A MULTIRATE APPROACH TO DDS ALGORITHM
VERSUS TAYLOR-SERIES EXPANSION TECHNIQUE
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ABSTRACT
We show that a look-up table (LUT)-based direct digital synthesizer (DDS) can be treated as a multirate algorithm. Two approaches to exploiting the LUT are considered and their performance compared. First of them applies a fractional delay filter in the flash-Farrow structure. The second approach uses phase rotation based on the Taylor-series expansion method. Experiments show that both approaches used in a quadrature DDS are capable of yielding the spurious free dynamic range SFDR ≥ 80 dBc. However, the flash-Farrow structure exhibits twice smaller numerical complexity expressed in terms of the number of multiplications than the structure of a Taylor-series based rotator of the same order N=2.

ABBREVIATIONS AND NOTATION
DDS direct digital synthesizer
FDF fractional delay filter
FIR finite impulse response
LUT look-up table
QDDS quadrature DDS
QO quadrature oscillator
ROM read only memory
SFDR spurious free dynamic range
R real numbers’ set
I integer numbers’ set

\[ \text{int}(x) = \lfloor x + \frac{1}{2} \rfloor \in \mathbb{I} \quad \forall x \in \mathbb{R} \] is the integer part of x (rounds x to the nearest integer towards minus infinity)

\[ \text{fra}(x) = x - \text{int}(x) \in [-1/2,1/2) \quad \forall x \in \mathbb{R} \] is the fractional part of x.

1. INTRODUCTION
Quadrature direct digital synthesizer (QDDS) plays an extremely important role especially in modern digital communications. It offers many advantages including precise and rapid manipulation of its output frequency, the ability to tune with extremely fine frequency resolution and to switch rapidly between frequencies.

Most QDDSs use a read only memory (ROM) look-up table (LUT) for function generation [1]. The concept of QDDS using a LUT of M samples of one period of a complex sinusoid (further called the base waveform) having the normalized angular frequency \( \omega_M = 2\pi / M \) [rad/Sa], where Sa stands for a sample, resolves to one relationship:

\[ s[n] = \exp(j\varphi[n]) = \exp(j\varphi_p[n]) ; n = 0,1,2,... \] (1)

Here \( s[n] \) stands for the desired complex signal (sequence) to be generated by the QDDS for a given input instantaneous frequency \( \omega[n] \). The angular frequency \( \omega[n] \) can be constant: \( \omega[n] = \omega_c \) with \( \varphi_c \in (0,\pi) \) or varying in time, with \( \omega[n] \in (-\pi,\pi) \). The accumulate

\[ \varphi[n] = \begin{cases} \varphi_0, & n = 0 \\ \varphi_0 + \sum_{k=1}^{\infty} \omega[k], & n = 1,2,3,... \end{cases} \quad \varphi[n] \in (-\infty,\infty)\forall n \] (2)

stands for the theoretical, numerically intractable, unwrapped phase at the output of an phase accumulator (PA, see Fig.1) driven by \( \omega[n] \) at the QDDS input, while

\[ \varphi_p[n] = 2\pi \text{fra}(\varphi[n]/2\pi), \quad \varphi_p[n] \in [-\pi,\pi)\forall n \] (3)

stands for the instantaneous phase used in the computation practice, wrapped to the interval \( \varphi_p[n] \in [-\pi,\pi) \) (principal phase wind). The sequence \( s[n] \) (1) can otherwise be written as

\[ s[n] = \exp(j\omega M \tau_p[n]) \] (4)

where

\[ \tau_p[n] = \varphi_p[n]/\omega_M, \quad \varphi_p[n] \in [-M/2,M/2) \] (5)

is a (virtual) phase-time in units determined by the time interval between samples of the base complex sinusoid stored in the LUT.

The delay \( \tau_p[n] \) can be decomposed into integer and fractional parts in the following way

\[ \tau_p[n] = m_p[n] + d[n] \] (6)

The integer part of \( \tau_p[n] \) for addressing the LUT is

\[ m_p[n]^\Delta = \text{int}(\tau_p[n]) \] (7)

where \( m_p[n] \in [-M/2,...,-1,0,1,...,(M-1)/2] \).
The fractional part of (6) for the correction of the result obtained from the LUT on the basis of (7), further called the fractional (subsample) time-delay [15], is

\[ d[n] = \text{fra}(\tau_p[n]), \quad d[n] \in [-1/2, 1/2) \]  

Consequently the sequence \( s[n] \) in (4) can be written as

\[ s[n] = \exp(j\omega_M (m_p[n] + d[n])) \]  

or equivalently in the factorized form used in Fig.1 as

\[ s[n] = \exp(j\omega_M m_p[n]) \exp(j\omega_M d[n]) \]  

In many applications, e.g. for portable communications, the power consumption of frequency synthesizer is very limited. One way to minimize the power of the QDDS is to minimize the size of the LUT. For the QDDS it is sufficient to store in the LUT only \( 8 \) complex samples. Ideally, the spectra of these sinusoids, both having the initial phase equal to zero, are: \( 2\pi\delta(\omega' - \omega_M) \) and \( 2\pi\delta(\omega'' - \omega_M) \). Here \( \omega', \omega'' \in [-\pi, \pi] \) are the spectral frequencies defined relative to the sampling intervals: for the LUT content \( \exp(j\omega_M m) \), thus for discrete-time enumerated using \( m \), and for the QDDS output \( \exp(j\omega_M n) \), thus for discrete-time enumerated using \( n \).

On the other hand, when \( \omega[n] \) is not a constant, thus when the QDDS is operating as an FM-modulator, the angular frequency \( \omega[n] \) becomes varying in time. Then we comply with a resampling with the time-variable ratio \( \omega[n]/\omega_M \).

In the second approach: B, based on (10), we deal with a phase rotation of the sample \( \exp(j\omega_M m_p[n]) \) from the LUT, by an angle (phase correction) equal to \( \omega_M d[n] \). This can be achieved, e.g., by using the Taylor-series expansion of the factor \( \exp(j\omega_M d[n]) \) [1], by direct linear interpolation [13], by CORDIC [14], or any other angle-rotation algorithm.

2. CLASSIFICATION

The importance of mathematically equivalent formulas (9) and (10) is twofold. They lead to two different approaches: A and B, to employing the LUT in the QDDS algorithm.

In the first approach: A, based on (9), a phase change of \( s[n] \) by \( \omega_M d[n] \) is performed, where \( d[n] \) is the fractional delay/advance relative to the integer phase address \( m_p[n] \) for the LUT. Conceptually this means a resampling of the base discrete-time complex sinusoid having a constant normalized angular frequency \( \omega_M \) into a complex sinusoid having the instantaneous angular frequency \( \omega[n] \). Such a resampling is done generally, i.e. for an arbitrary handlimited waveform, by means of a fractional delay filter (FDF) [15]. When \( \omega[n] = \omega_c \) is constant, the sampling rate conversion by a constant factor of \( \omega_c/\omega_M \) is accomplished. In this case the QDDS operating as a quadrature oscillator (QO), is in fact two-rate. While the normalized angular frequency of the complex sinusoid stored in the LUT is \( \omega_M \), the normalized angular frequency of a complex sinusoid synthesized by the QDDS is \( \omega_c \). Ideally, the spectra of these sinusoids, both having the initial phase equal to zero, are: \( 2\pi\delta(\omega' - \omega_M) \) and \( 2\pi\delta(\omega'' - \omega_M) \). Here \( \omega', \omega'' \in [-\pi, \pi] \) are the spectral frequencies defined relative to the sampling intervals: for the LUT content \( \exp(j\omega_M m) \), thus for discrete-time enumerated using \( m \), and for the QDDS output \( \exp(j\omega_M n) \), thus for discrete-time enumerated using \( n \).

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Fig.1. An idealized block-scheme of the LUT-based QDDS. In practice all waveforms are quantized which leads to an finite precision algorithm.

Fig.2. The flash-Farrow structure for the maximally flat (Lagrangian) FDF of order \( N=2 \) (15), (16).

Fig.3. The structure of a rotator (17b) of order \( N=2 \) based on the Taylor-series expansion technique.
However, our choice here is the Taylor-series expansion method as a reference for a comparison with the approach A.

In practice of the approach A, for a digital implementation of the fractional delay \( d[n] \) by application of an FDF of FIR type and of length \( N = 3,5,7,\ldots \), an concurrent access to \( N \) samples stored in the LUT is required. Well tailored for this aim is an algorithm, see Fig.2, for which we have coined the name flash-Farrow.

In contradiction to that, the second approach, B, requires from the LUT only one sample:

\[
W_m^{\omega} = \exp(j \omega m) \tag{11}
\]

Thus (10) can be rewritten shortly as follows

\[
s[n] = W_m^{\omega} W_m^{d[n]} \tag{12}
\]

In the approach A we deal with a multirate task in general. In particular, for the QDDS operating as a QO, it resolves to a two-rate problem with different input and output sampling frequencies. This is the case of most often met practical multirates.

3. LUT-BASED DDS AS A MULTIRATE SYSTEM

A further clarification is due to the reader. Text-book examples inculcate us with a conviction that a typical multirate/two-rate system has to have an input and an output like an interpolator or decimator. But in the LUT-based DDS operating as a QO the input resides in the LUT and in fact there is only an output. Then a question arises: how to discern that it is two-rate? Indeed, when the DDS is of LUT-less type and computes every output sample continually on the basis of the same rule as, e.g., in [12], there is no place for two rates. To the contrary, when the DDS is LUT-based, then the LUT is a source from which the algorithm gathers the samples of the stored base sinusoid having the angular frequency \( \omega_M \) in order to convert it into samples of the output sinusoid having the angular frequency \( \omega_c \neq \omega_M \). Then the algorithm we are dealing with is a sampling frequency converter – a typical representative of digital signal processing field known under the name of “multi-rate DSP” (cf. chapter 10 in [6]). The simplest algorithm for decreasing the sampling frequency, thus for increasing the angular frequency of the base sinusoid to be resampled, is an ordinary down-sampler by a factor of \( k \), commonly denoted as \( \downarrow k \). The down-sampler is used in the LUT-based DDS only when the angular frequency \( \omega_c \) of the target sinusoid is equal to the multiple of the angular frequency \( \omega_M \) of the base sinusoid stored in the LUT:

\[
\omega_c = k \omega_M \tag{13}
\]

and when the initial phase is \( \phi_0 = m_0 \omega_M \), with \( k, m_0 \in \{-M/2,\ldots,-1,0,1,\ldots,M/2-1\} \) for \( M \) even. A resampling algorithm for the generation of such a sinusoid is simply a sampling rate converter which yields at the DDS output every \( k \)-th sample of the base complex sinusoid from the LUT. In this case \( d[n] = 0 \forall n \) and \( m_p[n] \), the address for the LUT, can be calculated according to the following relationship

\[
m_p[n] = M \text{frac}(m_0 + kn)/M, \quad n = 0,1,2,\ldots \tag{14}
\]

Substitution of (14) for (7) means that for the generation of sinusoids having the angular frequency (13) the phase accumulator is unnecessary.

In a more general case, when \( \omega_c \neq k \omega_M \) or \( \phi_0 \neq m_0 \omega_M \) hence \( d[n] \neq 0 \), the ordinary down-sampling is insufficient. Here the LUT-based DDS operating as a QO has to be considered a two-rate algorithm converting the angular frequency \( \omega_M \) of the base sinusoid stored in the LUT into the angular frequency \( \omega_c \) of the sinusoid at the QDDS output (cf. chapter 10.5 in [6]). As can be seen from (9), in order to obtain the desired complex sinusoid, e.g., \( s[n] = \exp(j (\omega_c n + \phi_0)) \) with \( \omega_c \neq k \omega_M \) or \( \phi_0 \neq m_0 \omega_M \), each sample of the base sinusoid from the LUT having the address \( m_p[n] \) has to be fractionally delayed by the phase time: \( -d[n] \), and driven to the output.

The first ones who proposed a technique of sampling frequency conversion by using an FDF were R.Lagadec and H.O. Kunz in 1981 [7]. Later on T. Ramstad [8], [9] applied this technique to sampling frequency conversion with an arbitrary ratio and F.M. Gardner [10] – to synchronization of sampling instants in a digital modem.

4. THE FARROW APPROACH TO QDDS


In [11] Fig.2 scaled horizontally in both sampling intervals shows sampling instants corresponding to a sinusoid stored in the LUT and the samples and sampling instants for the DDS output. The authors also marked the fractional time-delay in this figure. The Farrow structure of order \( N=3 \) was in [11] treated as an adjustable piece-wise parabolic interpolator with a given base-point with an address \( m_k \), which corresponds to \( m_p[n] \) here, and with the fractional time-delay \( \mu_k \), corresponding to \( d[n] \) here. Finding these two parameters is realized in every clock cycle. Starting from this we can now modify the Farrow structure for the QDDS so as to make it flash.

It is important to note that at every clock instant \( n \) two operations are performed. The value of \( d[n] \) for the Farrow structure is updated and the structure is provided with three neighboring samples from the LUT with a new central sample (base point) having the address
Both above mentioned operations have to be realized simultaneously. A flash memory used for the LUT enables reading from different locations at the same time. Hence the time-delay blocks, $z^{-1}$, in the Farrow structure can be removed. The result is further called the flash-Farrow structure.

Fig.2 presents the flash-Farrow structure for the maximally flat (Lagrangian) FDF [2], [15] of order $N=2$. We have used here the maximally flat FDF rather than that invented in [10] and used in [11] because it is numerically more efficient and in the QDDS performs equally well. The input to the flash-Farrow structure is

$$ \mathbf{w}[n] = \{W_{m_1}^{(m_1)[n]}[n], W_{m_2}^{(m_2)[n]}[n], W_{m_3}^{(m_3)[n]}[n]\} $$  

(15)

The algorithm for this structure is given by

$$ \hat{s}[n] = W_{m_1}^{(m_1)[n]}[n] + d[n](\frac{1}{2} W_{m_2}^{(m_2)[n]}[n] + \frac{1}{2} W_{m_3}^{(m_3)[n]}[n]) $$  

$$ + d[n](\frac{1}{2} W_{m_2}^{(m_2)[n]}[n] - W_{m_3}^{(m_3)[n]}[n]) $$

(16)

where $\hat{s}[n]$ stands for an approxmate of $s[n]$. On the other hand, for the Taylor-series expansion technique we have used the following algorithm

$$ \hat{s}[n] = W_{m_1}^{(m_1)[n]}[n](1 + j\omega_M d[n](1 + \frac{j\omega_M d[n]}{2})(1 + \frac{j\omega_M d[n]}{3})... (1 + \frac{j\omega_M d[n]}{N})) $$

(17)

In particular from (17) we get for $N=1$

$$ \hat{s}[n] = W_{m_1}^{(m_1)[n]}[n](1 + j\omega_M d[n]) $$

(17a)

and for $N=2$, shown in Fig.3

$$ \hat{s}[n] = W_{m_1}^{(m_1)[n]}[n](1 + j\omega_M d[n](1 + \frac{j\omega_M d[n]}{2})) $$

(17b)

5. EXPERIMENTS AND MAIN RESULTS

It was interesting to assess the potential of the approach A, with an application pioneered in [11] based on a FDF, and to compare the results with the approach B based on the Taylor-series expansion technique [1], known from providing good results. Firstly we have it done analytically by resolving the appropriate algorithms: (16) and (17), to the form (12) with the following expressions for the estimates of (11) easy to handle:

$$ \hat{W}_{m_1}^{(m_1)[n]}[n] = 1 - d^2[n](1 - \cos \omega_M) + j d[n]\sin \omega_M $$

(18)

for an FIR maximally flat FDF of order $N=2$, thus of length 3 and

$$ \hat{W}_{m_1}^{(m_1)[n]}[n] = 1 + j\omega_M d[n] $$

(19a)

for $N=1$, thus for two-term Taylor-series approximation and

$$ \hat{W}_{m_1}^{(m_1)[n]}[n] = 1 - \frac{1}{2} \omega_M^2 d^2[n] + j\omega_M d[n] $$

(19b)

for $N=2$, thus for three-term Taylor-series approximation. The results obtained in floating point arithmetic in MATLAB are presented in Fig.4. Fig.4a shows the relative complex error magnitude as a function of $d[n]$

$$ \text{RCME}(d[n]) = \frac{20 \log_{10} \left| \frac{\hat{H}_{M}[n] - H_{M}[n]}{W_{M}[n]} \right| }{\pi} $$

(20)

and Fig.4b presents the phase error

$$ \text{PE}(d[n]) = \frac{180}{\pi} \left( \text{arg} \hat{H}_{M}[n] - \text{arg} H_{M}[n] \right) $$

(21)

As we can see from Fig.4 the relative complex error magnitude for the maximally flat FDF-based approximation of order $N=2$ computed in floating point arithmetic in MATLAB lies in between the Taylor-series expansion approximants: of order $N=1$, meaning linear interpolation [1], and of order $N=2$. The phase error magnitude for the maximally flat FDF-based approximation is greatest here. Note that this comparison is independent of the angular frequency $\omega_c$ of the complex sinusoid to be synthesized by the QDDS.

Further on we have examined the performance of the QDDS with quantization, as a QO of a complex sinusoid having constant angular frequency $\omega_0 = 2\pi \omega_c$, with $\omega_c = 1023/4096$; a situation which gives rise to spurious frequencies [11]. We have used a LUT storing only $M/8=8$ complex samples of the base waveform whose principal period included $M=64$ samples. We
have simulated in MATLAB the QDDS for approach A with the FDF from Fig.2 and the QDDS for approach B, based on the rotator from Fig.3 with an algorithm using the Taylor-series expansion. Both were aimed at obtaining the spurious free dynamic range SFDR ≥ 80 dBc. The quantization precision for the FDF in the flash-Farrow structure from Fig.2 in bits was: $b_1 = b_2 = 11$, $b_3 = 10$, $b_4 = 9$, $b_5 = 11$, $b_6 = 10$, $b_7 = b_8 = 11$ and $b_9 = 6$ while for the rotator from Fig.3 using the Taylor-series expansion technique was: $b_1 = 11$, $b_2 = 8$, $b_3 = 9$, $b_4 = 11$, $b_5 = 10$, $b_6 = b_7 = 12$ and $b_8 = 6$.

Hence our aim was achieved by using the maximally flat FDF of 3 real-valued coefficients ($N=2$) and by the rotator of the same order $N=2$. The result for the former (approach A) is shown in Fig.5 where the spurious peak level relative to the main spectral line of the desired complex sinusoid is -81.2 dBc. Thus the given requirement for the SFDR, defined as the spurious peak level but with opposite sign, is fulfilled. The other approach, B, with the rotator gave SFDR=80.8 dBc.

Note that the number of arithmetic operations performed on real-valued numbers per one sample for approach A, cf. Fig.2, is: 4 multiplications, 10 summations and 4 scaling operations, and for approach B, cf. Fig.3, is: 8 multiplications, 4 summations and 2 scaling operations.

### 6. CONCLUSIONS

In this paper we have shown that a LUT-based QDDS can be considered a two-rate system. Two approaches to exploiting the LUT have been considered. First of them applies a maximally flat FDF in the flash-Farrow structure. The second approach uses phase rotation exploiting the Taylor-series expansion of the complex factor $\exp(j\omega_M d[n])$. A comparison of these two approaches has been presented. Experiments have shown that QDDS algorithms for both these approaches are capable of yielding the spurious free dynamic range SFDR ≥ 80 dBc. However, the flash-Farrow structure exhibits twice smaller numerical complexity expressed in terms of the number of multiplications per output sample than the structure of the Taylor-series based rotator of the same order $N=2$.

In our experiments we have used the $\pi$ representation: $\pi \cong 355/113$ (Zu Chongzi; 430-501 AD [16]), accurate to 6 decimal places.

### REFERENCES


