. For example, Fig. 4.8 shows calculation of the partial Gibbs derivative with respect to x_3 for f .

References

- [1] Burrus, C.S., Eschenbacher, R.W., "An in-place, in-order prime factor FFT algorithm", *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-29, 1981, 806-816.
- [2] Butzer, P.L., Engels, W., Wipperfurth, U., "An extension of the dyadic calculus with fractional order derivatives: general theory," *Comp. and Math., with Appls.*, 12B, No. 5/6, 1986, 1073-1090.
- [3] Butzer, P.L., Stanković, R.S., Eds., *Theory and Applications of Gibbs Derivatives*, Matematički institut, Beograd, 1990.
- [4] Butzer, P.L., Wagner, H.J., "Walsh-Fourier series and the concept of a derivative", *Applicable Analysis*, Vol. 3, 29-46, 1973.
- [5] Clarke, E.M., Zhao, X., Fujita, M., Matsunaga, Y., McGeer, R., "Fast Walsh transform computation with Binary Decision Diagram", in: Kebschull, U., Schubert, E., Rosentiel, W., Eds., *Proc. IFIP WG 10.5 Workshop on Applications of the Reed-Muller Expansion in Circuit Design*, 16.-17.9.1993, Hamburg, Germany, 82-85.

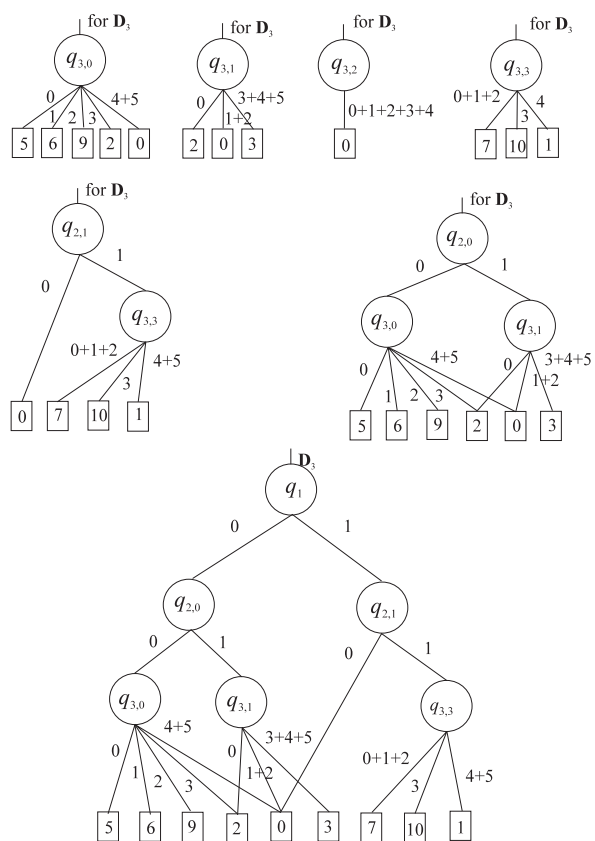


Figure 4.8: Calculation of the partial Gibbs derivative with respect to x_3 for f in Example 3.5.

- [6] Cohn-Sfetcu, S., Gibbs, J.E., "Harmonic differential calculus and filtering in Galois fields, *Proc. 1976 IEEE Conf. on Acoustics, Speech and Signal Processing*, Philadelphia, Pa., 148-153, 1976.
- [7] Gibbs, J.E., "Walsh spectrometry, a form of spectral analysis well suited to binary digital computation", *NPL DES Rept.*, Teddington, Middlesex, United Kingdom, July 1967.
- [8] Gibbs, J.E., "Walsh functions as solutions of a logical differential equation", *National Physical Lab.*, Teddington, England, DES Report, No. 1, 1969.
- [9] Gibbs, J.E. Millard, M.J., "Some methods of solution of linear ordinary logical differential equations", *NPL DES Rept.*, No. 2, Dec. 1969.
- [10] Gibbs, J.E., Simpson, J., "Differentiation on finite Abelian groups", *National Physical Lab.*, Teddington, England, DES Rept, No.14, 1974.
- [11] Gibbs, J.E., Stanković, R.S., "Matrix interpretation of Gibbs derivatives on finite groups", private communication 1989.
- [12] Gibbs, J.E., Stanković, R.S., "Why IWGD - 89? a look at the bibliography on Gibbs derivatives", in: Butzer, P.L., Stanković, R.S., Eds., *Theory and Applications of Gibbs Derivatives*, Matematički institut, Beograd, 1990, xi-xxiv.
- [13] Good, I.J., "The interaction algorithm and practical Fourier analysis", *J. Roy. Statist. Soc.*, ser. B, Vol. 20, 1958, 361-372, Addendum, Vol. 22, 1960, 372-375.
- [14] Karpovsky, M.G., "Fast Fourier transforms on finite non-Abelian groups", *IEEE Trans. Comput.*, Vol.C-26, No.10, 1028-1030, 1977.
- [15] Karpovsky, M.G., Trachtenberg, E.A., "Fourier transform over finite groups for error detection and error correction in computation channels", *Inform. Control*, Vol.40, No.3, 335-358, 1979.
- [16] Kečkić, J.D., "On some classes of linear equations", *Publ. Inst. Math. (Beograd)*, 24, (38), 1978, 89-97.
- [17] Le Dinh, C.T., Le, P., Goulet, R., "Sampling expansions in discrete and finite Walsh-Fourier analysis", *Proc. 1972 Symp. Applic. Walsh Functions*, Washington, D.C., 1972, 265-271.
- [18] Moraga, C., "Introduction to linear p -adic invariant systems, In: *Cybernetics and Systems Research*, Vol.2, 121-124, Ed., R. Trappl, Vienna: Electronic Science Publ., 1984.
- [19] Nussbaumer, H.J., *Fast Fourier Transform and Convolution Algorithms*, Springer-Verlag, Berlin, Heidelberg, New York, 1981.

- [20] Onneweer, C.W., "Fractional differentiation on the group of integers of a p-adic or p-series field, *Anal.Math.*, Vol.3, 119-130, 1977.
- [21] Schipp, F., Wade, W.R., "A fundamental theorem of dyadic calculus for the unit square", *Applicable Analysis*, Vol. 34, 1989, 203-218.
- [22] Stanković, R.S., "A note on differential operators on finite non-Abelian groups, *Applicable Analysis*, Vol.21, 31-41, 1986.
- [23] Stanković, R.S., *Differential Operators on Groups*, Ph.D.Thesis, Faculty of Elec. Engng., Univ. Belgrade, iii+127pp, 1986 (in Serbian, summary in English 8p).
- [24] Stanković, R.S., "A note on differential operators on finite Abelian groups", *Cybernetics and Systems*, 18, 221-231, 1987.
- [25] Stanković, R.S., "A note on spectral theory on finite non-Abelian groups", In: *Theory and Applications of Spectral Techniques*, Ed., C. Moraga, Forschungsbereich 268, ISSN 0933-6192, Dortmund University, 1988.
- [26] Stanković, R.S., "Gibbs derivatives on finite non-Abelian groups", in: Butzer, P.L., Stanković, R.S., Eds., *Theory and Applications of Gibbs Derivatives*, Matematički institut, Beograd, 1990, 269-297.
- [27] Stanković, R.S., "Matrix interpretation of fast Fourier transform on finite non-Abelian groups", *Res. Rept. in Appl. Math.*, YU ISSN 0353-6491, Ser. Fourier Analysis, Rept. No.3, April 1990, 1-31, ISBN 86-81611-03-8.
- [28] Stanković, R.S., "Matrix interpretation of the fast Fourier transforms on finite non-Abelian groups", *Proc. Int. Conf. on Signal Processing, Beijing/90*, 22.-26.10.1990, Beijing, China, 1187-1190.
- [29] Stanković, R.S., "Fast algorithms for calculation of Gibbs derivatives on finite groups," *Approx. Theory and Its Applications*, 7, 2, June 1991, 1-19.
- [30] Stanković, R.S., "Gibbs derivatives", *Numerical Functional Analysis and Optimization*, 15, 1-2, 1994, 169-181.
- [31] Stanković, R.S., Stanković, M.S., "Sampling expansions for complex-valued functions on finite Abelian groups", *Automatika*, 25, 3-4, 147-150, 1984.
- [32] Stanković, R.S., Stanković, M., "Calculation of Gibbs derivatives through decision diagrams", *Approximation Theory and Its Applications*, China, 1998.
- [33] Stanković, R.S., Stojić, M.R., "A note on the discrete Haar derivative", *Colloquia Mathematica Societatis Janos Bolyai 49. Alfred Haar Memorial Conference*, Budapest, Hungary, 897-907, 1985.

- [34] Stanković, R.S., Stojić, M.R., “A note on the generalized discrete Haar derivative”, *Automatika*, 28, 3-4, 117-122, 1987.
- [35] Stojić, M.R., Stanković, M.S., Stanković, R.S., *Discrete Transforms in Application*, Second Edition, Nauka, Beograd, 1993 in Serbian.
- [36] Thomas, L.H., “Using a computer to solve problems in physics”, In: *Application of Digital Computers*, Boston, Mass., Ginn, 1963.
- [37] Watari, Ch., “Multipliers for Walsh-Fourier series”, *Tôhoky Math. J.*, 2, 16, 239-251, 1964.
- [38] Weiyi Su, “Gibbs derivatives and their applications,” *Reports of the Institute of Mathematics, Nanjing University*, 91-7, Nanjing, P.R. China, 1991, 1-20.
- [39] Weiyi, Su, “Walsh analysis in the last 25 years”, *Proc. The Fifth Int. Workshop on Spectral Techniques*, 15.-17.3.1994, Beijing, China, 117-127.
- [40] Zelin, He, “The derivatives and integrals of fractional order in Walsh-Fourier analysis, with applications to approximation theory”, *J. Approx. Theory*, 39, 361-373, 1983.

Chapter 5

Linear Systems and Gibbs Derivatives on Finite Non-Abelian Groups

Linearity is a property very often used in providing mathematical models of physical phenomena. In that setting, the linear shift invariant systems and in particular, linear convolution systems on groups are efficiently used in mathematical modelling of real life systems. For example, in that general ground the linear time-invariant systems can be regarded as systems on the real group R . Similarly, the linear discrete-time-invariant systems are an example of systems defined on the additive group of integers Z . The use of some other groups different from R and Z offers some advantages in particular applications, see for example [13], [14].

Differential operators are used in linear systems theory to describe the change of state of a system. The systems on R described by linear differential equations with constant coefficients in terms of Newton-Leibniz derivative are probably the most familiar example. However, the group theoretic models of systems and Gibbs derivatives on groups, in particular on the dyadic and p -adic groups, have attained some considerable attention [18], [15], [8].

In what follows we will first give a short account of background to linear systems on groups and then discuss systems on finite non-Abelian groups described by discrete differential equations in Gibbs derivatives.

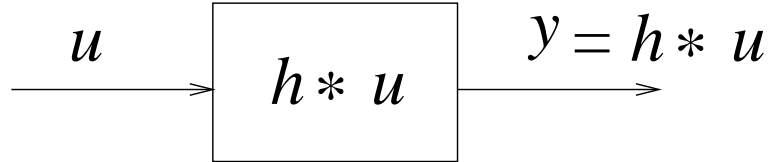


Figure 5.1: Linear shift-invariant system.

5.1 Linear shift-invariant systems on groups

In this section, we briefly discuss linear convolution systems with whose input and output signals are deterministic signals on groups, and point out the relationship between the Gibbs differentiators and linear convolution systems.

Definition 5.1. A linear invariant system S over a group G is defined as a quadruple $S = (U, Y, h, *)$ where the operation $*$ is defined for any $u \in U$, $y \in Y$ as follows:

$$y(t) = (h * u)(t) = \int_{x \in G} h(x^{-1} \circ t)u(x), \quad (5.1)$$

i.e., $*$ is the operation of group convolution of two functions h, u ; x^{-1} is the inverse of x in G and \circ denotes the group operation.

Wording differently, the system S consists of the set U of input signals and the set Y of output signals defined respectively, as the mappings $u : G \rightarrow X$ and $y : G \rightarrow Y$, and the impulse function h defined as the mapping $h : U \rightarrow Y$. If (5.1) is true for a given system S , and given $u \in U, y \in Y$, then that system computes the input/output pair (u, y) .

Fig. 5.1 shows a general model of of a linear shift-invariant system on groups.

Note that from the system theory point of view, S is a linear input/output system whose input and output are defined over an arbitrary group G . By using different groups, various systems studied by several authors can be obtained. For example, if G is the dyadic group, the dyadic systems were introduced by Pichler and further studied in a series of papers by this and by several other authors; see [12] for a bibliography up to 1989. For more recent result, we refer to [7], [8], [21].

The systems where input and output signals are modelled by functions mapping infinite cyclic group of integers into Galois field of order 2^q , $q \in N$,

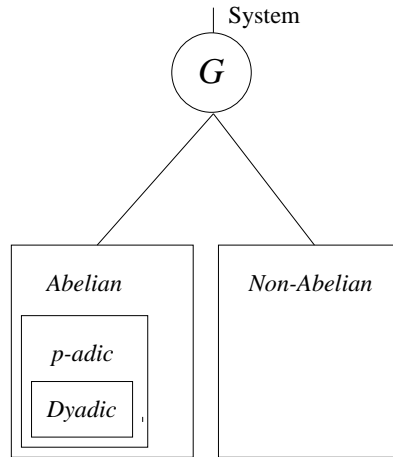


Figure 5.2: Classification of linear shift-invariant systems on groups.

$GF(2^q)$ were considered by Tsyppkin and Fadejev [32]. A generalization of the concept was given in [23] where it was shown that both cyclic and dyadic convolution systems on finite groups can be regarded as special classes of permutation-invariant systems. In a more general setting, the systems over locally compact Abelian groups were considered by Falb and Friedman [9]. Some aspects of the theory were extended also to non-Abelian groups by Karpovsky and Trachtenberg [13], [14].

Recall that systems over finite groups can be regarded as a special class of digital filters [28], [29] [30] or a special class of discrete-time systems with variable structure [16] over a finite interval $[0, g - 1]$, see also [31].

It may be said that in last few years the theory of linear invariant systems on groups has been well established by several authors, although a lot of further work is still needed. Regarding the applications of these systems note that they can be used as models of both information channels (for example to represent an encoder, a digital filter, or a Wiener filter if K is the field of complex numbers and u is a stochastic signal), and computation channels if K is a finite field. For example, different criteria for the approximation of linear time-invariant systems by linear convolution systems on groups were discussed in [31]. Fig. 5.2 shows a classification of linear shift-invariant systems with respect to the domain groups for input and output signals.

5.2 Linear shift-invariant systems on finite non-Abelian groups

In the case of systems on finite non-Abelian groups the Definition 5.1 can be stated as follows.

Definition 5.2. A scalar linear system A over a finite not necessarily Abelian group G is defined as a quadruple $(P(G), P(G), h, *)$ where the input-output relation $*$ is the convolution product on G ,

$$y = h * f, \quad f, h, y \in P(G),$$

i.e.,

$$y(\tau) = \sum_{x \in G} h(x) f(\tau x^{-1}), \quad \forall \tau \in G. \quad (5.2)$$

So, an ordered pair $(f, y) \in P(G) \times P(G)$ is exactly then an input-output pair of A if f and y fulfill equation (5.2). The function $h \in P(G)$ is the impulse response of A .

It is easy to show that the system A is invariant against the translation of input function. By that we mean that if y is the output to f , then $T^\tau y$ is the output to $T^\tau f$, for all $\tau \in G$. Therefore, we denote the system A as a linear translation invariant (LTI) system.

It is apparent that when G is the dyadic group, the Definition 5.2 reduces to the dyadic systems introduced in [17] and further studied in [18] and a series of papers of that and other authors. If G is the group Z_{p^n} we obtain the systems studied in [4], [6], and [15].

The dyadic and p -adic systems are closely related with Gibbs differentiators on the corresponding groups see, for example [15], [18]. A corresponding relationship can be established between LTI systems and Gibbs derivatives on finite non-Abelian groups [24].

First of all, note that (4.5) shows that the Gibbs differentiator D^k of order k is a LTI system having an impulse response h given by $h = \delta^{(k)}$, see [24].

The Gibbs discrete differential equation (4.6) can be interpreted as an input-output relation of a system A belonging to a linear combination of Gibbs derivatives on a finite non-Abelian group.

From (4.7), the general output function of this system is represented as the sum of the zero-input response of the system y_{zi} and the zero-state response y_{zs} has the form identical to (5.2). Therefore, we infer that the

scalar linear system A associated with (5.2) is a LTI system for which (4.7) represents an input-output-state relation and h is the impulse response of A to the unit impulse $\delta(x)$. Since h is the inverse Fourier transform of $H(w)$, the transfer function of A is given by (4.9).

5.3 Gibbs derivatives and linear systems

The relationship between linear convolution systems on locally compact Abelian and finite non-Abelian groups discussed above can be considered and summarized in a general setting as follows.

In a general ground the Gibbs differentiator of order k of a function $f \in K(G)$, which we denote by $D^k f$, is considered as the linear operator in $K(G)$ satisfying the relationship [27]

$$(F(D^k f))(w) = \varphi(w, k)(F(f))(w), \quad (5.3)$$

where F denotes the Fourier transform operator in $K(G)$.

In the most cases $\varphi(w, k) = w^k$, but in some cases a scaling factor should be applied, while in a few particular cases the function φ differs and is related to the order of group G . For example, in the case of the extended Butzer-Wagner dyadic derivative [1] $\varphi(w, k) = (w^*(w))^k$, where

$$w^*(w) = \sum_{i=0}^{\infty} (-1)^i w_i 2^i,$$

w_i being the coefficients in the dyadic expansion of $w \in P$. In the case of Gibbs derivatives on Vilenkin groups [35], [36], [37], the function $\varphi(w, k)$ is a function from the so-called symbol class $S_{\rho, \sigma}^m$ [35] defined as $\varphi(w, k) = \langle k \rangle^m$, $m \geq 0$, where $\langle x \rangle = \max\{1, \|x\|\}$.

It should want point out in attempting to determine a relationship between Gibbs derivatives and linear convolution systems that

1. Thanks to the relation (5.3) and the convolution theorem in the Fourier analysis on groups, the Gibbs differentiator of order k can be considered as a convolution operator and, therefore, can be identified with a linear convolution system whose impulse response function h is given in the transform domain by $(F(h))(w) = \varphi(w, k)$. For example, in the case of the Gibbs derivative on finite not necessarily Abelian groups, as well as in the case of dyadic and p -adic groups, $\varphi(w, k) = w^k$ by definition and, therefore, h is the k -th Gibbs derivative of the δ -function defined as $\delta(x) = 1$ for x equals the unit element of G , and $\delta(x) = 0$ otherwise.

2. A considerable class of linear systems on groups can be described by Gibbs differential equations in a way resembling the use of classical differential equations with constant coefficients in the linear system theory on the real group R . In the other words, a linear Gibbs differential on a group is defined as a polynomial in the Gibbs differentiator with real coefficients. The linear Gibbs differential operator form a subset of the group convolution operators realized by the corresponding subset of the group convolution systems.

As we noted above such linear systems over dyadic groups were discussed in [11], [22], and for the infinite dyadic groups in [18]. Recall that an extension of the theory to p -adic groups, the finite Vilenkin groups, was given in [6], [15]. A generalization to finite non-Abelian groups was given in [24], see also [26] and for p -adic systems with stochastic signals in [8].

Note that the use of systems modelled by Gibbs differential equations in the processing of two-dimensional signals was suggested in [19], [20].

5.3.1 Discussion

As is known, the dyadic derivative is especially adapted to functions having many jumps and possessing just a few and short intervals of constancy. Even functions having a denumerable set of discontinuities like the well-known Dirichlet function can be dyadically differentiable on $[0, 1]$. In the case of finite groups, the Gibbs derivatives also provide a mean to differentiate functions on those groups. In one word, through the family of Gibbs differentiators, the advantage of the use of differential calculus extends to the theory of systems whose input/output signals are piecewise constant, or discrete functions.

In order to point out some possible advantages of linear systems on groups modelled by the Gibbs differential equations, recall that the use of Fourier analysis in linear systems theory is based upon the convolution theorem and the relationship between the Newton-Leibniz derivative. Thanks to the first property, the Fourier transform maps the convolution into ordinary multiplication, while the second permits the translation of differential equations into the algebraic ones. As in many other areas, the application of Fourier analysis in linear systems theory is further supported by the existence of the fast Fourier transform, FFT, and related algorithms for efficient calculation of Fourier coefficients and some other parameters useful in practical applications.

The Gibbs derivatives possesses the most of the useful properties of Newton-Leibniz derivative, except the product rule and, therefore, their role in the theory of linear systems on groups can be compared to that of Newton-Leibniz derivative in classical linear systems theory on R . At the same time, the Gibbs derivatives are efficiently characterized by the Fourier coefficients on groups.

The matrices representing Gibbs derivatives are Kronecker product representable in the case of finite decomposable groups, Section 4.5, and, therefore, the fast algorithms for the calculation of the values of Gibbs derivatives on these groups can be defined, Section 4.6.

It may be said that the Gibbs differentiation shares some of very useful properties of both Fourier analysis and differential calculus.

Thanks to these properties the Gibbs derivatives could be very promising for the use in the theory of linear systems on groups. Some recent results and extensions of the theory are given in [7], [21].

References

- [1] Butzer, P.L., Engels, W., Wipperfurth, U., "An extension of the dyadic calculus with fractional order derivatives: general theory," *Comp. and Math., with Appls.*, 12B, No.5/6, 1986, 1073-1090.
- [2] Butzer, P.L., Wagner, H.J., "Walsh-Fourier series and the concept of a derivative", *Applicable Analysis*, Vol.1, No.3, 29-46, 1973.
- [3] Butzer, P.L., Stanković, R.S., Eds., *Theory and Applications of Gibbs Derivatives*, Matematički institut, Beograd, 1990.
- [4] Cohn-Sfetcu, S., "On the theory of linear dyadic invariant systems", *Proc. Symp. Theory and Applic. Walsh and other Non-sinus. Functions*, Hatfield, England, 1975.
- [5] Cohn-Sfetcu, S., *Topics on generalized convolution and Fourier transform: Theory and applications in digital signal processing and system theory*, Ph.D. Thesis, McMaster University, Hamilton, Ontario, Canada, 1976.
- [6] Cohn-Sfetcu, S., Gibbs, J.E., "Harmonic differential calculus and filtering in Galois fields," *Proc. IEEE Int. Conf. Acoust. Speech and Signal Processing*, Philadelphia, PA, 1976 April 12-14, 148-153.
- [7] Endow, Y., "Walsh harmonizable processes in linear system theory", *Cybernetics and Systems*, 1996, 489-512.

- [8] Endow, Y., Stanković, R.S., "Gibbs derivatives in linear system theory", in: R. Trappl, Ed., *Proc. Twelfth European Meeting on Cybernetics and Systems Research*, 5.-8.4.1994, Vienna, Austria.
- [9] Falb, P.L., Friedman, M.I., "A generalized transform theory for causal operators", *SIAM J. Control*, 1970, 452-471.
- [10] Gibbs, J.E., Simpson, J., "Differentiation on finite Abelian groups", *NPL DES Rept.*, No.16, July 1974.
- [11] Gibbs, J.E., Marshall, J.E., Pichler, F.R., "Electrical measurements in the light of system theory", IEE Conference on The Use of Computers in Measurement, University of York, September 24-27, 1973, ii+5pp.
- [12] Gibbs, J.E., Stanković, R.S., "Why IWGD- 89? A look at the bibliography of Gibbs derivatives," in: *Theory and Applications of Gibbs Derivatives*, P.L. Butzer, R.S. Stanković, Eds., Matematički institut, Beograd, 1990, *xi – xxiv*.
- [13] Karpovsky, M.G., Trachtenberg, E.A., "Some optimization problems for convolution systems over finite groups", *Inf. and Control*, 34, 1977, 227-247.
- [14] Karpovsky, M.G., Trachtenberg, E.A., "Statistical and computational performances of a class of generalized Wiener filters", *IEEE Trans. Inform. Theory*, Vol.IT-32, No.2, 1986, 303-307.
- [15] Moraga, C., "Introduction to linear p -adic systems," R. Trappl, Ed., *Cybernetics and Systems Research*, 2, North-Holland, 1984.
- [16] Nailor, A.W., "A transform technique for multivariable, timevarying, discrete-time linear systems", *Automatica*, 2, 1965, 211-234.
- [17] Pearl, J., "Optimal dyadic models of time-invariant systems", *IEEE Trans. Comput.*, Vol.C-24, 1975.
- [18] Pichler, F., "Walsh functions and linear system theory", Proc. Applic. Walsh Functions, Washington, D.C., 1970, 175-182.
- [19] Pichler, F., "Fast linear methods for image filtering", in: D.G. Lainiotis, N.S. Tzanues, Eds., *Applications of Information and Control Systems*, D. Reidel Publishing Company, 1980, 3-11.
- [20] Pichler, F., "Experiments with 1-D and 2-D signals using Gibbs derivatives", in: *Theory and Applications of Gibbs Derivatives*, P.L. Butzer, R.S. Stanković, Eds., Matematički institut, Beograd, 1990, 181-196.
- [21] Pichler, F., "Realizations of Priggogine's Λ -transform by dyadic convolution", in Trappl, R., Horn, W., (eds.), Austrian society for cybernetic Studies, ISBN 385206127X, 1992.

- [22] Pichler, F.R., Marshall, J.E., Gibbs, J.E., "A system-theory approach to electrical measurements", Colloquium on the Theory and Applications of Walsh Functions, The Hatfield Polytechnic, Hatfield, Hertfordshire, June 28-29, 1973.
- [23] Siddiqi, M.U., Sinha, U.P., "Permutation-invariant systems and their application in the filtering of finite discrete data", 1977.
- [24] Stanković, R.S., "Linear harmonic translation invariant systems on finite non-Abelian groups", R. Trappl, Ed., *Cybernetics and Systems Research* North-Holland, 1986.
- [25] Stanković, R.S., "A note on differential operators on finite non-Abelian groups," *Applicable Anal.*, 21, 1986, 31-41.
- [26] Stanković, R.S., "A note on spectral theory on finite non-Abelian groups," *3rd Int. Workshop on Spectral Techniques*, Univ. Dortmund, 1988, Oct.4-6, 1988.
- [27] Stanković, R.S., "Gibbs derivatives", *Numerical Functional Analysis and Optimization*, 15, 1-2, 1994, 169-181.
- [28] Trachtenberg, E.A., "Systems over finite groups as suboptimal filters: a comparative study", P.A. Fuhrmann, Ed., *Proc. 5th Int. Symp. Math. Theory of Systems and Networks*, Springer-Verlag, Beer-Sheva, Israel, 1983, 856-863.
- [29] Trachtenberg, E.A., "Fault tolerant computing and reliable communication: a unified approach", *Information and Computation*, Vol.79, No.3, 1988, 257-279.
- [30] Trachtenberg, E.A., Karpovsky, M.G., "Optimal varying dyadic structure models of time invariant systems", *Proc. 1988 IEEE Int. Symp. Circuits and Systems*, Espoo, Finland, June 1988.
- [31] Trachtenberg, E.A., "SVD of Frobenius matrices for approximate and multiobjective signal processing tasks", in: E.F. Deprettere, Ed., *SVD and Signal Processing*, Elsevier North-Holland, Amsterdam/New York, 1988, 331-345.
- [32] Tsympkin, Ya.Z., Faradjev, R.G., "Laplace-Galois transformation in the theory of sequential machines", *Dokl. Akad. Nauk USSR*, Vol.166, No.3, 1966, 507-573, in Russian.
- [33] Wade, W.R., "Decay of Walsh series and dyadic differentiation", *Trans. Amer. Math. Soc.*, Vol.277, No.1, 413-420, 1983.
- [34] Watari, Ch., "Multipliers for Walsh-Fourier series", *Tôhoky Math. J.*, 2, 16, 239-251, 1964.
- [35] Weiyi, Su., "Pseudo-differential operators in Bessov spaces over local fields", *J. Approx. Theory and Appls.*, Vol.4, No.2, 1988, 119-129.

- [36] Weiyi Su, "Gibbs derivatives and their applications," *Reports of the Institute of Mathematics, Nanjing University*, 91-7, Nanjing, P.R. China, 1991, 1-20.
- [37] Weiyi Su, "Pseudo-differential operators and derivatives on locally compact Vilenkin groups," *Science in China (Series A)*, Vol.35, No.7, 1992, 826-836.

Chapter 6

Hilbert Transform on Finite Groups

In theory of real variable functions the Hilbert transform is defined in the following way.

Definition 6.1. The Hilbert transform f of a function $f \in L^p, 1 \neq p \neq \infty$ is defined, see for example [2], [9], by

$$f(x) = v.p. \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{u} du \right), \quad (6.1)$$

where the notation $v.p.$ means that the integral is understood in the sense of Cauchy principal value.

The functions belonging to L^2 are the most widely exploited in practice, since they represent the finite energy signals. The following holds for these functions, see for example [2], [9].

Denote by $F(w)$ the Fourier transform of a function $f \in L^2$. Then, the Fourier transform of its Hilbert transform, $F(w)$, is given by

$$F(w) = -\text{sign}(w)F(w) \text{ a.e.}, \quad (6.2)$$

where

$$\text{sign}(w) = \begin{cases} -1, & w < 0, \\ 0, & w = 0, \\ 1, & w > 0. \end{cases} \quad (6.3)$$

The relation (6.2) can be regarded as an alternative definition of the Hilbert transform. The formula holds for $p = 1$ if $f \in L^1$, in which case

it holds everywhere. Note that there are functions from l^1 whose Hilbert transforms defined by (6.1) do not belong to l^1 . An example is $f(x) = \frac{1}{1+x}$ as is noted in [2].

Recall that

$$\begin{aligned} v.p. \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-iwx}{x} dx\right) \right) &= \lim_{\epsilon \rightarrow 0} \frac{-2i}{\sqrt{2\pi}} \int_{\epsilon}^{\infty} \frac{\sin(wx)}{x} dx \\ &= \{-i\text{sign}(w)\} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin y}{y} dy = \{-i\text{sign}(w)\} \sqrt{\frac{\pi}{2}}. \end{aligned}$$

From there it can be written at least formally:

$$F(w) = \mathbf{F} \left(\sqrt{\frac{2}{\pi}} \left(f * \frac{1}{\pi} \right) \right) = \{-\text{sign}(w)\} \mathbf{F}(w), \quad (6.4)$$

where \mathbf{F} is the Fourier transform operator, $*$ denotes the convolution product on R , and the convolution integral is understood in the sense of Cauchy principal value.

In the case of periodic functions with the period equal to 2π the convolution kernel used to define the corresponding Hilbert transform is $\{\cot \frac{x}{2}\}$.

The approach of defining the Hilbert transform in transform domain as the multiplication by a sign function, that is by employing the relation (6.2) was used as the starting point for the introduction of a discrete Hilbert transform, i.e., the Hilbert transform for functions on finite Abelian groups. It is important to note that definitions appearing in [3] and [1] [4], [5], [6] are based upon the differently defined sign functions and they coincide only in the case of cyclic groups.

Recalling that the real line R exhibits the structure of a locally compact Abelian group, it can be concluded that the Hilbert transform for real-variable functions and the discrete Hilbert transform can be considered uniformly as the Hilbert transform on Abelian groups. However, the above discussed group-theoretic approach of introducing the Hilbert transform on Abelian groups through the product in the transform domain by a suitably defined sign function, can hardly be used further to extend the concept to finite non-Abelian groups. That fact becomes obvious if we remind that unlike Abelian groups, the domain Γ of the Fourier transform S_f of a function f on a finite not necessarily Abelian group G may not have any algebraic structure suitable to define a multiplication in it which in turn can be mapped into a convolution in the group. Therefore, we have suggested in [8] the just opposite way, we have defined a Hilbert transform on a finite non-Abelian group as the pointwise multiplication of a given function

by a suitably defined sign function in the group. As was shown in [8], an analysis of the properties of thus defined transform justifies to consider it as a proper counterpart of the Hilbert transform on R or on finite Abelian groups. Therefore, we are encouraged to suggest this 'opposite' way to be actually a 'proper' way to define a Hilbert-like transform for functions on both Abelian and non-Abelian groups permitting the considerations of these two cases in a uniform way. Recall that the same approach was already used for Hilbert transform on R in some particular engineering applications as for example signal filtering.

In that way two aims are reached. First, the main properties of the "classically" defined Hilbert transform on Abelian groups are preserved by the "new" transform, and the concept is extended to finite non-Abelian groups. Further, as it will be shown below, the same approach can be used to introduce a Hilbert-like transform for functions mapping a given finite non-Abelian group into a finite field admitting the existence of a Fourier transform.

6.1 Some results of Fourier analysis on finite non-Abelian groups

For the sake of completeness of presentation in this section we disclose several further results of Fourier analysis on finite non-Abelian groups which are somewhat restricted counterparts of the corresponding results on finite Abelian groups. Recall again that the Fourier transform S_F is defined on Γ and, thus, cannot be regarded as a function on a group and, therefore, some of the properties valid on Abelian groups are non-existent when the group is no longer commutative.

Recalling the bijection V from non-Abelian group G of order g onto the subset $M = \{0, \dots, g-1\}$ of integers adopted in this monograph, note that the natural ordering " $<$ " in M induces an ordering upon G via the inverse mapping V^{-1} . We keep the symbol " $<$ " for the new ordering in G and define the following partition of G :

$$POS_V = \{x \in G | x < x'\}, \quad SYM_V = \{x \in G | x = x'\}, \quad NEG_V = \{x \in G | x' < x\},$$

where x' is the inverse of x in G , i.e., $x \circ x' = e$ and $(x')' = x$. The symbols \circ and e represent the group operation and the identity of G . Notice that both the set POS and NEG depend on the bijection V .

Among the functions from $P(G)$ we will not the following special classes.

Definition 6.2. Let $f \in P(G)$. Then, with respect to the ordering of G introduced by V , we say:

1. f is even iff $f(x) = f(x'), \forall x \in G$,
2. f is odd iff $f(x) = -f(x'), \forall x \in G$,
3. f is actual iff $f(x) = 0, \forall x \in G \setminus POS_V$,
4. f is coactual iff $f(x) = 0, \forall x \in POS_V$,
5. f is virtual iff $f(x) = 0, \forall x \in G \setminus NEG_V$,
6. f is covirtual iff $f(x) = 0, \forall x \in NEG_V$,
7. f is axial iff $f(x) = 0, \forall x \in G \setminus SYM_V$,
8. f is coaxial iff $f(x) = 0, \forall x \in SYM_V$.

Notice that if f is odd, then $f(x) = 0, \forall x \in SYM_V$, and if f is axial, then it is also even.

Definition 6.3. For all $x \in G$, the function sign_V is defined as follows

$$\text{sign}_V(x) = \begin{cases} 1, & x \in POS_V, \\ -1, & x \in NEG_V, \\ 0, & x \in SYM_V. \end{cases}$$

It becomes apparent that sign_V is a coaxial function.

Property 1. Let $f \in P(G)$. Then,

1. $f_e(x) = \frac{1}{2}(f(x) + f(x'))$ defines an even function,
2. $f_o(x) = \frac{1}{2}(f(x) - f(x'))$ defines an odd function,
3. $f(x) = f_e(x) + f_o(x)$.

The sign minus Property 1 as well as in Definition 6.2 is understood in the sense of subtraction in the field P . For the correctness of the notation, the derivation of the following properties will be restricted to the complex functions on G . The real functions will be considered as a subclass of these functions in which case we will use the notation $R(G)$. Note that the corresponding properties can be derived for function in finite fields, but the difference which should be appreciated is that in that case the notion of imaginary unity and, consequently, the complex conjugate do not exist

in the "classical" sense. Therefore, the concepts of Hermitean and skew-Hermitean matrix should be appropriately reformulated as it will be given in the corresponding section below.

Proof. The statement is obvious from corresponding definitions.

Property 2. Let $f \in R(G)$. Then f is odd iff its Fourier transform \mathbf{S}_f is skew-Hermitean, i.e., iff for each $0 \neq w \neq K-1$, $\mathbf{S}_f(w) = -\overline{\mathbf{S}_f^T(w)} = -\mathbf{S}_f^*(w)$, where \mathbf{S}_f^T denotes the transpose, $\overline{\mathbf{S}_f}$ the complex-conjugate, and \mathbf{S}_f^* the complex-conjugate transpose of \mathbf{S}_f .

Proof. Assume first that f is odd, i.e., $f(x) = -f(x'), \forall x \in G$. Then for $w = 0, \dots, K-1$ we have

$$\mathbf{S}_f(w) = r_w g^{-1} \sum_{x \in G} f(x) \overline{\mathbf{R}_w^T(x)} = r_w g^{-1} \sum_{x \in G} f(x') \overline{\mathbf{R}_w^T(x')}.$$

Since f is real we obtain

$$\mathbf{S}_f(w) = \left(-r_w g^{-1} \sum_{x \in G} f(x') \mathbf{R}_w \right)^T (x) = -\mathbf{S}_f^*(w).$$

Conversely, assume that \mathbf{S}_f is skew-Hermitean, i.e., $\mathbf{S}_w = -\mathbf{S}_f^*(w)$. Now, since f is real-valued, for each $x \in G$ we have

$$\begin{aligned} f(x) &= \sum_{w=0}^{K-1} \text{Tr}(\mathbf{S}_f(w) \mathbf{R}_w(x)) = \sum_{w=0}^{K-1} \overline{\text{Tr}(\mathbf{S}_f(w) \mathbf{R}_w(x))} \\ &= \sum_{w=0}^{K-1} \text{Tr}(\overline{\mathbf{S}_f(w) \mathbf{R}_w(x)}) = \sum_{w=0}^{K-1} \text{Tr}(\overline{\mathbf{S}_f(w)} \overline{\mathbf{R}_w(x)})^T \\ &= \sum_{w=0}^{K-1} \text{Tr}(\mathbf{S}_f^T(w) \mathbf{R}_w^T(x)) = - \sum_{w=0}^{K-1} \text{Tr}(\mathbf{S}_f(w) \overline{\mathbf{R}_w^T(x)}) \\ &= \sum_{w=0}^{K-1} \text{Tr}(\mathbf{S}_f(w) \mathbf{R}_w(x')) = -f(x'). \end{aligned}$$

Property 3. Let $f \in R(G)$. Then, f is even iff its Fourier transform \mathbf{S}_f is Hermitean, i.e., iff for each $0 \neq w \neq K-1$, $\mathbf{S}_f(w) = \mathbf{S}_f^*(w)$.

Proof. First assume that f is even, i.e., $f(x) = f(x'), \forall x \in G$. Then for all $w = 0, \dots, K-1$

$$\mathbf{S}_f(w) = r_w g^{-1} \sum_{x \in G} f(x) \overline{\mathbf{R}_w^T(x)} = r_w g^{-1} \sum_{x \in G} f(x') \overline{\mathbf{R}_w^T(x)}$$

$$\begin{aligned}
&= r_w g^{-1} \sum_{x \in G} \overline{(f(x') \mathbf{R}_w(x))} = \left(r_w g^{-1} \sum_{x \in G} f(x) \mathbf{R}_w^T(x) \right)^T \\
&= \overline{(\mathbf{S}_f(w))^T}.
\end{aligned}$$

Thus, \mathbf{S}_f is Hermitean.

Conversely, assume that \mathbf{S}_f is Hermitean, i.e., $\mathbf{S}_f(w) = \overline{\mathbf{S} - f^T(w)}$. Then, since $f \in R(G)$, we have

$$\begin{aligned}
f(x) &= \overline{f(x)} = \overline{\sum_{w=0}^{K-1} \text{Tr}(\mathbf{S}_f(w) \mathbf{R}_w(x))} = \sum_{w=0}^{K-1} \overline{\text{Tr}(\mathbf{S}_f(w) \mathbf{R}_w(x))}^T \\
&= \sum_{w=0}^{K-1} \text{Tr}(\mathbf{S}_f(w) \overline{\mathbf{R}_w^T(x)}) = \sum_{w=0}^{K-1} \text{Tr}(\mathbf{S}_f(w) \mathbf{R}_w(x')) \\
&= f(x').
\end{aligned}$$

Property 4. Let $f \in R(G)$ and $f(x) = f_e(x) + f_o(x)$. Then,

$$\begin{aligned}
\mathbf{S}_{f_e}(w) &= \frac{1}{2}(\mathbf{S}_f(w) + \mathbf{S}_f^*(w)), \\
\mathbf{S}_{f_o}(w) &= \frac{1}{2}(\mathbf{S}_f(w) - \mathbf{S}_f^*(w)),
\end{aligned}$$

for each $0 \neq w \neq K-1$.

Proof. because of the linearity of the Fourier transform

$$\begin{aligned}
\mathbf{S}_f(w) &= \mathbf{S}_{f_e}(w) + \mathbf{S}_{f_o}(w), \\
\mathbf{S}_f^*(w) &= \mathbf{S}_{f_e}^*(w) + \mathbf{S}_{f_o}^*(w),
\end{aligned}$$

for each $0 \neq w \neq K-1$ and with properties 2 and 3 we obtain

$$\mathbf{S}_f^*(w) = \mathbf{S}_{f_e}(w) - \mathbf{S}_{f_o}(w),$$

from where the assertion follows directly.

From there we have that \mathbf{S}_{f_e} is the Hermitean part of \mathbf{S}_f and similarly, \mathbf{S}_{f_o} is the skew-Hermitean part of \mathbf{S}_f , which we denote by $H(\mathbf{S}_f)$ and $sH(\mathbf{S}_f)$, respectively. This is a direct consequence of the linearity of Fourier transform and properties 2 and 3.

Property 5. The Fourier transform of the sign function is given by

$$\mathbf{S}_{\text{sign}}(w) = r_w g^{-1} \sum_{x \in POS_V} (\mathbf{R}_w^*(x) - \mathbf{R}_w(x)).$$

Proof.

$$\begin{aligned}
\mathbf{S}_{\text{sign}}(w) &= r_w g^{-1} \sum_{x \in G} \text{sign}(x) \mathbf{R}_w^*(x) \\
&= r_w g^{-1} \sum_{x \in POS_V} \mathbf{R}_w^*(x) - r_w g^{-1} x \in NEG_V \mathbf{R}_w^* \\
&= r_w g^{-1} \sum_{x \in POS_V} (\mathbf{R}_w^*(x) - \mathbf{R}_w^*(x')) \\
&= r_w g^{-1} \sum_{x \in POS_V} (\mathbf{R}_w^*(x) - \mathbf{R}_w(x)).
\end{aligned}$$

Corollary.

1. Let $f \in R(G)$ be covirtual. A covirtual function has a trivial decomposition in an actual function $g_a = g_e + g_o$ and as axial function g_{sym} . Then we have

$$sH(\mathbf{S}_f) = \mathbf{S}_{g_o}.$$

2. Let $f \in R(G)$ be coactual. A coactual function has a trivial decomposition in a virtual function $g_v = k_e + k_o$ and an axial function g_{sym} . Then we have

$$sH(\mathbf{S}_f) = \mathbf{S}_{k_o}.$$

Proof. Let $f \in R(G)$ be covirtual. Define

$$g_a(x) = \begin{cases} f(x), & \forall x \in POS_V \\ 0, & \text{otherwise,} \end{cases}$$

Obviously, g_a is actual and g_{sym} is even. Moreover, $f = g_a + g_{\text{sym}}$, i.e., $f = g_o + g_e + g_{\text{sym}}$. Since the sum of two even functions is even, then g_o represents the odd part of f and the assertion follows from property 2.

Similarly if f is a coactual function.

6.2 Hilbert transform on finite non-Abelian groups

As we noted above, the definition of the Hilbert transform based upon the relation (6.2) cannot be extended to functions defined on a non-Abelian group because the domain Γ of the Fourier transform \mathbf{S}_f may not exhibit a

suitable algebraic structure. For that reason we will use a reverse approach which leads to a definition holding uniformly for Abelian and non-Abelian groups.

Definition 6.4. The Hilbert transform $f^\#$ of a function $f \in P(G)$, where G is not necessarily an Abelian group, is defined under a given ordering bijection V as the linear operator $\mathcal{H} : P(G) \rightarrow P(G)$ given by

$$f^\#(x) = -i \text{sign}_V(x) f(x), \text{ for all } x \in G.$$

The main properties of thus defined Hilbert transform are given in the following theorem which justifies to consider it as a proper counterpart of the Hilbert transform on R and on finite Abelian groups.

Theorem 6.1. The chief properties of the Hilbert transform are

1. For each $f, h \in C(G)$, and $a, b \in C$
 $(af + bh)^\#(x) = af^\#(x) + bh^\#(x)$.
2. If $f \in R(G)$, then $f^\#$ is purely imaginary. Moreover, if f is even, then $f^\#$ is odd, and if f is odd, $f^\#$ is even.
3. For each $f \in C(G)$, $\overline{f^\#(x)} = -\overline{f(x)}$, where $\overline{f(x)}$ denotes the complex conjugate of $f(x)$.
4. For each $f, h \in C(G)$, $f^\#(x)h(x) = f(x)h^\#(x)$
5. The inverse Hilbert transform of a coaxial function $f \in C(G)$ is given by $f^\#(f^\#(x)) = -f(x)$. Notice that the actual and virtual functions are also coaxial functions, and it follows $f^\#(x)h(x) = -f(x)h^\#(x)$.
6. Let $f_a \in R(G)$ be actual. Then the Hermitean and skew-Hermitean parts of its Fourier spectrum \mathbf{S}_f are related by the Hilbert transform as shown below:
 For each $w \in \{0, \dots, K-1\}$

$$H(\mathbf{S}_{f_a}(w)) = (((isH(\mathbf{S}_{f_a})))^\perp)^\top, \quad (6.5)$$

$$sH(\mathbf{S}_{f_a}(w)) = i(((H(\mathbf{S}_{f_a})))^\perp)^\top. \quad (6.6)$$

Let $f \in R(G)$ be virtual. Then,

$$H(\mathbf{S}_{f_v}(w)) = -(((isH(\mathbf{S}_{f_v})))^\perp)^\top, \quad (6.7)$$

$$sH(\mathbf{S}_{f_v}(w)) = -i(((H(\mathbf{S}_{f_v})))^\perp)^\top. \quad (6.8)$$

7. Let $f \in R(G)$ be covirtual. Then equation (6.7) is also valid.
 Let $f \in R(G)$ be coactual. Then equation (6.9) is also valid.
8. Let $f \in R(G)$. Moreover, let $Ev(f)$ and $Od(f)$ denote the even and odd parts of f , respectively. These parts are related by the Hilbert transform as follows:
 If f is actual, then

$$Ev(f(x)) = i(Od(f(x))), \quad (6.9)$$

$$Od(f(x)) = i(Ev(f(x))), \quad (6.10)$$

and if f is virtual, then

$$Ev(f(x)) = -i(Od(f(x))), \quad (6.11)$$

$$Od(f(x)) = -i(Ev(f(x))). \quad (6.12)$$

9. If $f \in R(G)$ is covirtual, then (6.11) is also valid.
 If $f \in R(G)$ is coactual, then (6.13) is also valid.

Proof. The following parts of the proof are numbered as the assertions of the theorem.

1. This assertion follows directly from definition of the Hilbert transform.
2. Proof of the first part of the statement follows from definition of the Hilbert transform since the function sign is real. The second part follows from properties 5 and 3, respectively, since sign is an odd function.
3. $\overline{f} = -i\text{sign}(\cdot)(\overline{f}) = \overline{i\text{sign}(\cdot)f} = -(\overline{-i\text{sign}(\cdot)f}) = -\overline{f}$.
4. Proof follows directly from definition of the Hilbert transform.

5. From definition 2 follows that $\text{sign}^2(\cdot)$ is the identity coaxial function. It becomes apparent that since $\text{sign}(x) = \text{sign}^2(x) = 0$ for each $x \in SYM_V$, no inverse Hilbert transform exists for functions other than coaxial. Recall, however, that both actual and virtual functions are special kinds of coaxial functions.
6. Since f_a is axial, from property 1 it can be written as the sum of g_e and g_o . Then from property 6 we have

$$\begin{aligned} H(\mathbf{S}_{f_a}(w)) &= \mathbf{S}_{g_e}(w) = (\text{sign}(x)g_o(x))^\top \\ &= (-i\text{sign}(x)(ig_o(x)))^\top \\ &= ((ig_o(x))^\top)^\top = (((i\mathbf{S}_{g_o}(w))^\perp)^\top)^\top \\ &= (((isH(\mathbf{S}_{f_a}(w)))^\perp)^\top)^\top, \end{aligned}$$

and similarly

$$\begin{aligned} sH(\mathbf{S}_{f_o}(w)) &= \mathbf{S}_{g_o}(w) = (g_o(x))^\top \\ &= i(-i\text{sign}(x)g_o(x))^\top = i((g_e(x))^\top)^\top \\ &= i(((\mathbf{S}_{g_e}(w))^\perp)^\top)^\top = i(((H(\mathbf{S}_{f_a}(w)))^\perp)^\top)^\top. \end{aligned}$$

The proof is analogous in the case of virtual functions.

7. Let f be covirtual. Then f may be expressed as $f = f_a + g_{\text{sym}}$, where f_a is actual and g_{sym} takes the same values as f for each $x \in SYM_V$ and is zero otherwise. It becomes apparent that for each $x \in G$, $\text{sign}(x)g_{\text{sym}}(x) = 0$, since g_{sym} is axial (and therefore also even). Particularly, $(g_{\text{sym}})^\top = 0$.

It follows

$$f = f_a + g_{\text{sym}} = g_o + g_e + g_{\text{sym}},$$

where $Od(f) = g_o$ and $EV(f) = g_e + g_{\text{sym}}$.

From property 2 we have

$$\begin{aligned} sH(\mathbf{S}_f(w)) &= (Od(f))^\top = (g_o(x))^\top = i(-i\text{sign}(x)g_e(x))^\top \\ &= i(-i\text{sign}(x)g_e(x) - i\text{sign}(x)g_{\text{sym}}(x))^\top \\ &= i((g_e(x) + g_{\text{sym}}(x))^\top)^\top = i((EV(f))^\top)^\top \\ &= i(((H(\mathbf{S}_f(w)))^\perp)^\top)^\top. \end{aligned}$$

Similarly for f a coactual function.

8. Since f_a is actual, then $f_a = g_e + g_o = EV(f_a) + Od(f_a)$. From property 1 follows that $g_e(x) = \text{sign}(x)g_o(x)$, hence:

$$\begin{aligned} Ev(f_a(x)) &= \text{sign}(x)Od(f_a(x)) \\ &= (-i\text{sign}(x))(iOd(f_a(x))) = (iOd(f_a(x))) \end{aligned}$$

$$\begin{aligned} Od(f_a(x)) &= \text{sign}(x)Ev(f_a(x)) \\ &= (-i\text{sign}(x))(iEv(f_a(x))) = (iEv(f_a(x))) . \end{aligned}$$

9. See the proof of assertion 7.

Consider the following example for the illustration of the assertions of the theorem.

Example 6.1 Let G be the quaternion group Q_2 defined in Example 2.3. Note that the function f considered in this example is an actual function and, therefore, will be denoted here by f_a . In Table 6.1 we list the values of the sign_V function on G and illustrate the decomposition of the given f_a into an even function g_e and an odd function g_o by using assertion 1 of the Theorem 6.1. Their Fourier transforms \mathbf{S}_{g_e} and \mathbf{S}_{g_o} are given in Table 6.2. It is apparent that $\mathbf{S}_{g_e} = H(\mathbf{S}_{f_a})$ and $\mathbf{S}_{g_o} = sH(\mathbf{S}_{f_a})$ as is stated in assertion 6 of the Theorem 6.1.

Table 6.1. The even and odd parts of the test function.

x	x'	$\text{sign}_V(x)$	$f_a(x)$	$g_e(x)$	$g_o(x)$
0	0	0	0	0	0
1	3	1	α	$\frac{\alpha}{2}$	$\frac{\alpha}{2}$
2	2	0	0	0	0
3	1	-1	0	$\frac{\alpha}{2}$	$-\frac{\alpha}{2}$
4	6	1	β	$\frac{\beta}{2}$	$\frac{\beta}{2}$
5	7	-1	λ	$\frac{\lambda}{2}$	$\frac{\lambda}{2}$
6	4	-1	0	$\frac{\beta}{2}$	$-\frac{\beta}{2}$
7	5	-1	0	$\frac{\lambda}{2}$	$-\frac{\lambda}{2}$

$SYM_V = \{0, 2\}$, $POS_V = \{1, 4, 5\}$, $NEG_V = \{3, 6, 7\}$

Table 6.2. Fourier spectra of the test function.

w	$8\mathbf{S}_{g_e}(w)$	$8\mathbf{S}_{g_o}(w)$
0	$\alpha + \beta + \lambda$	0
1	$-\alpha + \beta - \lambda$	0
2	$\alpha - \beta - \lambda$	0
3	$-\alpha - \beta + \lambda$	0
4	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$2 \begin{bmatrix} -i\alpha & \beta + i\lambda \\ -\beta + i\lambda & i\alpha \end{bmatrix}$

6.3 Hilbert transform in finite fields

In this section we will consider the definition of Hilbert transform on finite non-Abelian groups for functions taking their values in a finite field admitting the existence of a Fourier transform.

Notice that some of the results from Section 6.2 have to be slightly modified so that they hold also in finite fields. However, some other no longer exists in this case. The main difference which should be appreciated is that the operation we denote by $*$ simply reduces to the transposition. In this setting the concepts of Hermitean and skew-Hermitean matrix can be reformulated.

Definition 6.5. For the Fourier spectrum \mathbf{S}_f of a function $f \in P(G)$ we define the field Hermitean part by $fH(\mathbf{S}_f) = \mathbf{S}_{f_e}$, and the field skew-Hermitean part by $fsH(\mathbf{S}_f) = \mathbf{S}_{f_o}$, where the functions f_e and f_o are defined in Property 1.

Definition 6.6. The Hilbert transform f of a function $f \in P(G)$, where G is a non-Abelian group, is defined under a given bijection V as the linear operator $\mathcal{H} : P(G) \rightarrow P(G)$ given by:

$$\mathcal{H}(f)(x) = \text{sign}(x)f(x), \quad \forall x \in G,$$

with $\text{sign}_V(\cdot)$ as in Definition 6.2.

It can be shown that except for Property 3, all other properties from Theorem 6.1 hold also in this case omitting simply the imaginary unit. Therefore, we have the following theorem.

Theorem 6.2. The chief properties of the Hilbert transform for functions belonging to $P(G)$ are

1. For each $f, h \in P(G)$, and $a, b \in P$
 $(af + bh)(x) = af(x) + bh(x)$.

2. If $f \in P(G)$, is even, then f^* is odd, and if f is odd, f^* is even.
3. For each $f, h \in P(G)$, $f^*(x)h(x) = f(x)h^*(x)$
4. The inverse Hilbert transform of a coaxial function $f \in P(G)$ is given by
 $f^*(f^*(x)) = -f(x)$. Notice that the actual and virtual functions are also coaxial functions, and it follows
 $f^*(x)h(x) = -f^*(x)h^*(x)$.
5. Let $f_a \in P(G)$ be actual. Then the Hermitean and skew-Hermitean parts of its Fourier spectrum \mathbf{S}_f are related by the Hilbert transform as shown below:
 For each $w \in \{0, \dots, K-1\}$

$$fH(\mathbf{S}_{f_a}(w)) = (((fsH(\mathbf{S}_{f_a})))^\perp)^\top, \quad (6.13)$$

$$fsH(\mathbf{S}_{f_a}(w)) = (((fH(\mathbf{S}_{f_a})))^\perp)^\top. \quad (6.14)$$

Let $f \in P(G)$ be virtual. Then,

$$fH(\mathbf{S}_{f_v}(w)) = -(((fsH(\mathbf{S}_{f_v})))^\perp)^\top, \quad (6.15)$$

$$fsH(\mathbf{S}_{f_v}(w)) = -(((fH(\mathbf{S}_{f_v})))^\perp)^\top. \quad (6.16)$$

6. Let $f \in P(G)$ be covirtual. Then equation (6.7) is also valid.
 Let $f \in P(G)$ be coactual. Then equation (6.9) is also valid.
7. Let $f \in P(G)$. Moreover, let $Ev(f)$ and $Od(f)$ denote the even and odd parts of f , respectively. These parts are related by the Hilbert transform as follows:
 If f is actual, then

$$Ev(f(x)) = (Od(f(x))) , \quad (6.17)$$

$$Od(f(x)) = (Ev(f(x))) , \quad (6.18)$$

and if f is virtual, then

$$Ev(f(x)) = -(Od(f(x))) , \quad (6.19)$$

$$Od(f(x)) = -(Ev(f(x))) . \quad (6.20)$$

8. If $f \in P(G)$ is covirtual, then (6.11) is also valid.
 If $f \in P(G)$ is coactual, then (6.13) is also valid.

It follows from the properties stated in this theorem that the transform introduced by Definition 6.5 can be regarded as the Hilbert transform for functions on finite non-Abelian groups into finite fields representing a proper counterpart of the Hilbert transform introduced by Definition 6.3 and, further, as a counterpart of the classical Hilbert transform in L^2 as well as the Hilbert transform on finite Abelian groups, see for example [1], [4].

Example 6.2 Let G be the group described in Example 4.2. In Table 6.3 the sign function, an actual function and its even and odd parts are shown. In Table 6.4 the Fourier spectrum of this function and in Table 6.5 of its field Hermitean and field skew-Hermitean parts are given. We define the field Hermitean and field skew-Hermitean parts of the Fourier spectrum respectively by $fH(\mathbf{S}_{f_a} = \mathbf{S}_{f_{ae}}$, and $fsH(\mathbf{S}_{f_a} = \mathbf{S}_{f_{ao}}$, where $f_{ae}(x) = \frac{1}{2}(f(x) + f(x'))$ and $f_{ao}(x) = \frac{1}{2}(f(x) - f(x'))$.

Note that $(\mathbf{S}_{f_a})^T = \mathbf{S}_{f_{ae}} - \mathbf{S}_{f_{ao}}$ and that formulas (6.15) and (6.16) are true.

Table 6.3. The even and odd parts of the test function.

x	x'	$\text{sign}_V(x)$	f_a	f_{ae}	f_{ao}
0	0	0	0	0	0
1	2	1	α	$\frac{\alpha}{2}$	$\frac{\alpha}{2}$
2	1	10	0	α	$-\frac{\alpha}{2}$
3	3	0	0	0	0
4	4	0	0	0	0
5	5	0	0	0	0
6	6	0	0	0	0
7	8	1	β	$\frac{\beta}{2}$	$\frac{\beta}{2}$
8	7	10	0	$\frac{\beta}{2}$	$-\frac{\beta}{2}$
9	9	0	0	0	0
10	10	0	0	0	0
11	11	0	0	0	0

$$POS_V = \{1, 7\}, \quad SYM_V = \{0, 3, 4, 5, 6, 9, 10, 11\}, \quad NEG_V = \{2, 8\}$$

Table 6.4. Fourier spectra of the test function.

w	\mathbf{S}_{f_a}
0	$\alpha + \beta$
1	$\alpha + \beta$
2	$\begin{bmatrix} 10\alpha + 10\beta & 6\alpha + 6\beta \\ 5\alpha + 5\beta & 10\alpha + 10\beta \end{bmatrix}$
3	$\alpha + 10\beta$
4	$\alpha + 10\beta$
5	$\begin{bmatrix} 10\alpha + \beta & 6\alpha + 5\beta \\ 5\alpha + 6\beta & 10\alpha + \beta \end{bmatrix}$

Table 6.5. Fourier spectra of the even and odd parts of the test function .

w	$\mathbf{S}_{f_{ae}}$	$\mathbf{S}_{f_{ao}}$
0	$\alpha + \beta$	0
1	$\alpha + \beta$	0
2	$\begin{bmatrix} 10\alpha + 10\beta & 0 \\ 0 & 10\alpha + 10\beta \end{bmatrix}$	$\begin{bmatrix} 0 & 6\alpha + 6\beta \\ 5\alpha + 5\beta & 0 \end{bmatrix}$
3	$\alpha + 10\beta$	0
4	$\alpha + 10\beta$	0
5	$\begin{bmatrix} 10\alpha + \beta & 0 \\ 0 & 10\alpha + \beta \end{bmatrix}$	$\begin{bmatrix} 10\alpha + \beta & 6\alpha + 5\beta \\ 5\alpha + 6\beta & 0 \end{bmatrix}$

References

- [1] Bljumin, S.L., Trakhtman, A.M., "Discrete Hilbert transform on a finite intervals", *Radiotekhnika in Elektronika*, No.7, 1977, 1390-1398, in Russian.
- [2] Butzer, P.L., Nessel, R.J., *Fourier Analysis and Approximation*, Birkhauser Verlag, Basel and Stuttgart, 1971.
- [3] Čížek, V., "Discrete Hilbert transform", *IEEE Trans.*, Vol. AU-18, No. 4, 1970, 340-343.
- [4] Moraga, C., Salinas, L., "On Hilbert and Chrestenson transform on finite Abelian groups", *Proc. 16th Int. Symp. on Multiple-Valued Logic*, Blacksburg, Virginia, U.S.A., 1986.
- [5] Moraga, C., Salinas, L., "On Hilbert and Chrestenson transforms on finite Abelian groups", *Scientia*, 1988.
- [6] Moraga, C., "On the Hilbert and Zhang-Hartley transform in finite Abelian groups", *Journal of China Institute of Communications*, (8), 3, 1987, 10-20.
- [7] Moraga, C., Stanković, R.S., "Hilbert transform in finite fields", *Proc. XXXVII Conf. for ETAN*, 20.-23.9.1993, Beograd, 59-64.

- [8] Stanković, R.S., Moraga, C., “Hilbert transform on finite non-Abelian groups”, in: Y. Baozong, Ed., *Proc. Int. Conf. on Signal Processing/Beijing'90*, International Academic Publishers, Beijing, China, 1990, 1179-1182.
- [9] Stanković, R.S., Stojić, M.R., Bogdanović, S.M., *Fourier Representation of Signals*, Naučna knjiga, Beograd, 1988, in Serbian.
- [10] Vlasenko, V.A., Shodin, O.I., *Microprocessors Systems for Quality Control of Digital Devices*, Tehnika, Kiev, 1990, in Russian.