DEVELOPMENT OF METHODS FOR SPECTRAL ESTIMATION OF NOISY HARMONIC SIGNAL PARAMETERS

Gamlet S. Khanyan

Central Institute of Aviation Motors named after P. I. Baranov,
2, Aviamotornaya st., Moscow, 111116, RUSSIA, Khanian@mail.ru

ABSTRACT

The paper deals with statement and solving a metrological problem that arises when one is developing spectral analysis applications in the field of measurement of the amplitude-phase-frequency characteristics of oscillation processes having different physical nature.

The paper is a summary aimed at describing parametric estimation methods elaborated by the author [1-4], in their interrelation and in comparison to other methods known in the art [5-12].

Exacting theoretical grounds are provided for deterministic part of the methods, i.e., where signal idealization is performed. At the same time empirical ideas put forward in order to allow for the stochastic nature of the processed data are being checked through experiments.

1. INTRODUCTION

Spectral analysis based on the discrete Fourier transform (DFT) is one of the digital signal processing methods, which has been thoroughly developed and has numerous fields of application. However, it is hardly possible to assume that all theoretical and practical issues involving this powerful varying processes study tool have been dealt with. The above is also true for the task of estimating parameters of harmonic oscillations mixed with more or less intense noise, which is confirmed by a large number of publications devoted to that subject from the middle of the previous century until the present time.

Among the papers, whose far-from-complete list is given in References, one should single out a monograph by C. Lanczos [5], in which the harmonic analysis tasks related to spectral estimation were specified before the appearance of fast Fourier transform (FFT) algorithm.

The well-known and much cited article by F. J. Harris [6] is devoted to the weighing window method – the most frequently applied means for smoothing spectral leakage in order to raise the peak and suppress lateral directional lobes of the amplitude spectrum. However, versatility of weighing windows and variety of related notions (equivalent band width, coherent gain, etc.) often give rise to ambiguous conclusions on that method’s universality [8, 11].

A large number of papers are dedicated to frequency measurement, namely, to determining the dimensionless fractional adjustment to the address of the maximum amplitude spectrum peak. Proposed solutions of that task can be classified in two directions – the adjustment determination proceeding from the relationships between the amplitude spectrum components [8-12] and the same adjustment estimation using relationships between the phase spectrum components. The latter direction is described in the review by M. L. Fowler [7], in the context of noise impact on the estimation error.

The present report is aimed at correlating methods elaborated by the author in [1-4]. Naturally, the content of those papers has a lot in common with essential methodological aspects of spectral estimation methods more or less known in the art, including those cited in the paper. Its distinctive peculiarity is the approach to the problem, which might be called analytical. Wherever possible, the method developed for practical use is to be substantiated by rigorous theory. Numerical modeling should be made use of to check the methods efficiency under real conditions in order to take into account distorting factors.

The main methods proposed in the current paper are the phase-differential method for estimating the noisy harmonic signal frequency and the spectral power density focusing method for smoothing spectral leakage.

Both proposed methods [2, 4] are based on an analytical research [1], which has made it possible to derive for the first time precise formulas of amplitude and phase spectra of a harmonic signal of finite duration. Due to those formulas complexity, application of their metrological properties for parameter measurement has only proved possible in the initial approximation, but sufficient enough to emphasize advantages of the formula-based methods.

An advantage of the phase-differential method is its resistance to intense noise impact, and of the focusing method – retention of the best spectral resolution typical for the rectangular window.

2. PARAMETER ESTIMATION ERRORS

Errors in the signal parameter estimation feature different origins and different consequence.

For instance, if one does not make use of specific Fourier transform results processing methods, then frequency can only be measured to a spectral resolution precision – at the amplitude peak address. In most practical cases such estimation is considered to be quite satisfactory. For instance, vibration of an aircraft engine body under cruising conditions can be regarded as
a random stationary process with harmonic components dominating the aerodynamic noise, their frequencies being multiple of rotor rotation frequencies and stable under normal conditions of engine operation. Therefore, the necessary accuracy of the rotor vibration analysis is ensured in practice by meeting the requirement for the harmonic frequencies during the process realization to vary by no more than a spectral resolution. At the same time, not every task of the engine’s dynamic strength and gas-dynamic stability can be solved by suitable choice of spectral resolution. Such hazardous phenomena as flutter of vanes, surging and combustion instability arise swiftly, and to prevent them one needs high accuracy of establishing diagnostic frequencies, amplitudes, and phase differences.

There are a number of causes which distort the sinusoidal amplitude, and their nature is complex. Apart from the superposition of noise components, the most significant of those causes is the spectral leakage due to the aliquant ratio of the signal frequency to the spectral resolution. The effect consists in the peak reduction as compared to the true amplitude value and in the appearance of false lateral components on both sides of the peak.

The amplitude estimation is also affected by the edge effect, i.e., superposition of the negative frequency domain onto the positive one as the peak approaches the left-hand or the right-hand spectrum edge.

For polyharmonic processes, oscillation interference is a major distorting factor. Although the Fourier transform is a linear operation with regard to the signal realization, the oscillation amplitude is a quadratic value, the superposition principle being limited by a certain critical distance between the frequencies of harmonics.

Errors due to phase distortion have the most complicated mathematical nature [3, 7].

3. THEORETICAL BACKGROUND

In most practical spectral analysis applications, the polyharmonic process mixed with noise and a constant component may be considered as a physically substantiated model:

\[ s(t) = c + \sum a_i \cos(2\pi f_i t + \varphi_i) + b(t); \quad T_{ini} \leq t < T_{fin}. \] (1)

The process is observed in a time window with set (known) initial \( T_{ini} \) and final \( T_{fin} \) time moments.

The digital realization of process

\[ s_n = s(t_n); \quad t_n = (n + \eta)\Delta t; \quad n = N_0, N_0 + 1, ..., N_0 + N - 1; \quad 0 \leq \eta < 1 \] (2)

is made up of \( N \) samples obtained for a sampling interval \( \Delta t = T/N \), where \( T = T_{fin} - T_{ini} \) is duration, and \( N_0 \) – number of the first sample of realization. Displacement \( \eta \) is introduced for parameterization of all digital realizations located within the specified time window. A fixed value of \( \eta \) corresponds to a certain discrete signal sampling.

It is assumed that process (1) is proceeding in a finite frequency band whose width is equal to half of sampling frequency \( F_s = 1/\Delta t \). Frequencies of harmonic components \( f_i \) need not be equidistant and need not be divisible by the spectral resolution \( \Delta f = 1/T \). All other process components, which do not have a deterministic harmonic nature, can be included in noise \( b(t) \), whose intensity is sufficiently large, but not big enough for the components with amplitudes \( a_i \) to “drown” against the background of peaks of the spectrum with randomly arising frequencies.

Noise described by normal distribution is the most natural one from the point of view of physics and has been thoroughly studied in mathematical statistics. In most studies it is that noise which is adopted as a suitable model of process \( b(t) \). In the current paper the type of probability distribution of a noise digital realization \( b_n \) is not of principal importance for testing the proposed algorithms for estimating frequencies and amplitudes of the signal harmonics. The only task set is a preliminary estimation of measurement error (its maximum value per the group of the treated process realizations in a numerical experiment), for which the uniformly distributed noise coming from the pseudo-random number generator is suitable.

3.1. View window index

In order to specify the process view window, it is convenient to introduce a dimensionless parameter – the window index

\[ h = \frac{T_{fin} + T_{ini}}{2T}, \] (3)

so that \( hT \) represents the middle of the analog realization of process (1). We can demonstrate that the initial sample number \( N_0 \) of realization (2) is expressed by index \( h \) and displacement \( \eta \) in the following way:

\[ N_0 = \left\lceil -\left( h - \frac{1}{2}\right)N + \eta \right\rceil. \] (4)

Here and further on brackets stand for an integer part (floor) of the expression they contain.

In this fashion, the process view window is characterized by three parameters \( h, \eta \) and \( N_0 \). Those parameters determine radically the process phase spectrum structure, whereas their influence on the amplitude spectrum type is negligible. Those two features formed the basis of the phase-differential frequency determination method developed in [4].

Generally speaking, the \( \eta \) index is a characteristic of the analog process realization \( s(t) \), whose spectral analysis is performed through its expansion into a Fourier series with an infinite number of coefficients

\[ S_m = \frac{1}{T} \int_{(h-1)T}^{(h+1)T} s(t)e^{-2\pi imt/T} dt; \quad m = 0, \pm 1, \pm 2, ... \] (5)
The spectral function of digital realization (2) is calculated with the help of N-point DFT

\[ S_m = \frac{1}{N} \sum_{n=0}^{N-1} s_n e^{-j2\pi mn/N}, \quad m = 0, 1, ..., N - 1, \] (6)

whose summing limits (allowing for the time resolution \( \Delta t \)) are within limits of integration (5). Transition from the analog realization to the digital one (accordingly, from the Fourier series to DFT) leads to modification of the index \( h \) – its replacement with parameter

\[ H = \frac{\sum_{n} + \sum_{n-N}^{N-1}}{2T} = \frac{N_0 + \eta}{N} + \frac{1}{2} - \frac{1}{2N}, \] (7)

which is the middle of the digital signal realization rated according to the window duration \( T \).

### 3.2. Calculation of amplitude and phase spectra

Spectra of amplitudes \( A_m \) and initial phases \( \Phi_m \) of digital realization \( s_n \) are usually linked to spectral function \( S_m \) by the following definitions:

\[ A_m = 2 |S_m|, \quad \Phi_m = \arg S_m; \quad m = 0, 1, ..., M, \] (8)

where \( M = [N/2] \) is the length of discrete spectrum (a dimensionless analog of Nyquist frequency).

According to the theory of complex numbers, the amplitude spectrum is expressed through real and imaginary parts of the spectral function as:

\[ A_m = 2 \sqrt{\text{Re} S_m^2 + \text{Im} S_m^2}. \] (9)

The phase spectrum formula has two branches. The first, main branch is obtained for \( \text{Re} S_m \times \text{Im} S_m \neq 0 \) and describes phase angles not multiple to \( \pi/2 \):

\[ \Phi_m = \text{arctan} \frac{\text{Im} S_m}{\text{Re} S_m} + \frac{\pi}{2} (1 - \text{sgn} \text{Re} S_m) \text{sgn} \text{Im} S_m. \] (10)

The second branch – for angles multiple to \( \pi/2 \), is obtained for \( \text{Re} S_m \times \text{Im} S_m = 0 \) and can be written as:

\[ \Phi_m = \frac{\pi}{2} (1 - \text{sgn} \text{Re} S_m) \text{sgn} \text{Im} S_m - \text{sgn} \text{Re} S_m. \] (11)

In practice DFT (6) is performed by the FFT method, and it is not difficult to program formulas (9)-(11), which are a result of spectral processing of the \( s_n \) data sequence entered in the PC, since those formulas do not contain any mathematical peculiarities (division by zero, etc.).

However, theoretical research of the signal \( s_n \), whose model is specified by an analytical expression, encounters the difficulty of spectra \( S_m, A_m \) and \( \Phi_m \) calculation by formulas (6), (9)-(11). Problems arise when special cases of the signal parameters are being considered separately or combined. So, the main difficulty of the phase spectra calculation is to determine the sign of a real and imaginary part of the spectral function \( S_m \). To overcome that difficulty, we propose a mathematical technique used for deriving the harmonic signal phase spectrum formula [1]. It consists in presenting the real non-negative amplitude in a complex form, which is obligatorily governed by the system of relationships

\[ A_m = 2S_m e^{-j\Phi_m} = |A_m| = \text{Re} A_m \geq 0, \quad \text{Im} A_m = 0 \] (12)

comprising a non-strict inequality concerning the real part and an equality concerning the imaginary part \( A_m \).

A more detailed system (12) has the following form:

\[ \begin{cases} \text{Re} S_m \cos \Phi_m + \text{Im} S_m \sin \Phi_m \geq 0 \\ \text{Im} S_m \cos \Phi_m - \text{Re} S_m \sin \Phi_m = 0. \end{cases} \] (13)

Then, the main phase value (arc tangent) is determined from equality (13), and the multiple \( \pi \) complement is found by solving the inequality. Both added phases have a complicated dependence on the signal parameters, as a result of which the expression for phase determination contains addsends multiple to \( 2\pi \). Nevertheless, the phase, as is the case in formulas (10)-(11), can be always considered to be contained between \( \pm \pi \), if we make use of the formula for reducing the arbitrary angle \( \Phi \) to its principal value \( \phi \):

\[ \phi = \Phi + 2\pi \frac{\pi - \Phi}{2\pi}, \quad -\infty < \Phi < +\infty, \quad -\pi < \phi \leq \pi. \] (14)

### 4. HARMONIC SIGNAL OF FINITE DURATION

Before studying such a general process as (1), we shall consider in details its elementary component – a harmonic oscillation of amplitude \( a_0 \), frequency \( f_0 \) and phase \( \phi_0 \):

\[ s(t) = a_0 \cos(2\pi f_0 t + \phi_0); \quad a_0 > 0, \quad 0 \leq f_0 \leq F/2, \quad -\pi < \phi_0 \leq \pi. \] (15)

It will be more convenient to conduct further analysis if we adopt a two-parameter presentation of the signal frequency as a sum

\[ f_0 = (m_0 + \mu_0) \Delta f; \quad m_0 = \left\lfloor -f_0 \tau + 1/2 \right\rfloor; \quad 0 \leq m_0 + \mu_0 \leq N/2; \quad -1/2 < \mu_0 \leq 1/2, \] (16)

where \( m_0 \) is an integer closest to the dimensionless frequency \( f_0 \tau \), and \( \mu_0 \) is a fractional adjustment to the bin. According to a further analysis, \( \mu_0 \) is a critical parameter determining the spectrum structure and, as well as the principal object of measurement in techniques for all other signal parameters estimation.

The spectral function of digital realization

\[ s_n = a_0 \cos \left( 2\pi \frac{(m_0 + \mu_0)(n + \eta)}{N} + \phi_0 \right) \] (17)

of signal (15) is a result of DFT (6) calculation

\[ S_m = \frac{a_0}{2} \sum_{j=1}^{N} e^{j(2\pi/m_0 - j\mu_0)(n + \eta)} a_{j\mu_0}, \] (18)

where parameter \( H \) is the modified index \( h \), which is determined just by formula (7).

Function \( a_{j\mu_0} \) of discrete frequency variable \( m \) with \( j=1 \) in (18) describes the essential part of the spectrum, which was determined for positive frequencies, and with \( j=-1 \) – the mirror part of the spectrum (in negative frequency domain). For \( \mu_0 \neq 0 \) it has the following form:
that solution \( \Psi_m \) has two branches, which differ by the value of discriminant \( \mu_0 \sin 2 \psi_0 \).

The main branch corresponds to \( \mu_0 \sin 2 \psi_0 \neq 0 \) and has the following form:

\[
\Psi_m = \psi_s + \pi \left( \frac{\psi_m - \psi_s}{\pi} - \frac{2 \mu_0 \sin \mu_0}{\pi} + m_0 - m \right),
\]

where the main phase value is specified by expressions

\[
\psi_s = \arctan \left( \frac{\tan \frac{m \pi}{2}}{\tan \frac{m_0 \pi + \mu_0}{N}} \right), \quad m \neq N/2;
\]

\[
\psi_s = \pi/2, \quad m = M = N/2.
\]

The second branch of the formula corresponds to \( \mu_0 \sin 2 \psi_0 = 0 \):

\[
\Psi_m = \psi_s + \pi \left| \mu_0 \right| \left( \left| \mu_0 \right| \sin (m_0 - m) - m_0 - m \right).
\]

It covers all specific cases of combinations of the harmonic signal parameters, including those of them which cause the amplitude \( A_m \) to be equal to zero:

\[
\begin{align*}
\mu_0 &= 0, \quad m \neq m_0; \\
m &= 0, \quad \psi_0 = \pi/2 + \pi K; \\
m &= M = N/2, \quad \psi_0 = \pi K; \quad K = 0, \pm 1, ...
\end{align*}
\]

The final expression for the phase spectrum samples \( \Phi_m \) through the signal parameters \( \varphi_0, \mu_0, m_0 \) and parameters of its view window \( h, \eta \) is obtained when one uses intermediate notations (22), (7), (4).

4.2. Initial approximation

Components \( \alpha_m \) and \( \alpha_{-m} \) make an unequal contribution to spectral function \( S_m \). In the vicinity of the signal dimensionless frequency \( m \approx m_0 \) with \( m_0 \) sufficiently removed from the spectrum edges 0 and \( M \), the value of the mirror component \( \alpha_{-m} \) is negligibly small as compared to the essential component \( \alpha_m \).

Thus, for the spectrum length \( M = 128 \), in the interval \( 25 < m_0 < M - 25 \), the relation of those components is \( |\alpha_m/\alpha_{-m}| < 1\% \). Then, assuming \( \alpha_{-m} = 0 \), we obtain the initial approximation of formulas (23)-(26). The complex amplitude (21) in this case is dramatically simplified, and the system (12) for phase spectrum yields exactly the same solution as (26), but without the requirement to satisfy the condition \( \mu_0 \sin 2 \psi_0 = 0 \).

Formula (26) for the combination of parameters

\[
h = 0, \pm 1, \ldots; \quad \eta = 1/2; \quad H = h; \quad N = 2M
\]

having the most significant meaning can be reduced to the following explicit form

\[
\Phi_m = \varphi_0 + \pi(2h \mu_0 + [\mu_0 \sin(m_0 - m)] + m_0 - m).
\]

The amplitude spectrum in the initial approximation

\[
A_m = a_0 | \alpha_m | = \frac{a_0}{\pi} \sin \frac{\pi \mu_0}{m_0 + \mu_0 - m}, \quad \mu_0 \neq 0;
\]

\[
A_m = a_0 \delta_m, \quad \mu_0 = 0
\]
can be derived with additional simplification – passing into limit \( N \to \infty \) in the denominator of the expression for \( \alpha_m \). Formula (30) can be used for \( N \geq 256 \) not only in the immediate vicinity of the signal frequency, but within the entire frequency range [1]. Although, when \( m \) moves away from \( m_0 \), the relation \( |\alpha_m/\alpha_{-m}| \) approaches unity, both these values \( \alpha_m \) are so small as compared to unity, that it makes no difference which of them to make use of.

5. SPECTRAL LEAKAGE

Let us consider the amplitude spectrum (30) in the immediate vicinity of the dimensionless signal frequency:

\[
A_{m_s} = a_0 \sin \frac{\pi \mu_0}{\pi h}, \quad A_{m_{s1}} = a_0 \sin \frac{\pi |\mu_0|}{\pi(1 + \mu_0)}.
\]

We can see that the amplitude spectrum maximum is reached at the bin \( m = m_0 \), and adjacent components are positioned on the left and on the right of the maximum in an asymmetrical way: with \( \mu_0 > 0 \), the right neighbor is bigger than the left one, with \( \mu_0 < 0 \) – it is the opposite thing. If \( \mu_0 = 1/2 \), the right neighbor is not distinguishable from the maximum peak, whose value is \( 2a_0/\pi = 0.64a_0 \), i.e., it is understated as compared to the true value of the signal amplitude by 36% only. With \( \mu_0 < 0 \), the maximum peaks is equal to the signal amplitude \( a_0 \), and adjacent components are equal to zero.

The amplitude drop is widely known as the spectral leakage [6]. We can see from (31) that it is due to a single parameter – correction \( \mu_0 \) to the dimensionless signal frequency.

Leakage also considerably distorts the phase at the amplitude spectrum peak (peak phase):
There is no distortion in a separate case $\mu_0=0$, where no leakage effect is present, or it is totally absent when the process is viewed in the window with index $h=0$, i.e., within $-T/2 \leq t < T/2$ time limits. When the process is observed in the window with index $h=1/2$, i.e., within $0 \leq t < T$ time limits, the phase distortion due to the leakage effect reaches the value of $\pi \mu_0$. With the maximum spectral leakage ($\mu_0=1/2$) that means measurement of the harmonic signal phase may result in a sinusoid being mistaken for cosinusoid!

Therefore, the most natural window for viewing the process is the window with index $h=0$, when the middle of process realization is assumed to be the initial moment of time. However, in practice spectral analysis is conducted “by default” in the viewing window with index $h=1/2$ and shift $\eta=0$. In this case $N_0=0$, and Fourier transform (6) is an ordinary DFT.

Technique for conducting spectral analysis in the window with index $h=0$, which makes it possible to eliminate the spectral leakage influence on phase measurement, is described in detail in paper [3]. A calculation method called alternating consists in changing places of the first ($0 \leq n < M$) and second ($Mn < n$) halves of the digital realization (2) of process (1) before performing DFT and other digital signal processing operations.

6. ASSESSING SIGNAL PARAMETERS

Relationships (31)-(32) make it possible to solve in principle the fundamental metrological spectral analysis task – to measure frequency, amplitude, and phase of a harmonic signal following the DFT data.

To determine the signal frequency

$$ f_0 = (m_0 + \mu_0) \Delta f $$

for a specified spectral resolution $\Delta f$, one needs to find two parameters – address of the maximum peak $m_0$ and correction to the address $\mu_0$. The first of those parameters is sought by sorting out and comparing the amplitude spectrum components $A_m$. The correction is found by excluding the sine from expressions (31) for known adjacent components:

$$ \mu_0 = \frac{A_{m+1} - A_{m-1}}{A_{m+1} + A_{m-1}}. $$

With the help of the known correction from the maximum peak formula (31) we find the harmonic signal amplitude:

$$ a_0 = \frac{\pi \mu_0}{\sin \pi \mu_0} A_m. $$

With specified window index $h$, the initial phase of signal is found from (32) by the peak phase value:

$$ \varphi_0 = \Phi_m - 2 \pi h \mu_0. $$

Accuracy of formulas defining frequency (33), restoring amplitude (35), and compensating the phase (36), depends on how precisely the correction $\mu_0$ was determined. Papers [2-4] show that formula (34) features precision which is high enough for a clean and slightly noisy harmonic signal. However it possesses an obvious edge effect, i.e., error increases as $m_0$ approaches the spectrum edges, and when the parameter $\mu_0$ itself approaches zero. Apart from that, error grows when the length of digital realization $N$ diminishes.

6.1. Amplitude techniques for frequency correction

Formula (34), whose derivation is given in the present paper, is known in the art and is used for specifying the harmonic signal frequency both in the pure state (for the raw DFT data), and in combination with weighing windows or by padding the digital realization with zeros [8].

From the methodological point of view, formula (34) should be considered as a part of whole family of the “amplitude technique” formulas, whose right sides comprise the amplitude spectrum components. In some modifications of those formulas the maximum peak is used along with adjacent components or in place of one of the adjacent components [9, 11]. Frequently $A_m$ is considered to signify samples not of the amplitude spectrum, but their squares [12], sometimes logarithms [10] or real [8] and/or imaginary parts of DFT [5]. This variety of approaches has arisen due to various areas of spectral analysis application and is justified by the fact that there is an opportunity to choose a suitable data processing technique for an adequate treatment of the physics of the phenomenon that has given rise to them.

All amplitude methods have in common an approximately similar sensitivity of the peak amplitude to noise. The nature of that sensitivity is also approximately the same – the root-mean-square value of a random amplitude measurement error grows in proportion to the noise intensity (noise-to-signal ratio). This conclusion is based on the results of numerical modeling [2] and is of qualitative nature. A more detailed investigation of the noise influence, along with determining the law of probability distribution with regard to the frequency adjustment, was carried out in paper [12].

6.2. Phase-differential method

Paper [4] sets forth a new phase-differential method of frequency determination, which is characterized by a high accuracy and is free of the drawbacks of amplitude methods cited above.

The essence of the method is that the correction $\mu_0$ is determined by the samples of the phase spectrum of two consecutive $N$-point digital realizations of process $s(t)$ with indices $h=0$ and $h=1$:

$$ s_0^{(n)} = s(n \Delta t - T/2), \quad s_1^{(n)} = s(n \Delta t + T/2); $$

$$ n = 0, 1, ..., N - 1. $$

The couples of amplitude and phase spectrums

$$ A_m^{(h)} = 2 | S_m^{(h)} |, \quad \Phi_m^{(h)} = \text{arg} S_m^{(h)}; $$

$$ m = 0, 1, ..., M; \quad h = 0, 1 $$

(37)
calculated by means of DFT (6) and corresponding to the data (37) are processed in the following way. Sorting out and comparing amplitude values, we find addresses of peaks $m^{(0)}$ and $m^{(1)}$. In case of a pure harmonic signal these addresses coincide and are equal to a certain $m_0$ number. For a noisy signal this coincidence is assured, and for moderate noise it is almost always implemented in practice, if the absolute value of parameter $\mu_0$ is not too close to $1/2$. With $|\mu_0| \approx 1/2$, or in case of intensive noise $b(t)$, the maximum peak address may be shifted into one of neighboring addresses $m_0 \pm 1$. Such a «miss» by a whole bin will lead to the frequency measurement result being $f_0 \pm \Delta$ instead of $f_0$. To avoid this, additional information is required, and it is available in three components of the phase spectrum, which are located in the vicinity of the amplitude peak. It follows from formula (29) that two of those components are similar and differ from the third one by $\pi$. For $\mu_0 \neq 0$ the peak phase and one of its neighbors turn out to be the same:

$$\Phi_{m_0+\mu_0} = \Phi_0 + 2\pi \mu_0 + \pi;$$

$$\Phi_{m_0} = \Phi_{m_0+\mu_0} = \Phi_0 + 2\pi \mu_0.$$  

(39)

With $\mu_0 = 0$ the adjacent components coincide:

$$\Phi_{m_0} = \Phi_0,$$

$$\Phi_{m_0+1} = \Phi_0 + \pi.$$  

(40)

The algorithm, which implements the phase-differential method, in each of the two phase spectrums (38) reveals two closest components of the three indicated in (39) or (40), checks out their difference from the third component by $\pi$, and determines presence or absence of the above-mentioned miss by $\pm 1$ bin. After establishing the true address of peak $m_0$, parameter $\mu_0$ is determined by the difference of peak phases of realizations with indices $h=1$ and $h=0$ excluding $\Phi_0$ from the two consecutive applications of formula (32):

$$\mu_0 = \frac{\Phi^{(1)}_{m_0} - \Phi^{(0)}_{m_0}}{2\pi}.$$  

(41)

### 6.3. SPD focusing method

The amplitude restoration formula (35) has limited application. It has been established that it is operating reliably for a monochromatic signal, as well as for a polyharmonic signal, whose harmonic components are spaced by a considerable distance from each other (over 32 bins). If the components’ frequencies are close or there is noise, these factors affect the amplitude calculation accuracy $\Delta_0$ stronger than the leakage effect.

Under these conditions, it is advisable to estimate amplitude by the method of spectral power density (SPD) focusing, which is described in paper [2]. The method’s concept was prompted by the content of the Parseval equality theorem, known in the series theory, and suggests quadratic amplitude summation in the vicinity of each local spectrum maximum. Efficiency of this idea is borne out by the fact that the quadratic sum of three components (31)

$$a = \sqrt{A_{m_0-1}^2 + A_{m_0}^2 + A_{m_0+1}^2} = a_0 \sin \mu_0 \sqrt{1 + 3\mu_0^2}$$  

(42)

already contains, depending on the $\mu_0$ value, from 92.5% ($|\mu_0| = 1/2$) to 100% ($\mu_0 = 0$) of the $a_0$ amount.

The method has been implemented as a fast algorithm, which is universal in relation to the structure of the spectral data. Its execution time is approximately an order less than the time of the fast Fourier transform processing. The algorithm breaks the SPD data array into adjacent sectors whose borders are addresses of local spectrum minimums, calculates the sum of such SPD components on each sector, and then zeroes the components, and puts the sum in the place of the local spectrum maximum belonging to that sector.

The algorithm operation diagram applied to the process model (1) is shown in Figure 1 (Snm are SPD samples or squares of the amplitude spectrum $A_m$ samples).

### 7. NUMERIC MODELLING RESULTS

Numeric experiments were conducted in order to verify the developed methods of spectral estimation of noisy harmonic signal parameters.

We modeled couples of $N$-point digital realizations with indices $h=0$ and $h=1$ of $T = 1$ s duration each:

$$s_n^{(h)} = d^{(i)} \cos(2\pi f^{(i)} t_n + \phi^{(i)}) + b^{(i)}r_n; \quad |r_n| \leq 1;$$

$$t_n = n\Delta t + (h-1/2)T; \quad n = 0, 1, ..., N-1; \quad l = 1, ..., L.$$  

(43)

The parameters of the process (43) in the three experiments described below changed according to a program as functions of number $l$ of the next couple of realizations. The harmonic oscillation amplitude was set constant for all $L$ couples of realizations: $d^{(i)} = 1$, so that parameter $b^{(i)}$ represented the noise intensity – ratio of the amplitude (half span) of the noisy process component to the signal amplitude. Noise $r_n$ was gener-
ated by uniformly distributed pseudo-random numbers

procedure. The initial harmonic oscillation phase
gchanged by 1° after the next couple of realizations per

formula \( \phi(l) = \pi(l-1)/180 \).

7.1. Signal frequency measurement experiment

In that experiment, with the realization length \( N=256 \)
and total number of realization couples \( L=500 \), the di-
mensionless signal frequency increased according to the
linear law \( f^{(l)} = N/8 + (l-1)/L \), changing from 32 to 33
bin, i.e., in the band with a spectral resolution width

\( \Delta' = 1 \text{ Hz} \), thus ensuring the passing of the entire range

of possible values of parameter \( \mu_0 \) (at first, from 0 to 1/2
with \( m_0=32 \), then from -1/2 to 0 with \( m_0=33 \)).

Noise intensity grew according to a stepwise law

\( b^{(l)} = [10(l-1)/L]/4 \) – augmenting by 0.25 units every 50
realization couples (during the first 50 couples a pure si-
nusoid was generated).

Figure 2 demonstrates the nature of the modeled pro-
cess (43). It shows amplitude spectrums of slightly and
intensively noisy signals obtained at the initial and final
stages of the given experiment – for \( l = 75 \) and \( l = 450 \),
respectively.

The oscillation frequency was calculated by three
methods: the phase-differential method (41), the ampli-
tude method (34), and without correction (with accuracy
to spectral resolution). Frequency characteristics ob-
tained in that manner are given in Figure 3.

Figure 3. Phase-differential method
as compared to the amplitude method

Measurement data obtained through the phase-
differential method practically lie on the programmed
frequency line, whereas the amplitude method data expe-
rience fluctuations towards both sides of that line, which
reach the maximum measurement error values of ±1 bin
determined by the structure of formula (34), while the
noise intensity increases. In a similar case, the maximum

phase-differential method error did not exceed 0.035
bin. Frequency measured with a spectral resolution pre-
cision is governed by a well-known law: it jumps when
passing the bin middle.

Such a weak noise sensitivity of the phase-
differential method is hard to explain even proceeding
from exact formulas (24)-(26) analysis. At the same time
the method application along with those formulas
modeling shows promise for solving the task of the sig-
nal separation from noise.

7.2. Comparison of amplitude methods

Conditions under which that experiment is conducted are
similar to those of the previous experiment. Frequency in
(43) also changed within one bin – this time from 16
to 17 Hz, with \( N=128 \) and \( L=256 \). Noise intensity grew
according to a linear law \( b^{(l)}=0.625(l-1)/L \).

The amplitude (in the same series of realizations)
was also assessed by three methods: the SPD focusing
method, the rectangular window application (i.e., with-
out adjustments), and the application of a weighing
function

\[ w_n = \gamma(a + \beta \cos 2\pi n / N); \quad a + \beta = 1; \quad \gamma^2 = a^2 + \beta^2 / 2 \quad (44) \]

whose parameters \( a=0.54, \beta=0.46 \) correspond to the
Hemming window, and the scaling factor \( \gamma \) ensures that
the RMS value of the weighed signal \( w_n s_n \) is equal to the
effective amplitude \( 0.707 a_0 \) of the raw signal \( s_n \) for the
entire family of weighing windows (44). The processing
results are given in Figure 4.

Figure 4. SPD focusing method
as compared to the weighing window method

The compared amplitude estimation methods are no-
ticeably different in terms of the error caused by spectral
leakage. It can be seen that weighing to a great extent
smooths that negative effect, which is maximal for the
rectangular window. The problem is only the proper
choice of the scaling factor \( \gamma \). Focusing eliminates spec-
tral leakage almost completely, without influencing
spectral resolution, which, as is well known, is the best
at the rectangular window.

Random error of the compared methods is approxi-
mately the same – in terms both of its value, and the na-
ture of noise dependence (the amplitude fluctuation
grows in proportion to noise intensity).
7.3. Edge effect investigation

The experiment was made in order to confirm another advantage of the phase-differential method – its weak sensitivity to the edge effect. A pure harmonic signal \( (b_0=0) \) was modeled with a small \( (N=64) \) and critical \( (N=8) \) digital realization length (in the latter case the amplitude spectrum includes but three samples). The dimensionless frequency passed the entire frequency range from zero to Nyquist frequency \( M = N/2 \) with a step \( M/L \) during \( L = 128 \) realization couples. Diagrams in Figure 5 show the measurement error – difference between the rated and measured frequencies in bins.

![Figure 5](image)

Figure 5. Impact of the edge effect and realization length on frequency measurement accuracy:

I – phase-differential method; II – amplitude method

It can be seen that the digital realization length \( N=64 \) is critical for the amplitude method (34), chosen for comparison with the phase-differential one. Oscillation of the methodological error is so great that one can only speak of the frequency measurement in the central part of the spectrum. This phenomenon (edge effect) can be explained by the fact that for the given and smaller values of \( N \) the mirror component \( \alpha_m \) in the exact formula for the amplitude spectrum (23) cannot be neglected.

As for the phase-differential method, the edge effect is considerable only in the region of the first and last bin of the spectrum, no matter what the digital realization length \( N \) is. That means that the phase-differential method can be used to measure the frequency of a wave, which is made up of just one oscillation.

8. ACKNOWLEDGMENTS

The study was carried out with support provided by Russian Foundation for Basic Research (RFBR grant № 05-08-17946).

9. CONCLUSIONS

Methods for spectral estimation of noisy harmonic signal parameters have been elaborated, which are based on a rigorous analytical research [1] of amplitude and phase spectrums of signal.

Numeric experiments have been conducted, which confirmed the methods’ efficiency in a wide range of varying signal and noise parameters.

A comparison to other methods for spectral estimation of signal parameters [6, 8-12] has been performed.

The phase-differential method of frequency estimation is characterized by accuracy, which considerably exceeds spectral resolution, and by low susceptibility to noise, edge effects and digital realization length. Therefore, it can be used to pick out high- and low-frequency oscillations, as well as for transient processes treatment – under conditions of the signal frequency varying fast in time.

The SPD focusing method for the harmonic signal amplitude estimation to a great extent smoothes off the spectral leakage effect, without exercising any impact on spectral resolution.

The developed spectral estimation methods do not require significant calculation resource, do not contradict each other, and can be applied jointly in various fields of digital signal processing.

10. REFERENCES


