Remarks on History of Abstract Harmonic Analysis

Radomir S. Stanković, Jaakko T. Astola1, Mark G. Karpovsky2
Dept. of Computer Science, Faculty of Electronics, 18 000 Niš, Serbia
1Tampere International Center for Signal Processing
Tampere University of Technology, Tampere, Finland
2Dept. of Electrical and Computer Engineering, Boston University
8 Saint Marry’s Street, Boston Ma 02215, USA

ABSTRACT
This paper reviews the development of abstract harmonic analysis, a mathematical discipline extending the classical Fourier analysis to more general settings.

1 INTRODUCTION
Fourier analysis represents a complex signal as the superposition (mostly linear combination) of simple signals that reflect the structure of the group $G$ on which the initial signal has been defined.

In this paper, we provide some of the highlights concerning developments of abstract harmonic analysis. We do not claim any originality of the exposition nor ability to write in the strict manner of an historian of mathematics. We simply want to collect a few facts from the history of abstract harmonic analysis.

2 TRIGONOMETRIC SERIES
The theory of trigonometric series can be dated back to the beginning of 18th century. Mathematicians of that time had been using trigonometric series, in particular for various astronomical calculations.

In 1729, Euler formulated and began to work on interpolation, the problem of determining function values in an arbitrary point $x$ if its values for $x = n$, where $n$ is an integer, are known.

In 1747, he applied the method he disclosed to a function $\phi$ derived from analysis of movement of planets, and represented $\phi$ in the form of a trigonometric series. This method derived in 1729, Euler published in 1753 [11]. This article actually contains what is now called the Fourier series, and Euler provided also formulae to determine the coefficients in the series by the integral of the function considered. Therefore, it may be stated justifiably that the trigonometric series of a function has been presented for the first time in 1750 to 1751.

In 1754, d’Alambert [9] considered the problem of representation of the reciprocal value of the mutual distance of two planets as a function of their position as a series in cosine functions. In this article, d’Alambert also provided formulae for determination of the coefficients of this series in terms of finite integrals.

In [11], [12], Euler derived trigonometric series of some functions in a way completely different from that he previously used. Similar results have been derived at about the same time by Lagrange [25] and Daniel Bernoulli [3].

It is interesting to notice that neither Euler nor Lagrange commented the interesting feature that a non-periodic function among the functions was considered. From a letter by Lagrange to d’Alambert dated on August 15, 1768 [26], it can be concluded that they realized this fact in the context of some other problems.

In 1757, Clairaut derived a cosine series representation of a function derived in a study of the movement of the Sun.

In 1777, in solving some astronomical problems, Euler determined the coefficients of the series representation of a trigonometric function by a method equivalent to that used nowadays [13].

In the research work discussed above and related publications there are examples of trigonometric series of various classes of functions. However, the essential question about the representation of an arbitrary function by a trigonometric series, remained unsolved until the work by J. B. Fourier, whose main ideas are contained in his book [14].

3 FOURIER AND HIS WORK
The fundamental contributions by Fourier are in mathematical physics. He studied the flow of heat between two regions of different temperatures. This question was very important in producing guns, therefore, a very important problem also for military authorities. The same problem had been discussed already by Sir Isaac Newton, who had provided an estimation of the temporal rate of cooling in terms of the difference between the temperature of an object and his environment. However, Newton had not determined the spatial rate of change, since it depends on several factors such as, for instance, the geometric shape of the object, the heat conductivity of the object, and the initial distribution of the temperature on its boundary. Fourier provided a solution of the problem by showing that the initial distribution of the temperature should be expressed as
Fourier viewed non-periodic functions as a limiting case of periodic functions with the period approaching infinity. In this case, the Fourier series is replaced by the Fourier integral that represents a continuous distribution of sine waves over all frequencies.

In 1807, Fourier completed his work "Propagation of Heat in Solid Bodies" and presented it before the French Academy of Science on December 12. The claim that any function defined in a finite closed interval can be represented in the form of a series of sine and cosine functions with suitably assigned coefficients, had not been completely accepted. The skeptical Referees were Laplace, Lagrange, Monge, and Lacroix. However, to encourage the author to continue and improve his research results, the Academy assigned as a subject for the award scheduled in 1812 the subject The Mathematical Theory of the laws of the propagation of heat and the comparison of the results of this theory with exact experiments. Fourier submitted a revised paper in 1811. The group of experts, among which were the previous Referees, awarded the memoir by Fourier. However, they criticized the lack of mathematical rigor and rejected the paper for publication in the Memoirs of the Academy.

However, in 1817, Fourier was elected to the Academy of Sciences, and became the permanent Secretary of the Division of Mathematical Sciences in 1822 when he published his award memoir Théoréme analytique de la chaleur, (Analytical Theory of Heat), which is widely considered as his major contribution to the mathematical physics. In 1826, Fourier became a member of the French Academy, and in 1826 followed Laplace in the position of the President of the Council for Improving the Polytechnic School. In 1828, Fourier was appointed as a member of a committee of French government for encouragement of literature.

The attitude of Fourier to the research and mathematics can be expressed by his often quoted remark Profound study of nature is the most fertile source of mathematical discoveries. However, exactly that was a source of much criticism of Fourier’s mathematical work by some mathematicians, for instance, Lagrange, Poisson, and Biot.

4 FURTHER DEVELOPMENT OF FOURIER ANALYSIS

The results by Fourier were expressed by Dirichlet and Riemann with stronger precision and formalism. The work by Dirichlet has been published in the Journal of the École Polytechnique from 1813 to 1823 and in Memoirs de l’ Académie for 1823. He also studied the Fourier integral.

Cauchy had shown that the work by Poisson on the convergence of Fourier series was non-rigorous. However, Dirichlet wrote that the proof provided by Cauchy "does not include certain functions for which the convergence is incontestable".

In 1799, Parseval published a formula for the sum of squares of the coefficients of a trigonometric series in terms of integrals, which is now called the Parseval theorem. This theorem can be viewed as a particular case of the Plancharel theorem.

The introduction of the Lebesgue integral in his PhD thesis in 1902, and discussed further in the book in 1904, provided foundations for formulation of the Riesz-Fischer theorem in 1907, showing that any square-summable sequence \( \{c_n\} \), for \( n \in \{-\infty, \infty\} \) is the sequence of Fourier coefficients of an \( L^2 \) function on the interval \( (-\pi, \pi) \), thus, Fourier coefficients provide an isometric linear mapping between two \( L^2 \) spaces.

In 1910, Plancharel proved a result which is called the Plancharel formula, which shows that the Fourier transform is an isometric mapping of \( L^2 \) into \( L^2 \).

5 ABSTRACT HARMONIC ANALYSIS

Harmonic analysis cannot be separated from theory of group representations, which are used as a basis replacing the role of exponential functions in classical Fourier analysis. In other words, harmonic analysis is an extension of the classical Fourier analysis derived by replacing the real line \( R \) by an arbitrary group \( G \). In this respect, there a distinction between the cases of Abelian and Non-Abelian groups.

The Fourier analysis on an Abelian group \( G \) is defined in terms of the corresponding group characters.
However, multiplicative characters are not sufficient to extend the Fourier analysis to non-Abelian groups and in this case group representations are required. They can be viewed as generalizations of multiplicative characters by increasing the dimension. Notice that for Abelian groups all the representations are single-dimensional and reduce to group characters.

For finite groups, generalizations are possible by allowing to replace the complex field by any field the characteristic of which is relatively prime to the order of the group $G$. The modular theory due largely to R. Brauer, removes this restriction to the characteristic of the field.

The group $G$ and the vector space $V$ are often topologized and the group action is normally assumed to be continuous. When $G$ topologized, for discussion of abstract harmonic analysis, the following topological groups should be distinguished compact, Locally compact, and Non-compact groups.

Abstract harmonic analysis is a branch of harmonic analysis that extends the definition of the Fourier transforms for functions defined on various groups.

5.1 Group Characters

In 1882, Heinrich Weber introduced multiplicative characters for an arbitrary finite Abelian group $G$.

The definition of a group character was discussed in the late 1870s by Dedekind [2]. He defined a character on a finite Abelian group $G$ to be a homomorphism from $G$ to the multiplicative group of nonzero complex numbers, and the orthogonality relations had been previously discovered. Dedekind also defined what he called the group determinant and noticed that it can be factored nicely, when the group is Abelian. Dedekind conjectured that this factorization can be extended to non-Abelian groups. In that respect, in 1896, Dedekind communicated with Frobenius, who had published in the same year a paper on group characters and presented these results to the Berlin Academy on July 16, 1896. It should be noticed that in this paper Frobenius did not relate the group characters to the group representations. However, Frobenius continued this research based on a paper by Dedekind from 1885, further supported by the communication with Dedekind that began on April 12, 1896 by a letter of Dedekind to Frobenius. In this communication, many interesting results can be found. For instance, in a letter by Frobenius to Dedekind on April 26, 1896, Frobenius presented the irreducible characters for the alternating groups $A_4$, $A_5$, the symmetric groups $S_4$, $S_5$, and the group $PSL(2,7)$ of order 168.

Due to this work, in 1897, Frobenius introduced the notion of group characters. After studying the work of Molien [30], [31], and reformulation some of these results in terms of matrices, Frobenius showed that the group characters defined by him in 1897 are traces of irreducible representations. In a letter to Dedekind on February 1924, Frobenius said that Molien investigated non-commutative multiplication and obtained general results from which the properties of group determinants can be derived as special cases.

It is interesting to notice that Molien studied the results by Frobenius in group theory and applied them to investigate polynomial invariants of finite groups. In particular, Molien studied how many times a given irreducible representation of a finite group appears in a complete reduction of the representation of the group on the vector space of homogeneous polynomials of degree $n$ over the complex numbers. In 1898, Molien introduced a generating function to compute the number of times the irreducible character occurs.

In 1898, Frobenius introduced the notion of induced characters and the tensor product of characters, and a theorem called now the Frobenius Reciprocity Theorem.

In 1900 and 1901, Frobenius completely determined characters of the symmetric and alternating groups, respectively, published in two separated papers. Further advent in application of group characters, Frobenius provided by studying the structure of groups called nowadays the Frobenius groups.

The theory of groups characters developed by Frobenius, was nicely presented by Burnside in [4].

5.2 Group Representations

In their work started in 1904 and 1905 respectively, Burnside and I. Schur [34], Vol. 1, discussed matrix representations, i.e., homomorphism into the group of invertible matrices of a given dimension. In their approach, group representations were complex-valued column vectors and the linear transformations are viewed as matrices.

Burnside is often credited as a founder of the theory of finite groups and his work complemented and sometimes compete with the work by Frobenius.

Schur was a student of Frobenius and made a considerable contributions individually and together with Frobenius.

In 1925, Schur returned to the group representation theory due to the development in theoretical physics and provided a complete description of the rational representations of the general linear group.

However, Emmy Noether replaced the matrices by linear transformations of a vector space, and therefore, her definition of the group representations is equal to that used nowadays. This approach to the definition of group representations is necessary when infinite-dimensional representations are required, as for instance, the Lie groups.

Burnside pointed out that in the case of finite groups every finite-dimensional representation is equivalent to a representation by unitary matrices and the complete reducibility follows from the unitarity. Burnside also

\[2\text{Notice that a multiplicative character } \chi_w(x) \text{ is a representation on the single-dimensional space } C \text{ of complex numbers, and the action by an element } g \in G \text{ is the multiplication by } \chi_w(g).\]
pointed out that if \( Q \) is a mapping between irreducible representations in two spaces \( V_1 \) and \( V_2 \), then \( Q = 0 \) or \( Q \) is invertible. Schur had shown that if \( V_1 = V_2 \), then \( Q \) is a scalar. Schur also proved the orthogonality of inequivalent irreducible unitary representations of finite groups.

Frobenius introduced the notion of induced representations as a way to define a representation \( R \) of a group \( G \) from a representation \( R_t \) of a subgroup \( G_t \) of \( G \).

6 HARMONIC ANALYSIS ON FINITE GROUPS

The harmonic analysis on finite groups is performed in terms of irreducible unitary representations, or their characters, for non-Abelian and Abelian groups, respectively. This approach has been developed first for the symmetric and alternating groups in the work by Frobenius and Young, who introduced the Young diagrams for manipulating with irreducible representations.

6.1 Finite Abelian groups

Notice that when \( G \) is an Abelian groups, the set of group characters \( \chi_w(x) \) forms a multiplicative group \( \Gamma \) isomorphic to \( G \). Therefore, a function \( f(x) \) on a finite Abelian group \( G \) of order \(|G| = g \) can be represented as

\[
f(x) = g^{-1} \sum_{w \in \Gamma} S_f(w)\chi_w(x),
\]

where \( \Gamma = \{ \chi_w(x) \}, x \in G \) is the set of characters of \( G \), and

\[
S_f(w) = \sum_{x \in G} f(x)\chi_w(x)^{-1}.
\]

6.2 Finite Non-Abelian groups

In the case of finite non-Abelian groups the Fourier transform is defined in terms of finite-dimensional irreducible unitary representations \( R_w(x), x \in G \), as

\[
f(x) = \sum_{w=0}^{K-1} Tr(S_f(w)R_w(x)),
\]

where \( K \) is the number of equivalence classes of unitary irreducible representations which classes form the dual object \( \Gamma \) for \( G \), and \( Tr(Q) \) denotes the trace of a square matrix \( Q \).

The Fourier coefficients are \( (r_w \times r_w) \) matrices, where \( r_w \) is the dimension of the representation \( R_w \),

\[
S_f(w) = r_w g^{-1} \sum_{u=0}^{g-1} f(u)R_w(u^{-1}),
\]

where \( g \) is the order of \( G \).

Finite groups are compact groups, and definition of Fourier transform is a simplified version of the Fourier transform for arbitrary compact groups.

7 COMPACT NON-ABELIAN GROUPS

Extensions of Fourier analysis to compact non-Abelian groups are due to the Peter-Weyl theorem formulated by H. Weyl and his student and associate F. Peter, first for the case of non-Abelian Lie groups \([32]\). The main contribution consists in the observation that not the finiteness of a group ensures existence of main properties of the Fourier representations, but existence of an averaging procedure over the group \([19]\). In other words, it is required the existence of an invariant integral that assigns a finite volume to the group. In this case, the Haar integral plays an important role.

In the case of non-Abelian groups it is necessary to distinguish the left and right invariance. For instance, an integral on a topological group \( G \) is the right invariant if \( \int_G f(xa)dx = \int_G f(x)dx \), for all \( a \in G \).

It is proved by Haar in 1933 that a right invariant integral exists for locally compact groups. This integral is now called the Haar integral. Notice that local compactness is implied by the existence of a right invariant integral as shown by Andre Weil in his book \([35]\).

The main idea by Peter and Weyl, which provides possibility to extend the abstract harmonic analysis, was to use an infinite dimensional representation and its decomposition by means of spectral theory for bounded operators on Hilbert space \([32]\).

In short, for compact non-Abelian groups the Peter-Weyl theorem explains determination of harmonics as representatives of each equivalence class of representations. From each equivalence class of representations, a representation is selected as a harmonic to define an analogue to the classical Fourier transform.

More precisely, the Peter-Weyl theorem for compact groups shows that the Fourier series of a function \( f \) on \( G \) is

\[
f(x) = \sum_{w \in \Gamma} r_w \sum_{i,j=0}^{r_w-1} S_f^{(i,j)}(w)R_w^{(i,j)}(x),
\]

where \( \Gamma \), the dual object of \( G \), is a collection of all equivalence classes of irreducible unitary representations \( R_w \) of \( G \).

The Fourier coefficients are determined as

\[
S_f^{(i,j)}(w) = \langle f, R_w^{(i,j)} \rangle = \int_G f(x)(R_w^{(i,j)})^{-1}(x)dx.
\]

In the case of compact Abelian groups, by the Schur lemma, the irreducible representations are single-dimensional and, thus, the dual object \( \Gamma \) is the dual group of all continuous homomorphisms \( \chi_w \) of \( G \) into the unit circle. Then, the Fourier series for \( f \) on \( G \) is

\[
f(x) = \sum_{\chi \in \Gamma} S_f(\chi)\chi_w(x),
\]

and the Fourier coefficients are numbers

\[
S_f(w) = \int_G f(x)\chi_w^{-1}(x)dx.
\]
From 1923 to 1938 Weyl developed the theory of compact groups in terms of matrix representations. In the case of compact Lie groups, he proposed a fundamental character formula.

There are compact groups that are not Lie groups, however, the representation theory and, therefore, harmonic analysis on such groups are highly incomplete.

8 LOCALLY COMPACT ABELIAN GROUPS

To discuss the harmonic analysis on locally compact groups, recall that the real line $R$ is a locally compact Abelian group. The Fourier integral $S_f$ of a function $f$ on the real line $R$, defined for all real numbers $w$ by

$$S_f(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx,$$

is an example of the Fourier transform on locally compact Abelian groups. The constant $1/2\pi$ can be viewed as the normalization of the Haar integral on $R$.

Notice that for $f$ integrable over the real line, i.e., $f \in L^1$, the spectrum $S_f$ is well defined. However, the integrability of $f$ does not imply the integrability of $S_f$, with integrability understood in the Lebesgue sense. Therefore, generalized methods of summability are required.

If $f \in L^2$, i.e., $f$ is both integrable and square integrable, then $S_f$ is also square-integrable and $f$ is equal to the Fourier integral in the means-square sense, i.e., the Plancherel formula is valid

$$\int_{-\infty}^{\infty} |S_f(w)|^2 dw = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$  

The classical Fourier analysis on $R$ has been extended to an arbitrary locally compact Abelian group $G$ due to the Pontryagin duality, also called Pontryagin-van Kampen duality which can be briefly summarized as follows.

For a locally compact Abelian group, the set of unitary multiplicative characters under the pointwise multiplication expresses the structure of a locally compact Abelian group $G$. This group, when topologized with the topology of uniform convergence of compact sets is the dual group for $G$. The group $\hat{G}$ has also a dual group, called dual dual $\hat{\hat{G}}$ [21], since there is a canonical continuous homomorphism of $G$ into $\hat{\hat{G}}$, i.e., if $x \in G$, then the corresponding member of $\hat{\hat{G}}$ evaluated on a character $\chi_w \in \hat{\hat{G}}$ has the value of $\chi_w(x)$.

The Pontryagin duality states that this homomorphism $G \to \hat{\hat{G}}$ is a homeomorphism, i.e., a topological isomorphism, of $G$ onto $\hat{\hat{G}}$.

This result has been exploited by Andre Weil [35] to define the Fourier transform pair for functions $f$ on $G$ as

$$S_f(w) = \int_G f(x) \overline{\chi_w(x)} dx,$$

where $dx$ and $dw$ are suitably normalized Haar integrals on $G$ and $\hat{G}$, respectively.

Thus, the inversion formula is valid for integrable continuous function $f$ whose Fourier transforms are integrable.

The foundations for the theory of locally compact Abelian groups and their duality has been established by Lev Semenovich Pontryagin in 1934. In his approach it was exploited the structure theory and assumed that the group is second countable and either compact or discrete. This was imposed to cover the general locally compact Abelian groups by E.R. van Kampen in 1935 and Andre Weil in 1953.

It has been shown by Rudin that the duality theorem and harmonic analysis can be establish without referring to the structure theory [33].

9 NON-COMPACT NON-ABELIAN GROUPS

For compact groups, either Abelian or non-Abelian, the Fourier transform has been defined in terms of finite-dimensional irreducible unitary representations. In the case of compact Abelian groups, the representations are single-dimensional. For locally compact Abelian groups, the representations are again single-dimensional. However, for locally compact non-Abelian groups, the irreducible infinite-dimensional representations are required.

Notice that, in general, a non-compact group $G$ may have representations that are not unitarizable in a Hilbert space.

It may be said that for non-Abelian groups which are not compact, there is no a general theory that would preserve at least some of the properties of the classical Fourier transform, as for example, the Plancherel theorem. However, many particular cases are considered, for example, $SL(n,F)$, in which case the representations of infinite dimensions are used.

In a series of publications, Harish-Chandra discussed extensions of harmonic analysis to noncompact real semi-simple Lie groups, providing also the Plancherel theorem in 1952 [20]. This work was preceded by the research done by Gelfand and Raikov in 1943, pointing out that in principle, there should exists a sufficient number of irreducible representations to perform harmonic analysis on locally compact groups [18].

For instance, G. W. Mackey used the notion of induced representations to deal with measure-theoretic foundations for infinite-dimensional representation theory [28], [29].

The work by A.A. Kirilov, started in his doctoral thesis in 1962, [22] provided a basis for the work by L. Auslander and C.C. Moore [1], and B. Konstant for extension of harmonic analysis to some solvable groups.
Some brief reviews of these results can be found in [19] and [23].

References